MEC-E8001

Finite Element Analysis

2023

WEEK 3: DISPLACEMENT ANALYSIS

2 DISPLACEMENT ANALYSIS

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LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems related to displacement FEA:

- ☐ Engineering paradigm in FEM, elements and nodes, nodal quantities and sign conventions.

- ☐ Displacement analysis of simple structures by using the virtual work expressions of the elements.

- ☐ Calculations of the element contributions of force, solid, beam, and plate elements out of virtual work density of the model and element approximation.

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

PREREQUISITE: FUNDAMENTAL LEMMA OF VARIATION CALCULUS

The fundamental lemma of variation calculus in one form or another is an important tool in FEM. The lemma tells how to deduce the equilibrium equations of a structure using a virtual work expression and the principle of virtual work:

- $u, v \in \mathbb{R}$: $vu = 0 \ \forall v \quad \Leftrightarrow \quad u = 0$

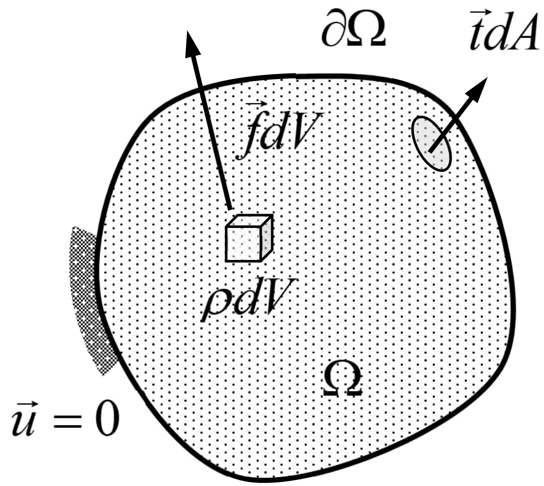
- $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$: $\mathbf{v}^{\mathrm{T}}\mathbf{u} = 0 \ \forall \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} = 0$ 🡐

- $u, v \in C^0(\Omega)$: $\int_\Omega uv d\Omega = 0 \ \forall v \quad \Leftrightarrow \quad u(x, y, \ldots) = 0 \ \text{ in } \ \Omega$

In mechanics of the materials, variable or function v is (usually) chosen as the kinematically admissible variation of displacement δu.

2.1 LINEAR ELASTICITY

Assuming equilibrium of a solid body (a set of particles) inside domain Ω, the aim is to find displacement $\vec{u}$ of the particles, when external forces or boundary conditions are changed in some manner:



Equilibrium equations $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in Ω,

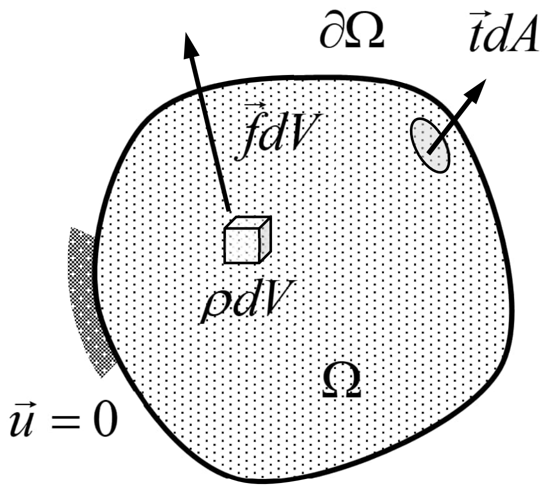
Hooke's law $\vec{\vec{\sigma}}=\dfrac{E}{1+\nu}(\dfrac{\nu}{1-2\nu}\vec{\vec{I}}\nabla\cdot\vec{u}+\vec{\vec{\varepsilon}})$ in Ω,

Boundary conditions $\vec{n}\cdot\vec{\vec{\sigma}}=\vec{t}$ or $\vec{u}=\vec{g}$ on $\partial\Omega$,

The balance law of angular momentum is satisfied 'a priori' by the form of Hooke's law.

PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta\vec{u}$ is a concise representation of the boundary value problem. In terms of virtual work densities δw^{int} , δw_V^{ext} , and δw_A^{ext}



Internal forces: $\delta W^{\text{int}} = \int_\Omega \delta w_V^{\text{int}} dV$

External forces: $\delta W^{\text{ext}} = \int_\Omega \delta w_V^{\text{ext}} dV + \int_{\partial\Omega} \delta w_A^{\text{ext}} dA$

Although the two representations are equivalent, principle of virtual work combines the equations in a way which is the key for multiple important applications in mechanics. Finite element method is just one of them.

DENSITY EXPRESSIONS

Virtual work densities (virtual work per unit volume or area) of the internal forces, external volume forces, and external surface forces are

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

The terms of the expressions consist of work conjugate pairs of kinematic and kinetic quantities.

GENERALIZED HOOKE'S LAW

The model $g(\vec{\sigma},\vec{u})=0$ for isotropic homogeneous material can be expressed, e.g., in its compliance form as

Strain-stress: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \dfrac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} = [E]^{-1}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$ and $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \dfrac{1}{G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$

Strain-displacement: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix}$ and $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$

Above, E is the Young's modulus, ν the Poisson's ratio, and $G = E/(2+2\nu)$ the shear modulus. Strain and stress are symmetric (the matrix of components is symmetric).

2.2 DISPLACEMENT ANALYSIS

- Model the structure as a collection of elements (solid, plate, beam). Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

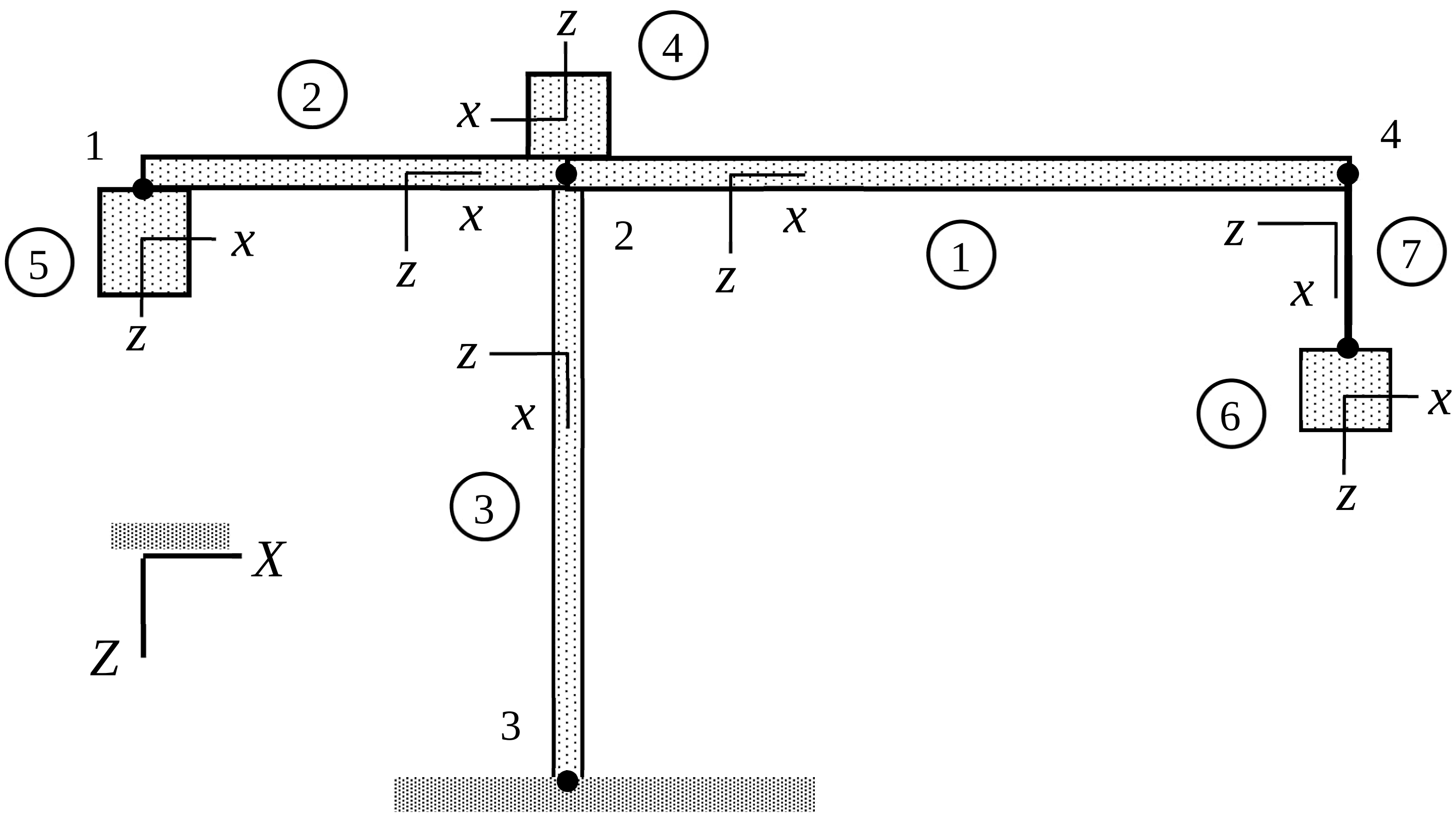
- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get the "standard" form $\delta W = -\delta \mathbf{a}^{\mathrm{T}} (\mathbf{Ka} - \mathbf{F}) = 0$.

