Analysis, Random Walks and Groups

Exercise sheet 3: Solutions

These model solutions are the same models used by Tuomas in Manchester, but some exercises are omitted, attached into another exercises or separeted into separate exercises. I (Kai) might have commented somewhere with red color if I think it is in place. Corrections and improvements are welcome.

Homework exercises: Return these for marking to Kai Hippi in the tutorial on Week 4. Contact Kai by email if you cannot return these in-person, and you can arrange an alternative way to return your solutions. Remember to be clear in your solutions, if the solution is unclear and difficult to read, you can lose marks. Also, if you do not know how to solve the exercise, attempt something, you can get awarded partial marks.

1. (5pts)

We say a probability distribution μ on $\mathbb{Z}_p$ has a **spectral gap** if

$$|\widehat{\mu}(k)| < 1, \quad \text{for all } k \in \mathbb{Z}_p \setminus \{0\}.$$

Find the Fourier transform of the probability distribution

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$$

in $\mathbb{Z}_4$ (all the values of $\widehat{\mu}(k)$) and use this to prove that μ does not have a spectral gap. Prove then that

$$\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$

in $\mathbb{Z}^4$ has a spectral gap.

Notice also that μ is supported on a subgroup $\{0, 2\}$ so it cannot be ergodic. Notice also that the support $\operatorname{spt}\nu = \{0, 1\}$ which is not a coset of a proper subgroup of $\mathbb{Z}_4$ so ν is ergodic. Later we will see that in general spectral gap implies ergodicity, so ν is ergodic.

Solution 1.

The Fourier transform of μ at $k \in \mathbb{Z}_4$ is

$$\widehat{\mu}(k) = \sum_{t \in \mathbb{Z}_4} \mu(t)e^{-2\pi ikt/4} = \frac{1}{2}e^0 + \frac{1}{2}e^{4\pi ik/4} = \frac{1}{2}(1 + e^{k\pi i}).$$

We see that

$$e^{0\pi i} = 1, e^{1\pi i} = -1, e^{2\pi i} = 1, e^{3\pi i} = -1.$$

Thus

$$\widehat{\mu}(0) = 1, \widehat{\mu}(1) = 0, \widehat{\mu}(2) = 1, \widehat{\mu}(3) = 0.$$

Hence

$$|\widehat{\mu}(k)| = 1$$

for some $k \in \mathbb{Z}_4 \setminus \{0\}$ (value $k = 2$). In particular, μ does not have a spectral gap.

The Fourier transform of ν at $k \in \mathbb{Z}_4$ is

$$\widehat{\nu}(k) = \sum_{t \in \mathbb{Z}_4} \mu(t) e^{-2\pi i k t/4} = \frac{1}{2} e^0 + \frac{1}{2} e^{2\pi i k/4} = \frac{1}{2}(1 + e^{k\pi i/2}).$$

We see that

$$e^{0\pi i/2} = 1, e^{1\pi i/2} = i, e^{2\pi i/2} = -1, e^{3\pi i/2} = -i.$$

Thus

$$\widehat{\nu}(0) = 1, \widehat{\nu}(1) = \frac{1}{2}(1+i), \widehat{\nu}(2) = 0, \widehat{\nu}(3) = \frac{1}{2}(1-i).$$

Hence

$$|\widehat{\nu}(1)| = \frac{\sqrt{2}}{2} < 1, |\widehat{\nu}(1)| = 0 < 1, |\widehat{\nu}(3)| = \frac{\sqrt{2}}{2} < 1$$

Thus

$$|\widehat{\nu}(k)| < 1, \quad \text{for all } k \in \mathbb{Z}_p \setminus \{0\}$$

so ν has a spectral gap.

