MS-C1081 – Abstract Algebra
2022–2023 (Period III)
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Problem set 3

Deadline: Tue 31.1.2023 at 10am

During the exercise sessions you may work on the Warm-up problems, and the TAs will present solutions to those. Submit your solutions to Homework 1,2,3 and to the Fill-in-the-blanks problem on MyCourses before the deadline. Remember that the Homeworks and the Fill-in-the-blanks go to separate return boxes, and each return box accepts a single pdf file.

Warm-ups

Warm-up 1. Consider the group $(\mathbb{Z}_2 \times S_3, *)$, where the operation is defined componentwise, that is, for all $\overline{a}, \overline{b} \in \mathbb{Z}_2$ and for all $\sigma, \tau \in S_3$,

$$(\overline{a}, \sigma) * (\overline{b}, \tau) := (\overline{a} + \overline{b}, \tau \circ \sigma).$$

Use the following notation and Cayley table:

$$\mathrm{id} := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad \alpha := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \qquad \beta := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\gamma := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \delta := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \varepsilon := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

$*$	$(\overline{0},\mathrm{id})$	$(\overline{0},\alpha)$	$(\overline{0},\beta)$	$(\overline{0},\gamma)$	$(\overline{0},\delta)$	$(\overline{0},\varepsilon)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\alpha)$	$(\overline{1},\beta)$	$(\overline{1},\gamma)$	$(\overline{1},\delta)$	$(\overline{1},\varepsilon)$
$(\overline{0},\mathrm{id})$	$(\overline{0},\mathrm{id})$	$(\overline{0},\alpha)$	$(\overline{0},\beta)$	$(\overline{0},\gamma)$	$(\overline{0},\delta)$	$(\overline{0},\varepsilon)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\alpha)$	$(\overline{1},\beta)$	$(\overline{1},\gamma)$	$(\overline{1},\delta)$	$(\overline{1},\varepsilon)$
$(\overline{0},\alpha)$	$(\overline{0},\alpha)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\delta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\beta)$	$(\overline{0},\gamma)$	$(\overline{1},\alpha)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\delta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\beta)$	$(\overline{1},\gamma)$
$(\overline{0},\beta)$	$(\overline{0},\beta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\delta)$	$(\overline{0},\gamma)$	$(\overline{0},\alpha)$	$(\overline{1},\beta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\delta)$	$(\overline{1},\gamma)$	$(\overline{1},\alpha)$
$(\overline{0},\gamma)$	$(\overline{0},\gamma)$	$(\overline{0},\delta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\alpha)$	$(\overline{0},\beta)$	$(\overline{1},\gamma)$	$(\overline{1},\delta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\alpha)$	$(\overline{1},\beta)$
$(\overline{0},\delta)$	$(\overline{0},\delta)$	$(\overline{0},\gamma)$	$(\overline{0},\alpha)$	$(\overline{0},\beta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\mathrm{id})$	$(\overline{1},\delta)$	$(\overline{1},\gamma)$	$(\overline{1},\alpha)$	$(\overline{1},\beta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\mathrm{id})$
$(\overline{0},\varepsilon)$	$(\overline{0},\varepsilon)$	$(\overline{0},\beta)$	$(\overline{0},\gamma)$	$(\overline{0},\alpha)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\delta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\beta)$	$(\overline{1},\gamma)$	$(\overline{1},\alpha)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\delta)$
$(\overline{1},\mathrm{id})$	$(\overline{1},\mathrm{id})$	$(\overline{1},\alpha)$	$(\overline{1},\beta)$	$(\overline{1},\gamma)$	$(\overline{1},\delta)$	$(\overline{1},\varepsilon)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\alpha)$	$(\overline{0},\beta)$	$(\overline{0},\gamma)$	$(\overline{0},\delta)$	$(\overline{0},\varepsilon)$
$(\overline{1},\alpha)$	$(\overline{1},\alpha)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\delta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\beta)$	$(\overline{1},\gamma)$	$(\overline{0},\alpha)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\delta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\beta)$	$(\overline{0},\gamma)$
$(\overline{1},\beta)$	$(\overline{1},\beta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\delta)$	$(\overline{1},\gamma)$	$(\overline{1},\alpha)$	$(\overline{0},\beta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\delta)$	$(\overline{0},\gamma)$	$(\overline{0},\alpha)$
$(\overline{1},\gamma)$	$(\overline{1},\gamma)$	$(\overline{1},\delta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\alpha)$	$(\overline{1},\beta)$	$(\overline{0},\gamma)$	$(\overline{0},\delta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\alpha)$	$(\overline{0},\beta)$
$(\overline{1},\delta)$	$(\overline{1},\delta)$	$(\overline{1},\gamma)$	$(\overline{1},\alpha)$	$(\overline{1},\beta)$	$(\overline{1},\varepsilon)$	$(\overline{1},\mathrm{id})$	$(\overline{0},\delta)$	$(\overline{0},\gamma)$	$(\overline{0},\alpha)$	$(\overline{0},\beta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\mathrm{id})$
$(\overline{1},\varepsilon)$	$(\overline{1},\varepsilon)$	$(\overline{1},\beta)$	$(\overline{1},\gamma)$	$(\overline{1},\alpha)$	$(\overline{1},\mathrm{id})$	$(\overline{1},\delta)$	$(\overline{0},\varepsilon)$	$(\overline{0},\beta)$	$(\overline{0},\gamma)$	$(\overline{0},\alpha)$	$(\overline{0},\mathrm{id})$	$(\overline{0},\delta)$