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the linear equation system $\mathbf{Ka} - \mathbf{F} = 0$.

- Solve the equations for displacements and rotations $\mathbf{a}$.

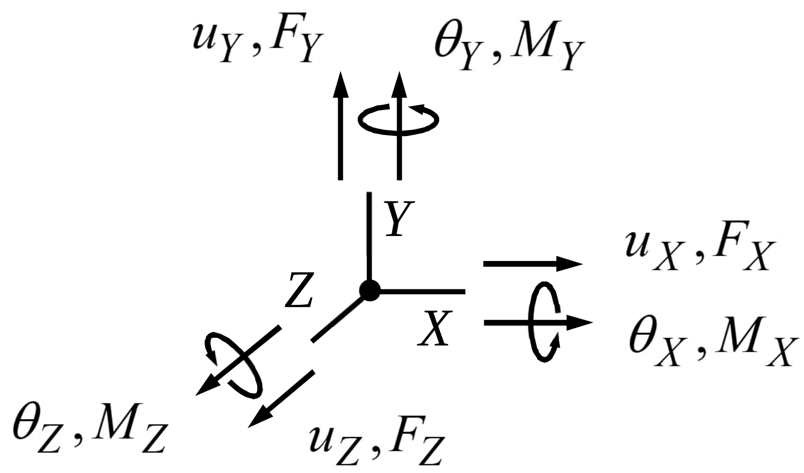
FINITE ELEMENT ANALYSIS

A complex structure is modelled as a collection of structural parts (or elements) modelled as rigid bodies, beams, plates, or solids. Elements are connected by nodes.



KINEMATIC AND KINETIC QUANTITIES

The primary quantities of analysis are displacements, rotations, forces and moments at the connection points of the structural parts. The components of the vector quantities (magnitude and direction) are taken to be positive in the directions of the coordinate axes.



Vector quantities are invariants in the sense $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = a_X\vec{I} + a_Y\vec{J} + a_Z\vec{K}$, and can be transformed from one coordinate system to another using the property.

SIGN CONVENTIONS AND NOTATIONS

Displacements, rotations, forces and moments are vector quantities whose components are positive in the directions of the chosen coordinate axes. The convention may differ from that used in mechanics of materials courses (be careful with that).

	Displacement	**Force**	**Rotation**	**Moment**
Material	u_x, u_y, u_z	F_x, F_y, F_z	$\theta_x, \theta_y, \theta_z$	M_x, M_y, M_z
Structural	u_X, u_Y, u_Z	F_X, F_Y, F_Z	$\theta_X, \theta_Y, \theta_Z$	M_X, M_Y, M_Z

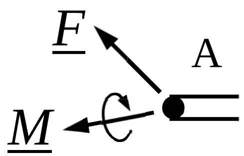
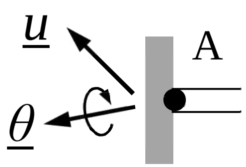
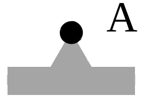
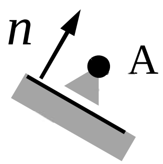
The basis vectors of the material and structural systems are $(\vec{i}, \vec{j}, \vec{k})$ and $(\vec{I}, \vec{J}, \vec{K})$, respectively!

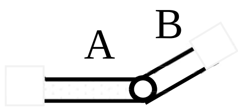
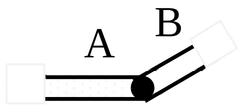
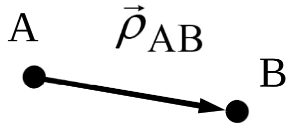
- In FE calculations, one needs to express displacement and rotation components in the material coordinate system in terms of those in the structural coordinate system. Expressing $\vec{i},\vec{j},\vec{k}$ (basis vectors of the material coordinate system) in terms of $\vec{I},\vec{J},\vec{K}$ (basis vectors of the structural coordinate system) and coordinate system invariance in form $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}=a_X\vec{I}+a_Y\vec{J}+a_Z\vec{K}$, one obtains

$$\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=\begin{bmatrix}i_X & i_Y & i_Z\\ j_X & j_Y & j_Z\\ k_X & k_Y & k_Z\end{bmatrix}\begin{Bmatrix}\vec{I}\\ \vec{J}\\ \vec{K}\end{Bmatrix} \text{ and } \begin{Bmatrix}a_x\\ a_y\\ a_z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=\begin{Bmatrix}a_X\\ a_Y\\ a_Z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{I}\\ \vec{J}\\ \vec{K}\end{Bmatrix}\Rightarrow$$

$$\begin{Bmatrix}a_x\\ a_y\\ a_z\end{Bmatrix}=\begin{bmatrix}i_X & i_Y & i_Z\\ j_X & j_Y & j_Z\\ k_X & k_Y & k_Z\end{bmatrix}\begin{Bmatrix}a_X\\ a_Y\\ a_Z\end{Bmatrix} \text{ or } \begin{Bmatrix}a_X\\ a_Y\\ a_Z\end{Bmatrix}=\begin{bmatrix}i_X & i_Y & i_Z\\ j_X & j_Y & j_Z\\ k_X & k_Y & k_Z\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}a_x\\ a_y\\ a_z\end{Bmatrix}. \Leftarrow$$

INTERACTION MODELS

name	symbol	equations
force		$\vec{F}_{\mathrm{A}} = \underline{F}, \; \vec{M}_{\mathrm{A}} = \underline{M}$
fixed		$\vec{u}_{\mathrm{A}} = \underline{u}, \; \vec{\theta}_{\mathrm{A}} = \underline{\theta}$
joint		$\vec{u}_{\mathrm{A}} = 0, \; \vec{M}_{\mathrm{A}} = 0$
slider		$\vec{n} \cdot \vec{u}_{\mathrm{A}} = 0, \; \vec{F}_{\mathrm{A}} - (\vec{F}_{\mathrm{A}} \cdot \vec{n})\vec{n} = 0, \; \vec{M}_{\mathrm{A}} = 0$

joint		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}}, \ \vec{M}_{\mathrm{A}} = 0, \ \vec{M}_{\mathrm{B}} = 0$
fixed		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}}, \ \vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$
rigid		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}, \ \vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$

Interaction models define a kinematic quantity (displacements and rotations) or its work conjugate (forces and moments). In practice, only the kinematic conditions need to be imposed explicitly.

