Analysis, Random Walks and Groups

Exercise sheet 4

Homework exercises: Return these for marking to Kai Hippi in the tutorial on Week 5. Contact Kai by email if you cannot return these in-person, and you can arrange an alternative way to return your solutions. Remember to be clear in your solutions, if the solution is unclear and difficult to read, you can lose marks. Also, if you do not know how to solve the exercise, attempt something, you can get awarded partial marks.

1. (5pts)

(a) Define the subgroup $\Gamma := \{0, 2\} \subset \mathbb{Z}_4$. Let μ be any probability distribution on $\mathbb{Z}_4$ with support $\mathrm{spt}(\mu) = \Gamma$. Define the uniform measure on Γ by

$$\nu_\Gamma = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2.$$

Prove the following version of the Upper Bound Lemma:

$$d(\mu^{*n}, \nu_\Gamma) \leq \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \Gamma} |\widehat{\mu}(k)|^{2n}}$$

Hint: the proof of the regular upper bound lemma can help

(b) In the previous part (a), after how many convolutions is the total variation distance

$$d(\mu^{*n}, \nu_\Gamma) < \frac{1}{100}?$$

2. (5pts)

Prove the upper bound lemma in $\mathbb{Z}_2^d$.
Hint: the proof of the regular upper bound lemma can help

Further exercises: Attempt these before the tutorial, they are not marked and will be discussed in the tutorial. If you cannot attend the tutorial, but want to do the attendance marks, you can return your attempts to these before the tutorial to Kai. Here Kai will not mark the further exercises, but will look if an attempt has been made and awards the attendance mark for that week's tutorial.

3.

Let $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ in $\mathbb{Z}_4$. Find an upper bound for the mixing time $n_{\mathrm{mix}}(1/100)$ for μ, that is, after how many convolutions μ^{*n} is the total variation distance

$$d(\mu^{*n}, \lambda) < \frac{1}{100}?$$

4.

Prove the lower bound lemma (Theorem 5.2. in the lecture notes):
Let $\mu : \mathbb{Z}_p \to [0,1]$ be a probability distribution. Then for all $n \in \mathbb{N}$ we have

$$d(\mu^{*n}, \lambda) \geq \frac{1}{2}\sqrt{\frac{1}{p}\sum_{k\in\mathbb{Z}_p\setminus\{0\}} |\widehat{\mu}(k)|^{2n}}.$$

Hint: the ideas of the upper bound lemma are useful here. Instead of using the Cauchy-Schwarz inequality, try to use some function to get from the $L1$-identity form into the inner product form and then use Plancherel's theorem.

5.

Let $\sigma_1,\ \sigma_1\sigma_2,\ \sigma_1\sigma_2\sigma_3,\ \ldots$ be the random walk on S_{52} driven by the probability distribution μ describing the weak Borel shuffle ("μ 'chooses' permutation σ_i at the i^{th} step randomly and attachs it to the end of the walk: $\sigma_i \ldots \sigma_{i-1} \to \sigma_1 \ldots \sigma_i$). Write down the formula for this measure μ. Then, let $e \in S_{52}$ be the identity permutation. Apply the right convolution $\mu *_R \mu$ in the group S_{52} to compute the probability

$$\mathbb{P}(\sigma_1\sigma_2 = e).$$

Hint: recall the first exercise sheet of the course for the weak Borel shuffle; right convolution can be found also from the lecture notes.