Given a probability distribution μ on $\mathbb{Z}_p$, recall that we defined the associated **transfer operator** as

$$T_\mu f(t) = \mu * f(t), \quad t \in \mathbb{Z}_p,$$

where $f : \mathbb{Z}_p \to \mathbb{C}$. A complex number $\lambda \in \mathbb{C}$ is an **eigenvalue** of T_μ if there exists a non-zero $\psi : \mathbb{Z}_p \to \mathbb{C}$ (called **eigenfunction** of T_μ) such that

$$T_\mu \psi(t) = \lambda \psi(t), \quad \text{for all } t \in \mathbb{Z}_p.$$

The **spectrum** $\sigma(T_\mu)$ of T_μ is then the collection of all eigenvalues

$$\sigma(T_\mu) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T_\mu\}$$

Notice here that $T_\mu f(t) = M_\mu f(t)$ when

$$M_\mu f(t) := f * \mu(t),$$

as $f * \mu = \mu * f$

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2. (5pts)

Given a probability distribution μ on $\mathbb{Z}_p$, prove that for each $k \in \mathbb{Z}_p$, the Fourier transform $\widehat{\mu}(k) \in \mathbb{C}$ is an eigenvalue of the transfer operator T_μ, that is, $\widehat{\mu}(k) \in \sigma(T_\mu)$.
Hint: Attempt to prove the function $\psi_k(t) = e^{2\pi i k t/p}, t \in \mathbb{Z}_p$, is an eigenfunction of T_μ with eigenvalue $\widehat{\mu}(k)$.

Solution 2.

For a fixed $k \in \mathbb{Z}_p$ denote

$$\psi_k(t) = e^{2\pi i k t/p}, \quad t \in \mathbb{Z}_p.$$

Notice that $\psi_k \neq 0$ for all $k \in \mathbb{Z}_p$. We claim that ψ_k is an eigenfunction of M_μ with eigenvalue $\widehat{\mu}(k)$.

By the convolution theorem we have for all $\ell \in \mathbb{Z}_p$ that

$$\widehat{\psi_k}(\ell)\widehat{\mu}(\ell) = \widehat{\psi_k * \mu}(\ell) = \widehat{M_\mu \psi_k}(\ell).$$

Fix $t \in \mathbb{Z}_p$. Hence by the Fourier series for $M_\mu \psi(t)$ at t we have that

$$M_\mu \psi(t) = \frac{1}{p} \sum_{\ell \in \mathbb{Z}_p} \widehat{M_\mu \psi_k}(\ell) e^{2\pi i \ell t/p} = \frac{1}{p} \sum_{\ell \in \mathbb{Z}_p} \widehat{\psi_k}(\ell) \widehat{\mu}(k) e^{2\pi i \ell t/p}. \tag{0.1}$$

Here we have that

$$\widehat{\psi_k}(k) = \sum_{t \in \mathbb{Z}_p} \psi_k(t) e^{-2\pi i k t/p} = \sum_{t \in \mathbb{Z}_p} e^{2\pi i k t/p} e^{-2\pi i k t/p} = \sum_{t \in \mathbb{Z}_p} 1 = p$$

and if $\ell \neq k$ we have

$$\widehat{\psi}(\ell) = \sum_{t \in \mathbb{Z}_p} \psi_k(t) e^{-2\pi i \ell t/p} = \sum_{t \in \mathbb{Z}_p} e^{2\pi i k t/p} e^{-2\pi i \ell/tp} = \sum_{t \in \mathbb{Z}_p} e^{2\pi i (k-\ell)t/p} = \frac{1 - e^{2\pi i (k-\ell)}}{1 - e^{2\pi i (k-\ell)/p}} = 0$$

by the exponential sum formula (which we can use as $k \neq \ell$). Hence by (0.1) we have

$$M_\mu \psi(t) = \frac{1}{p} \widehat{\mu}(k) p e^{2\pi i k t/p} + 0 = \widehat{\mu}(k) \psi_k(t)$$

so $\widehat{\mu}(k)$ is an eigenvalue of M_μ (i.e. $\widehat{\mu}(k) \in \sigma(M_\mu)$.

Further exercises: Attempt these before the tutorial, they are not marked and will be discussed in the tutorial. If you cannot attend the tutorial, but want to do the attendance marks, you can return your attempts to these before the tutorial to Kai. Here Kai will not mark the further exercises, but will look if an attempt has been made and awards the attendance mark for that week's tutorial.

3.

Conversely to Question 2, establish that if $\lambda \in \sigma(T_\mu)$, then $\lambda = \widehat{\mu}(k)$ for some $k \in \mathbb{Z}_p$.

In particular, together with Question 2, this proves that the spectrum agrees with the Fourier coefficients of μ:

$$\sigma(T_\mu) = \{\widehat{\mu}(k) : k \in \mathbb{Z}_p\}.$$

Solution 3.