BEAMS, PLATES AND SOLIDS

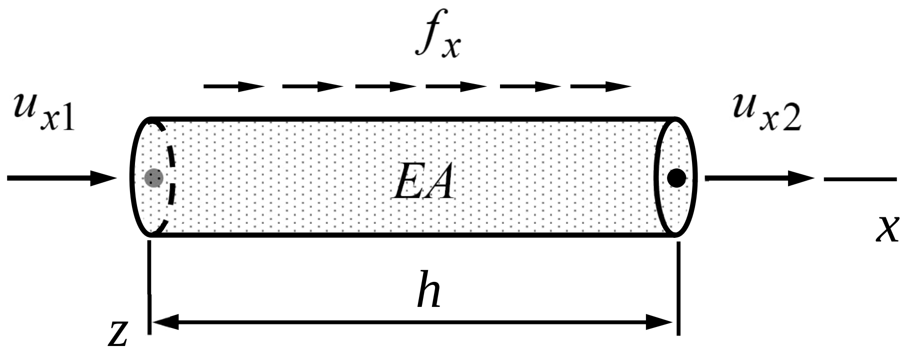
Elements of the structure may be modelled as rigid bodies, beams, plates, or solids or their simplified versions considering only the active loading modes, i.e., *bar*, *torsion*, and *bending* modes for the beam model and *thin slab* and *bending* modes for the plate model:

Beam: $\delta W = \delta W_{\text{bar}} + \delta W_{\text{tor}} + \delta W_{\text{xz-bnd}} + \delta W_{\text{xy-bnd}}$

Plate: $\delta W = \delta W_{\text{slb}} + \delta W_{\text{bnd}}$

The simple expressions above assume a clever positioning of material coordinate system and, thereby, uncoupling of the loading modes. Then one may treat the modes in the same manner as the elements of the structure (virtual work expression is obtained as the sum over the elements and the loading modes of them).

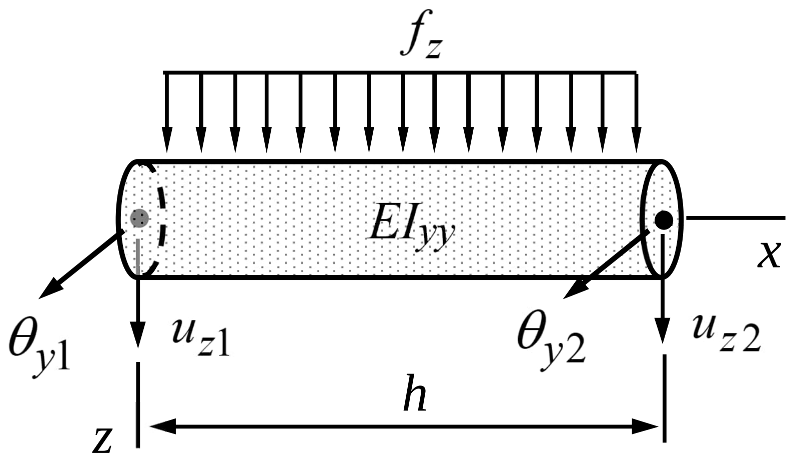
BAR MODE



$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right)$$

Above, f_x, E, and A are assumed to be constants. In terms of the unit vector in the direction of the x–axis $u_x = \vec{i} \cdot \vec{u} = i_X u_X + i_Y u_Y + i_Z u_Z$ and $\delta u_x = \vec{i} \cdot \delta\vec{u} = i_X \delta u_X + i_Y \delta u_Y + i_Z \delta u_Z$.

BENDING MODE



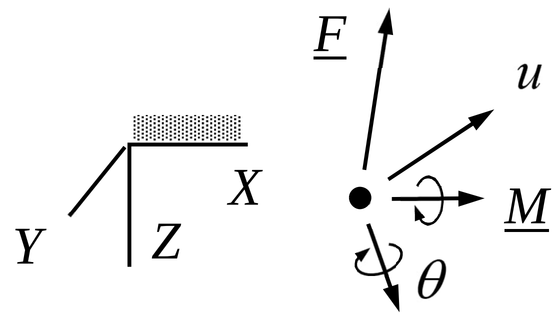
$$\delta W = -\left\{\begin{array}{c}\delta u_{z1}\\ \delta\theta_{y1}\\ \hdashline \delta u_{z2}\\ \delta\theta_{y2}\end{array}\right\}^{\mathrm{T}} \left(\frac{EI_{yy}}{h^3}\left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ \hdashline -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{array}\right]\left\{\begin{array}{c}u_{z1}\\ \theta_{y1}\\ \hdashline u_{z2}\\ \theta_{y2}\end{array}\right\} - \frac{f_z\, h}{12}\left\{\begin{array}{c}6\\ -h\\ \hdashline 6\\ h\end{array}\right\}\right)$$

Above, f_z , I_{yy} and E are assumed to be constants. In terms of the basis vectors of the $xyz-$ system $u_z = \vec{k}\cdot\vec{u}$, $\delta u_z = \vec{k}\cdot\delta\vec{u}$, $\theta_y = \vec{j}\cdot\vec{\theta}$, and $\delta\theta_y = \vec{j}\cdot\delta\vec{\theta}$.

FORCE ELEMENT

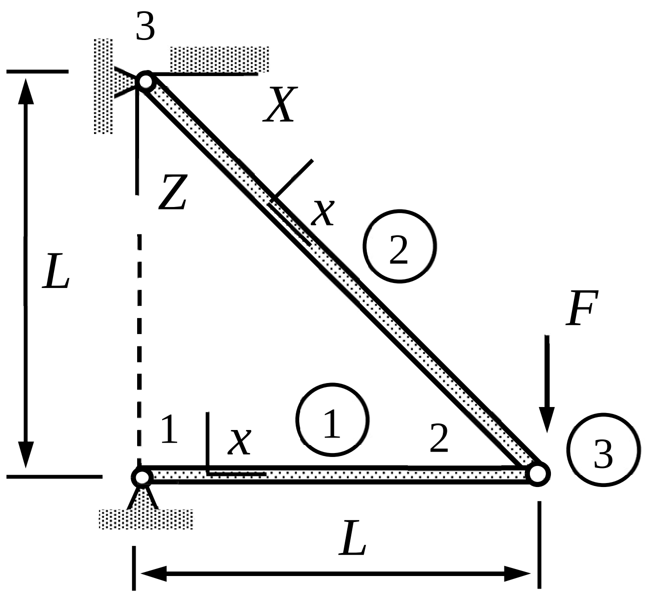
External point forces and moments are assumed to act on the joints. They are treated as elements associated with one node only. Virtual work expression is usually simplest in the structural coordinate system:

$$\delta W = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_X \\ \delta\theta_Y \\ \delta\theta_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}$$



Above, $\underline{F}_X$, $\underline{F}_Y$, $\underline{F}_Z$ and $\underline{M}_X$, $\underline{M}_Y$, $\underline{M}_Z$ are the given external force and moment components. A rigid body can be modeled as a particle at the center of mass connected to the other joints of the body by rigid links!

EXAMPLE 2.1 A bar truss is loaded by a point force having magnitude F as shown in the figure. Determine the nodal displacements. Cross-sectional area of bar 1-2 is A and that for bar 3-2 $\sqrt{8}A$. Young's modulus is E and weight is omitted.



Answer $\begin{Bmatrix} u_{X1} \\ u_{Z1} \end{Bmatrix} = \frac{LF}{EA} \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}$

- For element 1, the relationships between the nodal displacement components in the material and structural systems are $u_{x1}=0$ and $u_{x2}=u_{X2}$. Element contribution δW^1 to the virtual work expression of the structure is

$$\delta W^1=-\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix}-\begin{Bmatrix} 0 \\ 0 \end{Bmatrix})=-\frac{EA}{L}u_{X2}\delta u_{X2}.$$

- For element 2, $u_{x3}=0$ and $u_{x2}=(u_{X2}+u_{Z2})/\sqrt{2}$. Element contribution takes the form

$$\delta W^2=-\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2}+\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}}(\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2}+u_{Z2} \end{Bmatrix}-\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W^2=-\frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}).$$

- Virtual work expression of the point force follows from the definition of work

$$\delta W^3 = \delta u_{Z2} F \,.$$

- Virtual work expression of the structure is obtained as the sum of the element contributions. Then

$$\delta W = -\frac{EA}{L}\delta u_{X2} u_{X2} - \frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix}\right).$$ **”standard” form**

- Using the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus

$$\frac{EA}{L}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\u_{Z2}\end{Bmatrix}-\begin{Bmatrix}0\\F\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}u_{X2}\\u_{Z2}\end{Bmatrix}=\frac{LF}{EA}\begin{Bmatrix}-1\\2\end{Bmatrix}.$$

