VIIKKO 4 AV

M1

(a) $w = f(x,y,z) = xy + yz + zx$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$x = e^t \quad ; \quad x' = e^t$$

$$y = 2t^2 \quad ; \quad y' = 4t$$

$$z = e^{-t} \quad ; \quad z' = -e^{-t}$$

$$\frac{dw}{dt} = (y+z)\frac{dx}{dt} + (x+z)\frac{dy}{dt} + (x+y)\frac{dz}{dt}$$

$$= (2t^2 + e^{-t})e^t + (e^t + e^{-t})4t + (e^t + 2t^2)(-e^{-t})$$

$$= 2e^{-t}t\left(2 - t + e^{2t}(2+t)\right)$$

(b) $w = f(x,y) = \dfrac{2xy}{x^2+y^2}$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$x = 2t \quad ; \quad x' = 2$

$y = t^2 \quad ; \quad y' = 2t$

$$\frac{dw}{dt} = \left(-\frac{4x^2y}{(x^2+y^2)^2} + \frac{2y}{(x^2+y^2)} \right) \frac{dx}{dt}$$

$$+ \left(-\frac{4xy^2}{(x^2+y^2)^2} + \frac{2x}{(x^2+y^2)} \right) \frac{dy}{dt}$$

$$= \frac{2(t^2-4)}{(t^2+4)^2} \cdot 2 - \frac{4}{t}\frac{(t^2-4)}{(t^2+4)^2} \cdot 2t$$

$$= -\frac{4(t^2-4)}{(t^2+4)^2}$$

M2 $\quad h(x,y) = f(x\,g(y))$

$$\frac{\partial h}{\partial x} = g(y)\, f'(x\,g(y))$$

$$\frac{\partial h}{\partial y} = x\, g'(y)\, f'(x\,g(y))$$

$$\frac{\partial^2 h}{\partial x^2} = g(y)^2 f''(x\,g(y))$$

$$\frac{\partial^2 h}{\partial x\,\partial y} = \frac{\partial^2 h}{\partial y\,\partial x} = g'(y)\, f'(x\,g(y)) + x\, g'(y)\, g(y)\, f''(x\,g(y))$$

$$\frac{\partial^2 h}{\partial y^2} = x\, g''(y)\, f'(x\,g(y)) + x^2 g'(y)^2 f''(x\,g(y))$$

TEHTÄVÄ J1 Laske ketjusääntöä käyttäen $\dfrac{\partial w}{\partial s}$ ja $\dfrac{\partial w}{\partial t}$, kun

a) $w = x\ln(x^2+y^2), \quad x = s+t,\ y = s-t,$
b) $w = e^{x+2y}\sin(2x-y), \quad x = s^2+2t^2,\ y = 2s^2-t^2.$

RATKAISU

a) Lasketaan aluksi funktion $w(x,y) = w(x(s,t), y(s,t))$ osittaisderivaatat:

$$\frac{\partial w}{\partial x} = \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = \ln(2(s^2+t^2)) + \frac{(s+t)^2}{s^2+t^2}$$

ja

$$\frac{\partial w}{\partial y} = \frac{2xy}{x^2+y^2} = \frac{s^2-t^2}{s^2+t^2}.$$

Soveltamalla ketjusääntöä saadaan nyt

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\underbrace{\frac{\partial x}{\partial s}}_{=1} + \frac{\partial w}{\partial y}\underbrace{\frac{\partial y}{\partial s}}_{=1}\\ &= \ln(2(s^2+t^2)) + \frac{(s+t)^2}{s^2+t^2} + \frac{s^2-t^2}{s^2+t^2}\\ &= \ln(2(s^2+t^2)) + \frac{2s(s+t)}{s^2+t^2}\end{aligned}$$

ja

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{=1} + \frac{\partial w}{\partial y}\underbrace{\frac{\partial y}{\partial t}}_{=-1}\\ &= \ln(2(s^2+t^2)) + \frac{(s+t)^2}{s^2+t^2} - \frac{s^2-t^2}{s^2+t^2}\\ &= \ln(2(s^2+t^2)) + \frac{2t(s+t)}{s^2+t^2}.\end{aligned}$$

b) Funktion $w(x,y) = w(x(s,t), y(s,t))$ osittaisderivaatat ovat

$$\begin{aligned}\frac{\partial w}{\partial x} &= e^{x+2y}(\sin(2x-y) + 2\cos(2x-y)) \\ &= e^{5s^2}(\sin(5t^2) + 2\cos(5t^2))\end{aligned}$$

ja

$$\begin{aligned}\frac{\partial w}{\partial y} &= e^{x+2y}(2\sin(2x-y) - \cos(2x-y)) \\ &= e^{5s^2}(2\sin(5t^2) - \cos(5t^2)),\end{aligned}$$

joten ketjusääntöä soveltamalla tästä seuraa

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= e^{5s^2}(\sin(5t^2) + 2\cos(5t^2)) \cdot 2s \\ &\quad + e^{5s^2}(2\sin(5t^2) - \cos(5t^2)) \cdot 4s \\ &= 10se^{5s^2}\sin(5t^2)\end{aligned}$$

ja

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \\ &= e^{5s^2}(\sin(5t^2) + 2\cos(5t^2)) \cdot 4t \\ &\quad + e^{5s^2}(2\sin(5t^2) - \cos(5t^2)) \cdot (-2t) \\ &= 10te^{5s^2}\cos(5t^2).\end{aligned}$$

TEHTÄVÄ [redacted] Funktion f lineaarinen approksimaatio (tangenttitaso) saadaan kaavasta

J2

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Tässä tapauksessa linearisointi on luonnollista muodostaa pisteessä $(2, 2)$, jolloin $f(2, 2) = 2$,

$$f_x(2,2) = -\frac{24(2x + y)}{(x^2 + xy + y^2)^2}\bigg|_{(x,y)=(2,2)} = -1$$

ja

$$f_y(2,2) = -\frac{24(x + 2y)}{(x^2 + xy + y^2)^2}\bigg|_{(x,y)=(2,2)} = -1.$$

Siten

$$L(x, y) = 2 - (x - 2) - (y - 2) = 6 - x - y$$

ja approksimaatioksi saadaan

$$f(1.9, 2.1) \approx L(1.9, 2.1) = 6 - 1.9 - 2.1 = 2.$$

Vertailun vuoksi funktion f tarkka arvo pisteessä $(1.9, 2.1)$ on noin 1.99833.