1. Determine the order of all elements of $\mathbb{Z}_2 \times S_3$.
2. Show that there are subgroups of $\mathbb{Z}_2 \times S_3$ of all possible cardinalities compatible with Lagrange's theorem (i.e., the divisors of $\#G$, which is called the *order* of G).
3. Consider the cyclic subgroup $H := \langle(\overline{1}, \alpha)\rangle$. Compute all left and right cosets of H, and use this to determine whether H is a normal subgroup.
4. Is the cyclic subgroup $K := (\overline{1}, \varepsilon)\rangle$ normal? Do not use the definition of normality.
5. Describe explicitly the quotient G/K. Give an explicit isomorphism $G/K \cong \mathbb{Z}_2$.
6. Show that $G/(\{0\} \times S_3) \cong \mathbb{Z}_2$, by using the First Isomorphism Theorem.

Warm-up 2. Prove the second item in Remarks 2.23 of the lecture notes:

$$x \sim_H y \qquad \Leftrightarrow \qquad y \cdot x^{-1} \in H \qquad \Leftrightarrow \qquad y \in Hx.$$

Homework

Homework 1. Let X be a set. Recall that a partition of X is a collection of non-empty subsets of X which are pairwise disjoint and whose union is X. Define the map

$$\begin{aligned}\Phi\colon \{\text{equivalence relations on } X\} &\longrightarrow \{\text{partitions of } X\}\\ \sim &\longmapsto X/\!\sim .\end{aligned}$$

1. Show that Φ is well-defined, that is, $X/\!\sim$ is a partition of X. [2 points]
2. Show that Φ is injective. [2 points]
3. Show that Φ is surjective. [2 points]

Homework 2. Let G be a group. Let H and K be subgroups of G, with K normal in G and satisfying $K \subseteq H \subseteq G$.

1. Show that K is normal in H. [2 points]
2. Show that the set $H/K = \{hK \mid h \in H\}$ is a subgroup of $G/K = \{gK \mid g \in G\}$. [4 points]

Homework 3. Consider the group $G := \mathbb{Z}_3 \times S_3$ with operation defined component-wise. Let $\delta := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in S_3$ and define the cyclic subgroups $H_1 := \langle(\overline{0}, \delta)\rangle$ and $H_2 := \langle(\overline{1}, \mathrm{id}_{\{1,2,3\}})\rangle$.

1. Show that H_1 and H_2 are isomorphic, and both normal in G. [2 points]
2. Show that $G/H_1 \cong \mathbb{Z}_6$ and $G/H_2 \cong S_3$. [3 points]
3. Show that G/H_1 and G/H_2 are not isomorphic. [1 point]

Fill-in-the-blanks. Complete the proof of the following claim:

Claim. Let G be a finite group, and let $H \subset G$ be a subgroup with $\#H = \frac{\#G}{2}$. Then H is normal in G.

Proof. First of all, by Lagrange's theorem we know that

$$\#(G/\sim_H) = \#(G/\approx_H) = \frac{\#\underline{\qquad\quad}}{\#\underline{\qquad\quad}} = 2,$$

hence there are exactly two right cosets and exactly two left cosets. Next, we show the normality of H by showing that the left and right cosets coincide, that is, for all $x \in G$, we have ________=________. We do this in two parts.

- For all $x \in H$, we show that $Hx = H = xH$. This follows from ____________________ of Problem set 1.
- For all $x \in G \setminus H$, we show that $Hx = G \setminus H = xH$. An element $y \in Hx$ can be written as $y =$________, for some _____$\in H$. We want to show that $y \notin H$. Indeed, if we had $y \in H$, then, since H is a subgroup, we would also have $x =$_____$\cdot y \in H$, and this is a contradition, since $x \notin H$. Similarly for xH.

Since the cosets form a partition of G, the only options for Hx and xH are H or $G \setminus H$. And since any element $x \in G$ is either in H or in $G \setminus H$, this concludes the proof. □

[3 points]