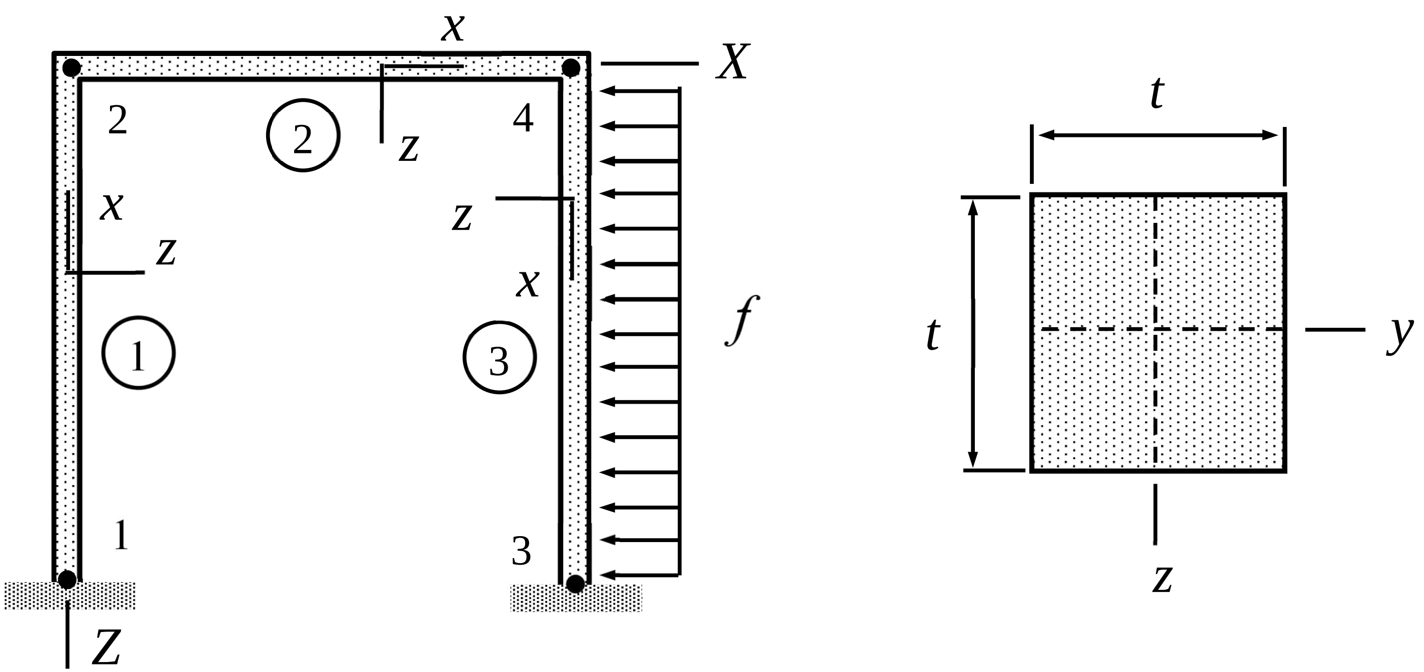
- The Mathematica description of the problem and solution are given by

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\{\{\mathbb{E}\}, \{2\sqrt{2}\,A\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\Theta_X, \Theta_Y, \Theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[2]\to -\frac{F\,L}{A\,\mathbb{E}},\ uZ[2]\to \frac{2\,F\,L}{A\,\mathbb{E}}\right\}$$

EXAMPLE 2.2 Consider the beam truss shown. Determine the displacements and rotations of nodes 2 and 4. Assume that the beams are rigid in the axial directions so that the axial *strain* vanishes. Cross-sections and lengths are the same and Young's modulus E is constant.



Answer $u_{X2} = u_{X4} = -\frac{3}{112}\frac{fL^4}{EI}$, $\theta_{Y2} = \frac{19}{1008}\frac{fL^3}{EI}$, and $\theta_{Y4} = \frac{5}{1008}\frac{fL^3}{EI}$

- Only the bending in *XZ*-plane needs to be accounted for. The displacement and rotation components of the structure are u_{X2}, θ_{Y2}, and θ_{Y4}. As the axial strain of beam 2 vanishes, axial displacements satisfy $u_{X4} = u_{X2}$.

$$\delta W^1 = -\left\{\begin{array}{c} 0 \\ 0 \\ \hdashline \delta u_{X2} \\ \delta\theta_{Y2} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ \hdashline u_{X2} \\ \theta_{Y2} \end{array}\right\}\right) \quad (u_{z2} = u_{X2}, \theta_{y2} = \theta_{Y2})$$

$$\delta W^2 = -\left\{\begin{array}{c} 0 \\ \delta\theta_{Y2} \\ \hdashline 0 \\ \delta\theta_{Y4} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ \theta_{Y2} \\ \hdashline 0 \\ \theta_{Y4} \end{array}\right\}\right) \quad (\theta_{y2} = \theta_{Y2}, \theta_{y4} = \theta_{Y4})$$

$$\delta W^3 = -\left\{\begin{array}{c} -\delta u_{X2} \\ \delta\theta_{Y4} \\ \hdashline 0 \\ 0 \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} -u_{X2} \\ \theta_{Y4} \\ \hdashline 0 \\ 0 \end{array}\right\} - \frac{fL}{12}\left\{\begin{array}{c} 6 \\ -L \\ \hdashline 6 \\ L \end{array}\right\}\right) \quad (u_{z4} = -u_{X2})$$

- Virtual work expression of the structure is

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Y4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix}\right).$$

- Principle of virtual work $\delta W = 0 \;\forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix} = 0 \Leftrightarrow \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} = \frac{fL^3}{1008EI}\begin{Bmatrix} -27L \\ 19 \\ 5 \end{Bmatrix}. \Leftarrow$$

- In the Mathematica code calculation, horizontal displacements of nodes 2 and 4 are forced to be same ($u_{X4} = u_{X2}$)

	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}}	Line[{1, 2}]
2	BEAM	{{E, G}, {A, I, I}}	Line[{2, 4}]
3	BEAM	{{E, G}, {A, I, I}, {-f, 0, 0}}	Line[{4, 3}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, 0}	{0, 0, 0}
4	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[4], 0}

$$\left\{uX[2] \to -\frac{3\,f\,L^4}{112\,E\,I},\ \theta Y[2] \to \frac{19\,f\,L^3}{1008\,E\,I},\ \theta Y[4] \to \frac{5\,f\,L^3}{1008\,E\,I}\right\}$$

2.3 ELEMENT CONTRIBUTIONS

Virtual work expressions for the solid, beam, plate elements combine virtual work densities representing the model and a case dependent approximation. To derive the expression for an element:

- ☐ Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$ and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection (if not available there, derive the expression in the manner discussed in MEC-E1050).

- ☐ Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- ☐ Integrate the virtual work density over the domain occupied by the element to get δW.

ELEMENT APPROXIMATION

Approximation of a function is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In displacement analysis, shape functions depend on (x, y, z) and the nodal values are parameters to be evaluated by FEM.

