MS-C1081 – Abstract Algebra **Problem set 4**
2022–2023 (Period III)
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During the exercise sessions you may work on the Warm-up problems, and the TAs will present solutions to those. Submit your solutions to Homework 1,2,3 and to the Fill-in-the-blanks problem on MyCourses before the deadline. Remember that the Homeworks and the Fill-in-the-blanks go to separate return boxes, and each return box accepts a single pdf file.

Warm-ups

Warm-up 1. Possible or not? For each point, give an example of a group G and a subgroup $H \lhd G$ satisfying the required properties, or explain why such an example does not exist:

1. G is infinite and G/H is finite,
2. G is infinite and G/H is infinite,
3. G is finite and G/H is infinite,
4. G is abelian and G/H is not abelian,
5. G is not abelian and G/H is abelian.

Warm-up 2 (Continuation of Homework 3 from Problem set 2)**.** Let G be a group and denote by 1 the identity of G. Consider the direct product $G^2 := G \times G$ and define

$$H := \{(x,y) \in G^2 \mid y = 1\} \qquad \text{and} \qquad K := \{(x,y) \in G^2 \mid x = y\}.$$

You already showed in Problem set 2 that H and K are both subgroups of G^2 and both isomorphic to G.

1. Show that H is normal in G^2.
2. Show that $G^2/H \cong G$.
3. Show that K is normal in G^2 if and only if G is abelian.

Warm-up 3. Think of $H = \mathrm{GL}_n(\mathbb{Q})$ as a subgroup of $G = \mathrm{GL}_n(\mathbb{R})$. Show that H is not normal in G if $n > 1$. (But H is normal in G if $n = 1$.)

Homework

Homework 1. Let $G = \mathrm{GL}_n(\mathbb{R}) = \{n \times n \text{ real matrices with non-zero determinant}\}$, equipped as usual with matrix multiplication.

1. Let $K = \left\{A \in G \mid \det(A) \in \mathbb{Q} \setminus \{0\}\right\}$. Show that K is a subgroup of G, and that it is normal in G. [3 points]
2. Show that $G/K \cong (\mathbb{R} \setminus \{0\}) \big/ (\mathbb{Q} \setminus \{0\})$.

   Hint. Consider $H = \mathrm{SL}_n(\mathbb{R}) = \{A \in G \mid \det(A) = 1\}$. Show that $G/H \cong \mathbb{R} \setminus \{0\}$ and $K/H \cong \mathbb{Q} \setminus \{0\}$ with the First Isomorphism Theorem. Conclude by using the Third Isomorphism Theorem. [3 points]

Homework 2. Let G be a group, and let H_1 and H_2 be subgroups of G.

1. Suppose that H_1 is normal in G and H_1 is isomorphic to H_2. Can we deduce that H_2 is normal in G?

   Hint. Look elsewhere in this sheet for inspiration. [2 points]

2. Suppose that H_1 and H_2 are isomorphic, and that both are normal in G. Is it true that G/H_1 and G/H_2 are isomorphic?

   Hint. Take $G = \mathbb{Z}$. [2 points]

3. Suppose f is an automorphism of G (that is, an isomorphism $G \to G$) that satisfies $f(H_1) = H_2$. Show that H_1 and H_2 are isomorphic. [2 points]

Homework 3. Let $(G, +)$ be an abelian group. Show that, for $x_1, \ldots, x_n \in G$,

$$\langle x_1, x_2, \ldots, x_n \rangle = \{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \mid a_1, \ldots, a_n \in \mathbb{Z}\}.$$

[6 points]

Fill-in-the-blanks. Complete the proof of the following claim:

Claim. Let G be a group, and let K be a normal subgroup of G for which the quotient G/K is abelian. All subgroups H of G with $K \subseteq H \subseteq G$ are normal in G.

Proof. Let H be a subgroup of G satisfying $K \subseteq H \subseteq G$. We will show that H is normal in G by using the normality criterion, that is, for all ________ and ________, we show that $x^{-1}hx \in H$. Recall that the set H/K, seen in Homework 2 of Problem set 3, is defined as $\{yK \mid y \in H\}$. If we show that the coset $x^{-1}hxK$ is in H/K, then we are done. Indeed

$$x^{-1}hxK \overset{(1)}{=} \_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \overset{(2)}{=} \_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \overset{(3)}{=} \_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \overset{(4)}{=} \_\_\_\_\_\_ \in H/K,$$

where in equality (1) we simply use the definition of the operation $\cdot$ in G/K, in (2) we use the abelianity of G/K, in (3) we use the fact that $x^{-1}K = (xK)^{-1}$, and in (4) we use the fact that eK is the identity of G/K, if e denotes the identity of G. □

[3 points]