Class exercises for Week 2. To be done in class. These exercises do not need to be returned, and they are not marked.

1. Compute the following limits or show they do not exist:

   (a) $\lim_{(x,y)\to(0,0)} \frac{y^2 - x^2}{x^2 + y^2}$

   (b) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$

   (c) $\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2 + y^2}$

2. Consider the function $f(x,y)$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Sketch the surface $z = f(x,y)$
   (b) Is the function continuous at $(0,0)$?
   (c) Compute $\partial f/\partial x(0,0)$ and $\partial f/\partial y(0,0)$. Note that due to the piecewise definition of the function, you must use the definition of the partial derivatives in order to compute them.
   (d) Find the equation of the tangent plane you would get by just plugging the data from part (c) in the tangent plane equation.
   (e) Does the plane you found in part (d) approximate the surface well in any small region around $(0,0)$?
   (f) Does the surface $z = f(x,y)$ have a tangent plane at $(0,0)$? Discuss/Explain.

3. Consider the function $f(x,y) = x^3 - 3xy^2 - 6y - 1$.

   (a) Compute all the 1st and 2nd order partial derivatives.
   (b) Find all the points on the surface where the tangent plane is horizontal.
   (c) Find the tangent plane to the surface at the point $(1,2)$.

4. One of the most crucial operators in Partial Differential Equations and thus in for example models of quantum mechanics is the *Laplacian* of a continuous function $f(x,y)$ with continuous 2nd order partial derivatives in $\mathbb{R}^2$. It is denoted by Δ or ∇^2 and defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

   The *polar coordinates* r, θ of $\mathbb{R}^2$ is given by the change of variable

$$x = r\cos\theta, y = r\sin\theta$$

   for $r > 0$ and $-\pi < \theta \leq \pi$. Then we can always write $f(x,y)$ depending on r and θ as follows:

$$f(x,y) = f(r\cos\theta, r\sin\theta)$$

   Show that we have the following change of variable formula for the Laplacian of f in the polar coordinates:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial f}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 f}{\partial \theta^2}.$$