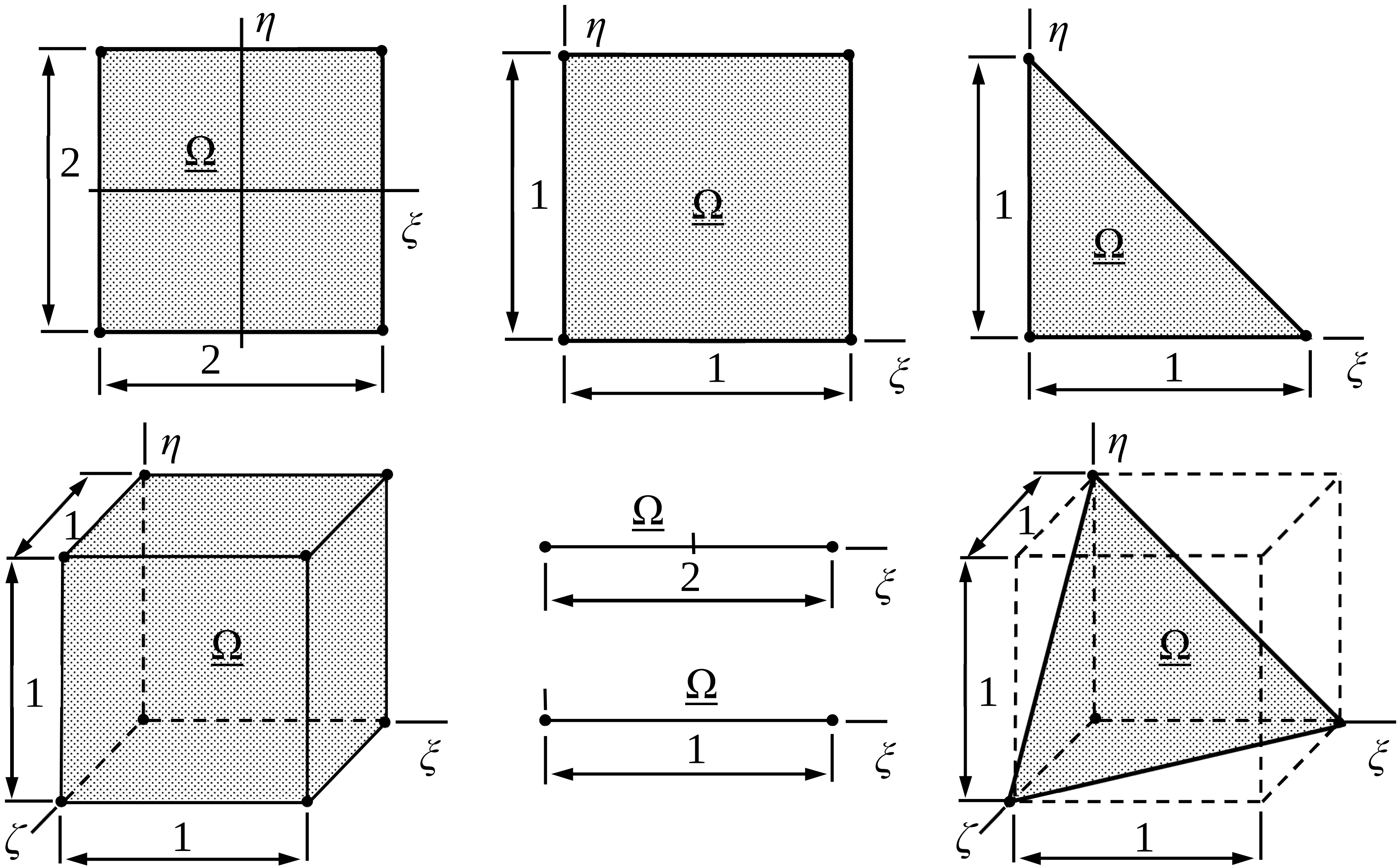
Approximation $\mathbf{u} = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

Shape functions $\mathbf{N} = \{N_1(x,y,z) \quad N_2(x,y,z) \quad \ldots \quad N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1 \quad \mathrm{a}_2 \quad \ldots \quad \mathrm{a}_n\}^{\mathrm{T}}$

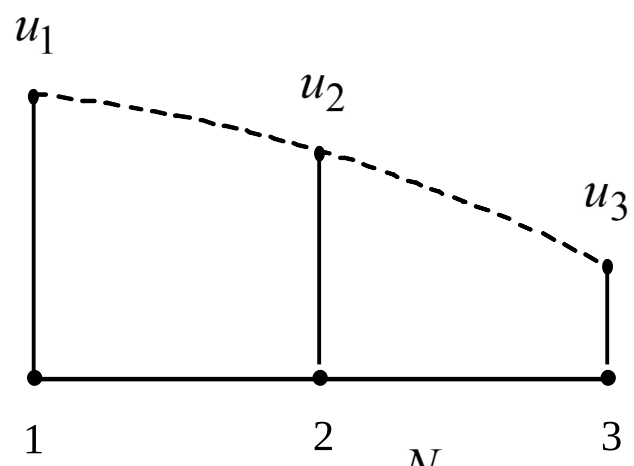
Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the Bernoulli beam model).

ELEMENT GEOMETRY



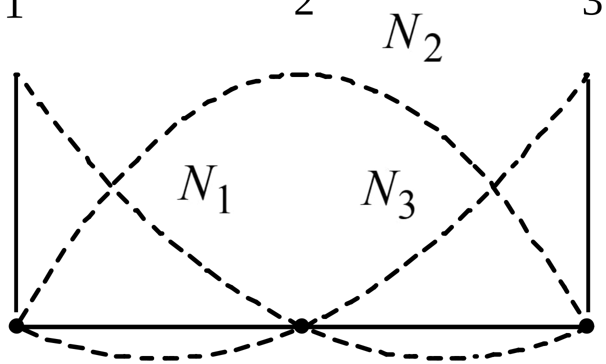
QUADRATIC SHAPE FUNCTIONS

Piecewise quadratic approximation is continuous in Ω and second order polynomial inside the elements. In a typical element Ω^e



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad u_2 \quad u_3\}^{\mathrm{T}}$



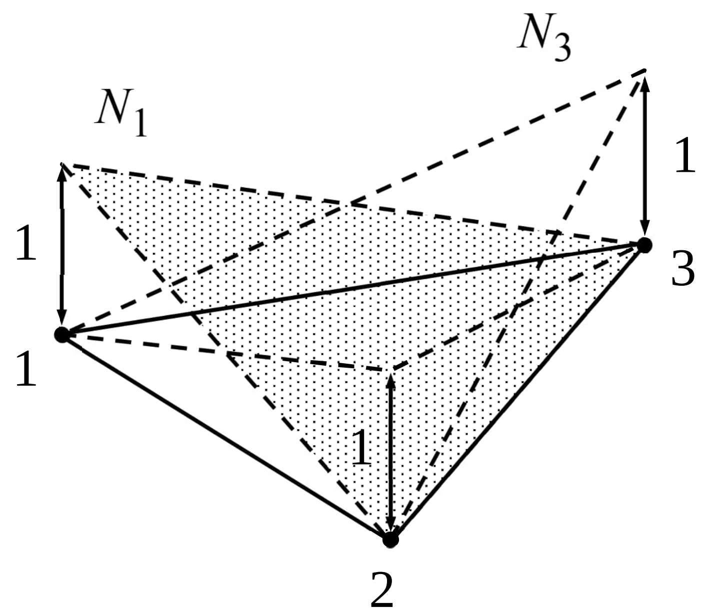
Shape functions: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}, \quad \xi = \dfrac{x}{h}$

More nodes can be used to generate higher order approximations!

LINEAR SHAPE FUNCTIONS

A piecewise linear approximation is continuous in Ω and linear inside each element of triangle shape. In a typical element

Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$



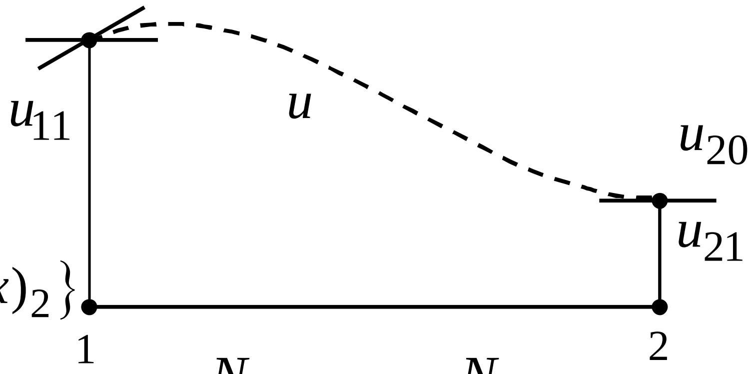
Nodal values: $\mathbf{a} = \{u_1 \quad u_2 \quad u_3\}^{\mathrm{T}}$

Shape functions: $\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$

Triangle element is the simplest element in two dimensions. Division of any 2D domain into triangles is always possible, which makes the element quite useful.

