Analysis, Random Walks and Groups

Exercise sheet 3

Homework exercises: Return these for marking to Kai Hippi in the tutorial on Week 4. Contact Kai by email if you cannot return these in-person, and you can arrange an alternative way to return your solutions. Remember to be clear in your solutions, if the solution is unclear and difficult to read, you can lose marks. Also, if you do not know how to solve the exercise, attempt something, you can get awarded partial marks.

1. (5pts) We say a probability distribution μ on $\mathbb{Z}_p$ has a **spectral gap** if

$$|\widehat{\mu}(k)| < 1, \quad \text{for all } k \in \mathbb{Z}_p \setminus \{0\}.$$

Find the Fourier transform of the probability distribution

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$$

in $\mathbb{Z}_4$ (all the values of $\widehat{\mu}(k)$) and use this to prove that μ does not have a spectral gap. Prove then that

$$\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$

in $\mathbb{Z}^4$ has a spectral gap.

Notice also that μ is supported on a subgroup $\{0, 2\}$ so it cannot be ergodic. Later we will see that in general spectral gap implies ergodicity, so ν is ergodic.

Given a probability distribution μ on $\mathbb{Z}_p$, recall that we defined the associated **transfer operator** as

$$T_\mu f(t) = \mu * f(t), \quad t \in \mathbb{Z}_p,$$

where $f : \mathbb{Z}_p \to \mathbb{C}$. A complex number $\lambda \in \mathbb{C}$ is an **eigenvalue** of T_μ if there exists a non-zero $\psi : \mathbb{Z}_p \to \mathbb{C}$ (called **eigenfunction** of T_μ) such that

$$T_\mu \psi(t) = \lambda \psi(t), \quad \text{for all } t \in \mathbb{Z}_p.$$

The **spectrum** $\sigma(T_\mu)$ of T_μ is then the collection of all eigenvalues

$$\sigma(T_\mu) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T_\mu\}$$

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2. (5pts) Given a probability distribution μ on $\mathbb{Z}_p$, prove that for each $k \in \mathbb{Z}_p$, the Fourier transform $\widehat{\mu}(k) \in \mathbb{C}$ is an eigenvalue of the transfer operator T_μ, that is, $\widehat{\mu}(k) \in \sigma(T_\mu)$.

Hint: Attempt to prove the function $\psi_k(t) = e^{2\pi i k/tp}, t \in \mathbb{Z}_p$, is an eigenfunction of T_μ with eigenvalue $\widehat{\mu}(k)$.

Further exercises: Attempt these before the tutorial, they are not marked and will be discussed in the tutorial. If you cannot attend the tutorial, but want to do the attendance marks, you can return your attempts to these before the tutorial to Kai. Here Kai will not mark the further exercises, but will look if an attempt has been made and awards the attendance mark for that week's tutorial.

3. Conversely to Question 2, establish that if $\lambda \in \sigma(T_\mu)$, then $\lambda = \widehat{\mu}(k)$ for some $k \in \mathbb{Z}_p$.

In particular, together with Question 2, this proves that the spectrum agrees with the Fourier coefficients of μ:

$$\sigma(T_\mu) = \{\widehat{\mu}(k) : k \in \mathbb{Z}_p\}.$$

4. Define the Laplace operator Δ for functions $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$\Delta f(t) = \frac{f(t \oplus 1) + f(t \ominus 1)}{2} - f(t), \quad t \in \mathbb{Z}_p.$$

In literature this would be called **graph Laplacian** associated to the graph formed by the group $\mathbb{Z}_p$. We say that $\psi : \mathbb{Z}_p \to \mathbb{C}$ is an **eigenfunction** of the Laplacian with eigenvalue $\lambda \in \mathbb{C}$ if

$$\Delta\psi(t) = \lambda\psi(t), \qquad \text{for all } t \in \mathbb{Z}_p.$$

Prove that the function

$$\psi_k(t) := e^{2\pi i k t/p}, \quad t \in \mathbb{Z}_p$$

is an eigenfunction of the Laplacian with eigenvalue $\lambda_k = \cos(2\pi k/p) - 1$.

5. Define the iteration $T_\mu^n f = T_\mu(T_\mu^{n-1} f)$ with $T_\mu^0 f = f$ for $n \geq 1$. Prove that the L^1 norm

$$\|T_\mu^n f\|_1 \leq \sqrt{p}\sqrt{\sum_{k \in \mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}}$$

for any $n \in \mathbb{N}$.