Analysis, Random Walks and Groups

Exercise sheet 2

January 17, 2023

Homework exercises: Return these for marking to Kai Hippi in the tutorial on Week 3. Contact Kai by email if you cannot return these in-person, and you can arrange an alternative way to return your solutions. Remember to be clear in your solutions, if the solution is unclear and difficult to read, you can lose marks. Also, if you do not know how to solve the exercise, attempt something, you can get awarded partial marks.

1. (5pts)

Let $\mu_\alpha = \alpha\delta_0 + (1-\alpha)\delta_1$ on $\mathbb{Z}_5$ for some $0 < \alpha \leq 1$. For which α is μ_α ergodic? Explain your answer. Compute $d(\mu_\alpha * \mu_\alpha, \lambda)$ as a function of α.

2. (5pts)

Prove that if $\mu, \nu : \mathbb{Z}_p \to [0,1]$ are probability distributions, then the entropy

$$H(\mu * \nu) \leq H(\mu) + H(\nu).$$

Hint: Use the convexity of $\varphi(x) = -x\, log(x)$ (you do not need to prove the convexity).

Further exercises: Attempt these before the tutorial, they are not marked and will be discussed in the tutorial. If you cannot attend the tutorial, but want to do the attendance marks, you can return your attempts to these before the tutorial to Kai. Here Kai will not mark the further exercises, but will look if an attempt has been made and awards the attendance mark for that week's tutorial.

3.

Prove the following identities for the convolution: for all $f, g, h : \mathbb{Z}_p \to \mathbb{C}$ we have:

(a) **Commutativity:** $f * g = g * f$

(b) **Associativity:** $f * (g * h) = (f * g) * h$

(c) **Linearity:** if $\alpha, \beta \in \mathbb{C}$, then $f * (\alpha g + \beta h) = \alpha f * g + \beta f * h$

4.

Let μ be a probability distribution on $\mathbb{Z}_4$, which is not a Dirac mass, and assume that the support

$$spt(\mu) = \{t \in \mathbb{Z}_4 \, : \, \mu(t) > 0\}$$

is a coset of a proper non-trivial subgroup of $\mathbb{Z}_4$. Is there a limit

$$\mu_\infty = \lim_{n \to \infty} \mu^{*n}?$$

What is it? No proofs necessary, just have a think how to maybe prove this.

5.

(a) Prove that for all $A, B \subset \mathbb{Z}_p$ the cardinalities

$$\max\{|A|, |B|\} \leq |A \oplus B| \leq |A||B|.$$

(b) Give examples of sets $A, B \subset \mathbb{Z}_p$ such that

$$|A \oplus B| = \max\{|A|, |B|\}.$$

(c) Give examples of sets $A, B \subset \mathbb{Z}_p$ which are not $\mathbb{Z}_p$ such that

$$|A \oplus B| = |A||B|.$$