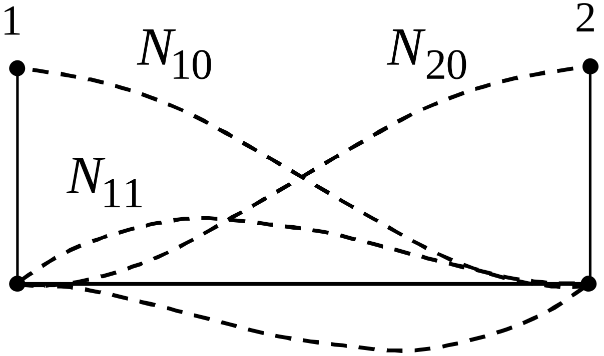
CUBIC SHAPE FUNCTIONS

Piecewise cubic approximation has continuous derivatives up to the first order in Ω and is a third order polynomial inside the elements.



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad (du/dx)_1 \;\vdots\; u_2 \quad (du/dx)_2\}$



Shape functions: $$\mathbf{N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ \hline N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}$$

In $xz-$plane bending $u = u_z$, $du/dx = -\theta_y$ and in $xy-$plane bending $u = u_y$, $du/dx = \theta_z$.

SOLID MODEL

The model does not contain assumptions in addition to those of linear elasticity theory.

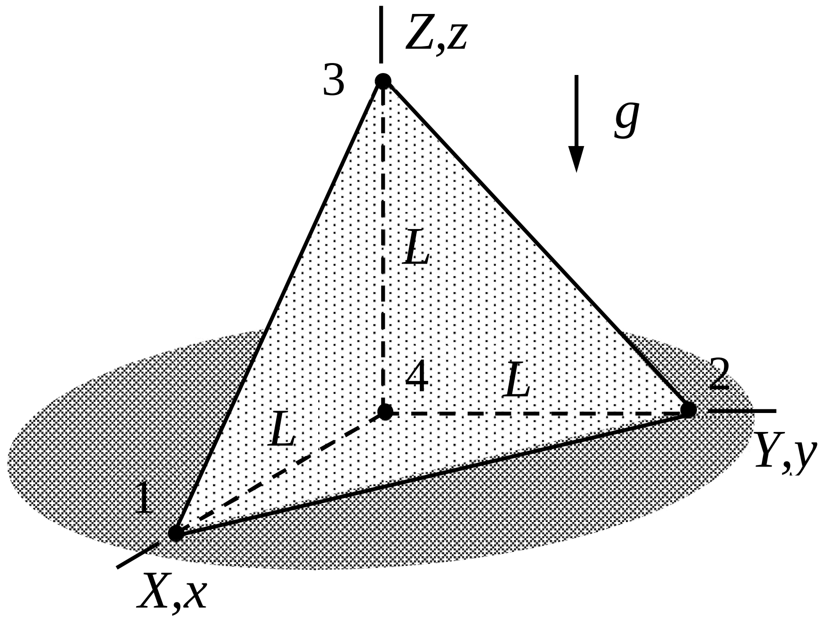
$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} \text{ in which } [E] = E \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1}.$$

The solution domain can be represented, e.g, by tetrahedron elements with linear interpolation of the displacement components $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$

EXAMPLE 2.3 A tetrahedron of edge length L, density ρ, and elastic properties E and ν is subjected to its own weight on a horizontal floor. Calculate the displacement u_{Z3} of node 3 with one tetrahedron element and linear approximation. Assume that $u_{X3} = u_{Y3} = 0$, and that the bottom surface is fixed.

Answer: $u_{Z3} = -\frac{1}{4}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}$



- Linear shape functions can be deduced directly from the figure $N_1 = x/L$, $N_2 = y/L$, $N_3 = z/L$, and $N_4 = 1 - x/L - y/L - z/L$. However, only the shape function of node 3 is needed as the other nodes are fixed. Approximations to the displacement components are

$$u = 0,\ \ v = 0,\ \text{ and }\ w = \frac{z}{L}u_{Z3},\ \text{ giving } \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 \ \text{ and } \frac{\partial w}{\partial z} = \frac{1}{L}u_{Z3}.$$

- When the approximation is substituted there, the virtual work densities of the internal and external forces simplify to

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta_{Z3} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L^2(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z3} \end{Bmatrix} = \frac{-E(1-\nu)}{(1+\nu)(1-2\nu)} \frac{u_{Z3}\delta u_{Z3}}{L^2}$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \frac{z}{L} \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix} = -\frac{z}{L}\rho g \delta u_{Z3}.$$

- Virtual work expressions are obtained as integrals of densities over the volume:

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} dV = \delta w_\Omega^{\text{int}} \frac{L^3}{6} = -\frac{1}{6} \frac{1-\nu}{(1+\nu)(1-2\nu)} E L u_{Z3} \delta u_{Z3},$$

$$\delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} dV = -\frac{L^3}{24} \rho g \delta u_{Z3}.$$

- Finally, principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ and the fundamental lemma of variation calculus imply

$$u_{Z3} = -\frac{1}{4}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}.$$

- For the Mathematica code of the course, the problem description is given by

	model	properties	geometry
1	SOLID	{{E, ν}, {0, 0, -g ρ}}	Tetrahedron[{1, 2, 3, 4}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{0, L, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, 0, L}	{uX[3], uY[3], uZ[3]}	{0, 0, 0}
4	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[3] \to 0,\ uY[3] \to 0,\ uZ[3] \to -\frac{g\, L^2 \left(-1+\nu+2\,\nu^2\right)\rho}{4\,E\,(-1+\nu)}\right\}$$

BEAM MODEL

In the beam model, the displacement and rotation components to be interpolated on a line segment of $x-$axis are $u(x)$, $v(x)$, $w(x)$, and $\phi(x)$. Virtual work densities are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u / dx \\ d^2\delta v / dx^2 \\ d^2\delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du / dx \\ d^2 v / dx^2 \\ d^2 w / dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w / dx \\ d\delta v / dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}.$$

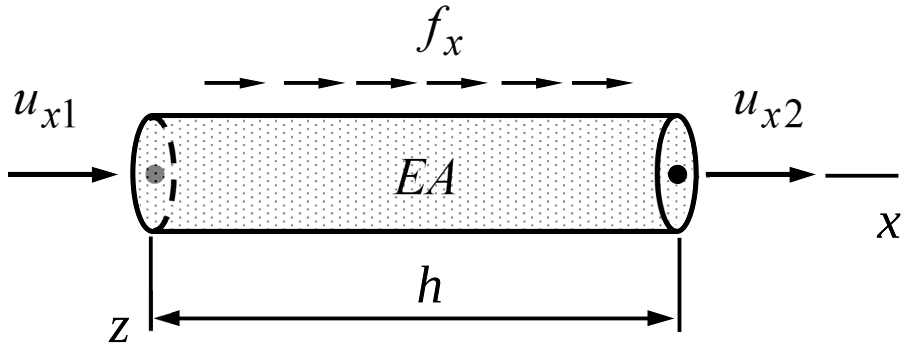
In what follows, the first and cross moments of the cross-section are assumed to vanish to disconnect the bar, torsion, and bending modes of the beam ($S_z = S_y = I_{yz} = 0$).

BAR MODE

Assuming a linear interpolation to $u(x)$ in terms of the end point displacements u_{x1}, u_{x2}, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$



Above, f_x, E, and A are assumed to be constants. The relationship between the axial displacement component and the displacement components in the structural coordinate system is $u_x = \vec{i} \cdot \vec{u} = i_X u_X + i_Y u_Y + i_Z u_Z$.

