A0001 13.12.2022

P1

(a) $Au = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ 1p

(b) Bv is not defined 1p

(c) $u^T A = (0 \; 0 \; 4)$ 1p

(d) $v^T B = (0 \; 3 \; -2$ 1p

(e) $u^T A u = 12$ 1p

(f) $v^T B A = (-3 \; 8 \; -7)$ 1p

P2

$A^T A = A A^T = I$; A orthogonal 3p

$x = A^T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ 2p

$x = \begin{pmatrix} \frac{2+\sqrt{2}}{\sqrt{3}} & -1-\sqrt{2} & -\frac{5+\sqrt{2}}{\sqrt{3}} \end{pmatrix}^T$ 1p

Alternate: Gauss 2p

U 2p

$x = \ldots$ 2p

P3

$$\begin{matrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{matrix} \quad \downarrow -1 \qquad \begin{matrix} 1 & 3 & 8 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{matrix} \quad \downarrow 1$$

$$\begin{matrix} 1 & 3 & 8 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{matrix} \qquad | : -1 \qquad \uparrow -3$$

$$\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix}$$ 2p

$$N(A) = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$ 2p

$$= N(A^T A)$$ 2p

Alternate: Form $A^T A$ 2p

Gauss 2p

$N(A^T A)$ 2p

P4 $PA = \begin{pmatrix} 2 & -1 & 4 \\ 6 & -2 & 9 \\ 0 & 0 & 1 \end{pmatrix}$ 2p

$L = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ 0 & & 1 \end{pmatrix}$ 2p $U = \begin{pmatrix} 2 & -1 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$ 2p

P5

(a)

$$\begin{matrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{matrix} \quad \downarrow \tfrac{1}{2}$$

$$\begin{matrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{matrix} \quad \downarrow \tfrac{2}{3}$$

$$\begin{matrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \end{matrix}$$

Rank $(A) = 3$ 2p

(b) $\det(A^T A) = 0$ 2p

(c) $\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3/2 & 0 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 5 & -3/2 & 0 \\ -3/2 & 13/4 & -4/3 \\ 0 & -4/3 & 25/9 \end{pmatrix} \Rightarrow \det AA^T = 30$ 2p

Alternate: (b) compute $A^T A$ 1p

det $(A^T A)$ with any method of choice 1p

(c) similarly

P6 $Ax = \lambda x, \quad \lambda \neq 0, \quad x \neq 0, \quad A^{-1}$ exists. 1p

$$A^{-1} A x = A^{-1} (\lambda x) \iff A^{-1} x = \frac{1}{\lambda} x$$ 2p

$$A^{-1} A^{-1} x = A^{-1} \left(\frac{1}{\lambda} x \right)$$

$$\iff \quad A^{-2} x = \frac{1}{\lambda^2} x$$ 2p

$$\Rightarrow \quad A^{-k} x = \frac{1}{\lambda^k} x \qquad \square$$ 1p