- First, element interpolant $u=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ and its variation $\delta u=\mathbf{N}^{\mathrm{T}}\delta\mathbf{a}=\delta\mathbf{a}^{\mathrm{T}}\mathbf{N}$ are substituted into the virtual work expression to get (here $\Omega=]0,h[$ and $d\Omega=dx$)

$$\delta W=\int_0^h\left(-\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x\right)dx \qquad \Rightarrow$$

$$\delta W=-\int_0^h \delta\mathbf{a}^{\mathrm{T}}\frac{d\mathbf{N}}{dx}EA\frac{d\mathbf{N}^{\mathrm{T}}}{dx}\mathbf{a}dx+\int_0^h \delta\mathbf{a}^{\mathrm{T}}\mathbf{N}f_x dx \qquad \Leftrightarrow$$

$$\delta W=-\delta\mathbf{a}^{\mathrm{T}}\left(\int_0^h \frac{d\mathbf{N}}{dx}EA\frac{d\mathbf{N}^{\mathrm{T}}}{dx}dx\mathbf{a}-\int_0^h \mathbf{N}f_x dx\right). \qquad \Leftarrow$$

- If the interpolant is taken to be linear, shape functions and the nodal values are given by

$$\mathbf{N}=\frac{1}{h}\begin{Bmatrix}h-x\\ x\end{Bmatrix},\quad \frac{d}{dx}\mathbf{N}=\frac{1}{h}\begin{Bmatrix}-1\\ 1\end{Bmatrix},\quad \mathbf{a}=\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},\text{ and } \delta\mathbf{a}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}$$

- If Young's modulus E, cross-sectional area A, and the distributed force f_x are constants, integration over the element domain gives (the expressions of the shape functions need to be substituted now)

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} (\int_0^h \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} EA \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} dx \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx) \quad \Leftrightarrow$$

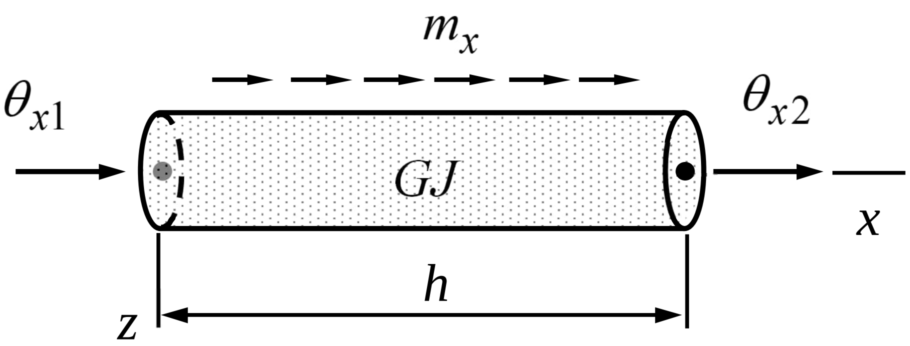
$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}). \quad \leftarrow$$

Derivation out of virtual work densities works also when Young's modulus E, cross-sectional area A, and the distributed force f_x are not constants. Also, approximation to axial displacement $u(x)$ may be chosen in various ways.

TORSION MODE

Assuming a linear interpolation to $\phi(x)$ in terms of the end point rotations θ_{x1} and θ_{x2}, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{GJ}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$



$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

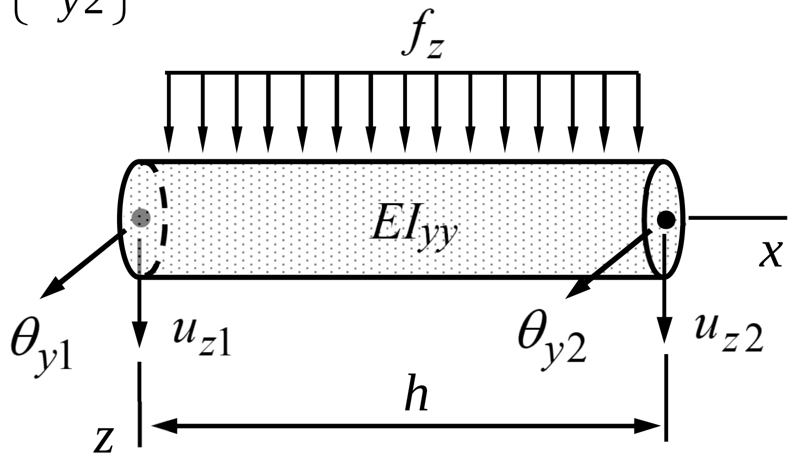
Above, m_x, E, and J are assumed to be constants. The relationship between the axial rotation component and the rotation components in the structural coordinate system is $\theta_x = \vec{i} \cdot \vec{u} = i_X \theta_X + i_Y \theta_Y + i_Z \theta_Z$.

BENDING MODE (*xz*-plane)

Assuming a cubic approximation to $w(x)$ in terms of the end point displacements u_{z1}, u_{z2} and rotations θ_{y1} and θ_{y2}, virtual work expressions of the internal and external forces

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$



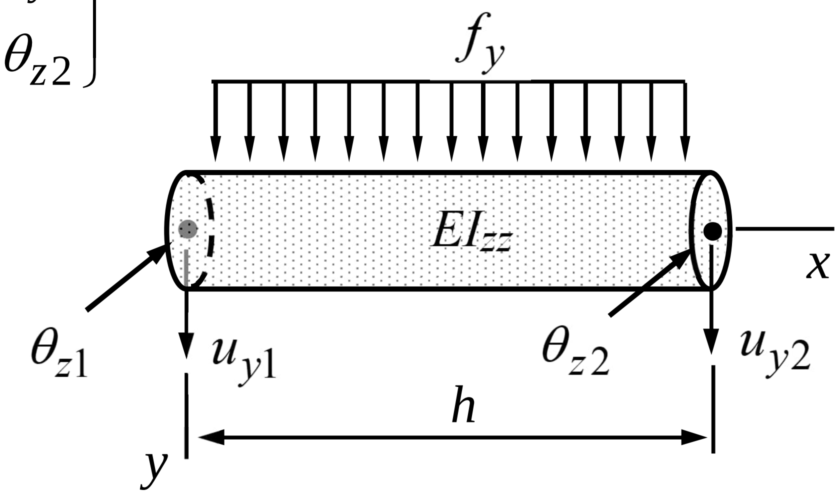
Above, f_z, I_{yy} and E are assumed to be constants.

BENDING MODE (*xy*-plane)

Assuming a cubic approximation to $v(x)$ in terms of point displacements u_{y1}, u_{y2} and rotations θ_{z1} and θ_{z2}, virtual work expressions of the internal and external forces

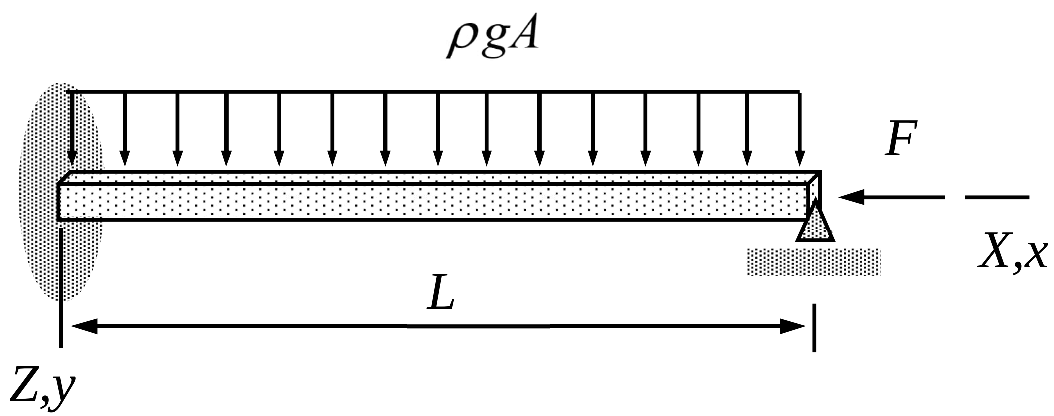
$$\delta W^{\text{int}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix} \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix}.$$



Above, f_y , I_{zz} and E are assumed to be constants.

EXAMPLE 2.4 The Bernoulli beam of the figure is loaded by its own weight $f = \rho g A$ and a point force F acting on the right end. Determine the displacement and rotation of the right end with the Mathematica code of MEC-E8001. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam cross-section properties A, I_{yy}, I_{zz}, and material properties E, ρ are constants.



Answer: $u_{X2} = \dfrac{FL}{EA}$ and $\theta_{Y2} = \dfrac{1}{48}\dfrac{\rho g A L^3}{EI_{zz}}$

- Bernoulli beam element of the Mathematica code requires the orientation of the y –axis unless y –axis and Y –axis are aligned. Orientation is given by additional parameter defining the components of $\vec{j}$ in the structural coordinate system:

	model	properties	geometry
1	BEAM	{{E, G}, {A, Iyy, Izz, {0, 0, 1}}, {0, f, 0}}	Line[{1, 2}]
2	FORCE	{-F, 0, 0}	Point[{2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$$\left\{uX[2] \to -\frac{F\,L}{A\,E},\ \theta Y[2] \to \frac{f\,L^3}{48\,E\,Izz}\right\}$$

PLATE MODEL

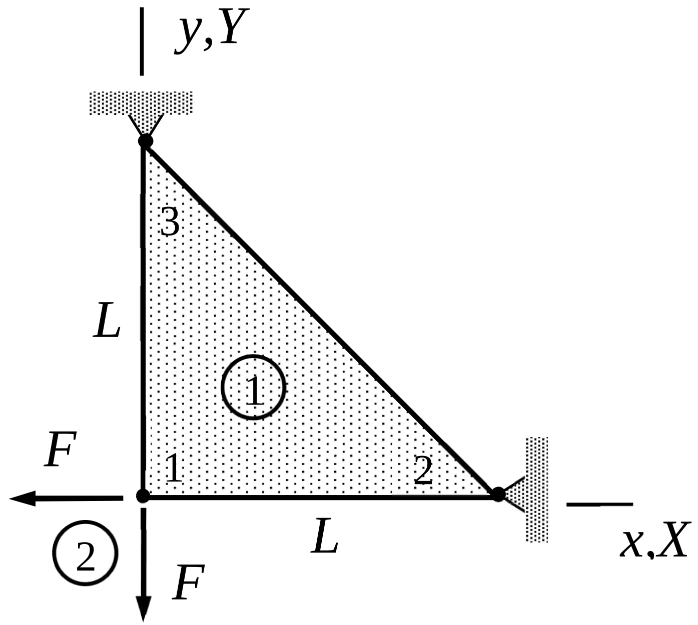
Virtual work densities combine the plane-stress and plate bending modes. Assuming that the material coordinate system is placed at the geometric centroid

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} - \begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \times$$

$$\times \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

Approximation to the displacement components $u(x,y)$, $v(x,y)$, $w(x,y)$ should be continuous and $w(x,y)$ should also have continuous derivatives at the element interfaces.

EXAMPLE 2.5. Consider the thin triangular structure shown. Young's modulus E, Poisson's ratio ν, and thickness t are constants. Distributed external force vanishes. Assume plane-stress conditions, $XY-$plane deformation and determine the displacement of node 1 when the force components acting on the node are as shown in the figure.



Answer: $\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\dfrac{F}{Et}\dfrac{(1+\nu)(1-2\nu)}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- Nodes 2 are 3 are fixed and the non-zero displacement components are u_{X1} and u_{Y1}. Linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{L - x - y}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial x \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u / \partial y \\ \partial v / \partial y \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work density of internal forces is given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix}.$$

- Integration over the triangular domain gives (integrand is constant)

$$\delta W^1 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta W^1 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{4} \frac{tE}{1-\nu^2} \begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work expression for the point forces follows from the definition of work

$$\delta W^2 = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ -F \end{Bmatrix}.$$

- Principle of virtual work in the form $\delta W = \delta W^1 + \delta W^2 = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{4}\frac{tE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) = 0 \quad \forall \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix} \quad \Rightarrow$$

$$\frac{1}{4}\frac{tE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{F}{tE}(1-\nu^2)\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The point forces acting on a thin slab should be considered as "equivalent nodal forces" i.e. just representations of tractions acting on some part of the boundary. Under the action of an actual point force, displacement becomes non-bounded. In practice, numerical solution to the displacement at the point of action increases when the mesh is refined.

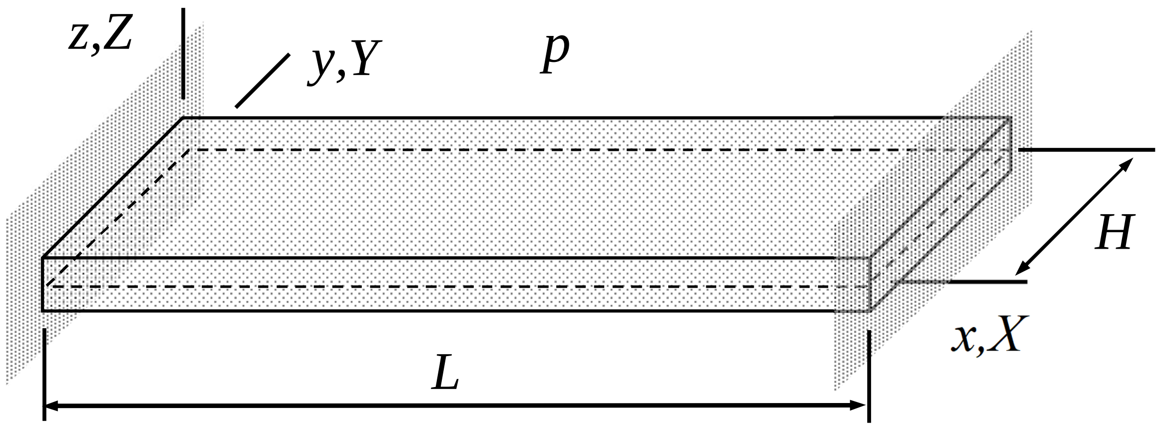
- In Mathematica code of the course, the problem description is given by

	model	properties	geometry
1	PLANE	{{E, ν}, {t}}	Polygon[{1, 2, 3}]
2	FORCE	{−F, −F, 0}	Point[{1}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{uX[1], uY[1], 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, L, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[1] \to \frac{F\,(-1+\nu)\,(1+\nu)}{t\,E},\ uY[1] \to -\frac{F - F\,\nu^2}{t\,E}\right\}$$

EXAMPLE 2.6 Consider a plate strip loaded by pressure p acting on the upper surface. Determine the deflection w at the center point according to the Kirchhoff model. Thickness, length and width of the plate are t, L, and H, respectively. Young's modulus E, and Poisson's ratio ν are constants. Use the one parameter approximation $w(x) = a_0(1 - x/L)^2(x/L)^2$.



Answer: $w = -\frac{1}{32}(\frac{L}{t})^3 \frac{Lp}{E}(1-\nu^2)$

- Approximation satisfies the displacement boundary conditions 'a priori' and contains a free parameter a_0 (not associated with any node) to be solved by using the principle of virtual work:

$$w = a_0(1-\frac{x}{L})^2(\frac{x}{L})^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = a_0\frac{2}{L^2}[1-6\frac{x}{L}+6(\frac{x}{L})^2] \text{ and } \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0.$$

- When the approximation is substituted there, virtual work densities (formulae collection) simplify to

$$\delta w_{\Omega}^{\text{int}} = -a_0 \delta a_0 \frac{Et^3}{3(1-\nu^2)}\frac{1}{L^4}[1-6\frac{x}{L}+6(\frac{x}{L})^2]^2,$$

$$\delta w_{\Omega}^{\text{ext}} = -\delta a_0 (1-\frac{x}{L})^2(\frac{x}{L})^2 p.$$

- Integrations over the domain $\Omega =]0, L[\times]0, H[$ give the virtual works of internal and external forces

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -a_0 \delta a_0 \frac{1}{15} \frac{HEt^3}{L^3(1-\nu^2)},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = -\delta a_0 \frac{1}{30} pLH \,.$$

- Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give finally $\forall \delta a_0$

$$\delta W = -\delta a_0 \left(\frac{1}{15} \frac{HEt^3}{L^3(1-\nu^2)} a_0 + \frac{1}{30} pLH\right) = 0 \quad \Leftrightarrow \quad a_0 = -\frac{1}{2} \frac{pL^4}{Et^3}(1-\nu^2). \quad \Leftarrow$$

- The problem can be solved numerically also by using the Reissner-Mindlin plate model and plate bending element of the Mathematica code. For example, assuming parameter values $p(L/t)^3/E = 10$, $\nu = 0.33$, $H/L = 0.3$, and $t/L = 0.01$ (a thin plate), the one parameter approximation to displacement gives $w/L = -0.278$ at the centerpoint whereas the solution on a regular (rough) mesh of about 300 unknown displacement/rotation components gives $w/L = -0.278$ (a fine mesh gives $w/L = -0.289$)