Let $\lambda \in \sigma(M_\mu)$. Then there exists $\psi : \mathbb{Z}_p \to \mathbb{C}$ which is non-zero such that

$$M_\mu \psi(t) = \lambda \psi(t), \quad \text{for all } t \in \mathbb{Z}_p. \tag{0.2}$$

Fix $k \in \mathbb{Z}_p$ such that $\widehat{\psi}(k) \neq 0$. Such k exists since $\psi \neq 0$: indeed, if $\widehat{\psi}(k) = 0$ for all $k \in \mathbb{Z}_p$ we would have by the Fourier series for all $t \in \mathbb{Z}_p$ that

$$\psi(t) = \frac{1}{p} \sum_{k \in \mathbb{Z}_p} \widehat{\psi}(k) e^{2\pi i k t/p} = 0$$

so $\psi(t) = 0$ for all $t \in \mathbb{Z}_p$. Thus we know that $\widehat{\psi}(k) \neq 0$ for this $k \in \mathbb{Z}_p$.

Taking now Fourier transform from both sides of (0.2) gives us

$$\widehat{\psi * \mu}(k) = \widehat{\lambda \psi}(k).$$

By the convolution theorem and the homogeneity of Fourier transform we have

$$\widehat{\psi}(k)\widehat{\mu}(k) = \lambda\widehat{\psi}(k).$$

Dividing $\widehat{\psi}(k) \neq 0$ gives us

$$\widehat{\mu}(k) = \lambda$$

as claimed.

4.

Define the Laplace operator Δ for functions $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$\Delta f(t) = \frac{f(t \oplus 1) + f(t \ominus 1)}{2} - f(t), \quad t \in \mathbb{Z}_p.$$

In literature this would be called **graph Laplacian** associated to the graph formed by the group $\mathbb{Z}_p$. We say that $\psi : \mathbb{Z}_p \to \mathbb{C}$ is an **eigenfunction** of the Laplacian with eigenvalue $\lambda \in \mathbb{C}$ if

$$\Delta\psi(t) = \lambda\psi(t), \quad \text{for all } t \in \mathbb{Z}_p.$$

Prove that the function

$$\psi_k(t) := e^{2\pi ikt/p}, \quad t \in \mathbb{Z}_p$$

is an eigenfunction of the Laplacian with eigenvalue $\lambda_k = \cos(2\pi k/p) - 1$.

Solution 4.

By definition of the Laplacian, we have

$$\Delta\psi_k(t) = \frac{1}{2}(\psi_k(t \oplus 1) + \psi_k(t \ominus 1)) - \psi_k(t) = \frac{1}{2}(e^{2\pi ik(t\oplus 1)/p} + e^{2\pi ik(t\ominus 1)/p}) - \psi_k(t)$$

We notice that

$$e^{2\pi ik(t\oplus 1)/p} = e^{2\pi i(kt\oplus k)/p} = \psi_k(t)e^{2\pi ik/p}$$

and

$$e^{2\pi ik(t\ominus 1)/p} = e^{2\pi i(kt\ominus k)/p} = \psi_k(t)e^{-2\pi ik/p}.$$

Hence

$$\frac{1}{2}(e^{2\pi ik(t\oplus 1)/p} + e^{2\pi ik(t\ominus 1)/p}) = \frac{e^{2\pi ik/p} + e^{-2\pi ik/p}}{2} \cdot \psi_k(t) = \cos(2\pi k/p)\psi_k(t)$$

so with $\lambda_k = \cos(2\pi k/p) - 1$ we have

$$\Delta\psi_k(t) = \lambda_k\psi_k(t)$$

for all $t \in \mathbb{Z}_p$. Thus ψ_k is an eigenfunction of the Laplacian with eigenvalue λ_k.

5. Define the iteration $T_\mu^n f = T_\mu(T_\mu^{n-1} f)$ with $T_\mu^0 f = f$ for $n \geq 1$. Prove that the L^1 norm

$$\|T_\mu^n f\|_1 \leq \sqrt{p}\sqrt{\sum_{k\in\mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}}$$

for any $n \in \mathbb{N}$.

Solution 5.

We have that

$$\|M_\mu^n f\|_1 = \sum_{t\in\mathbb{Z}_p} |f * \mu^{*n}(t)|$$

By Cauchy-Schwartz applied with $t \mapsto |f * \mu^{*n}(t)|$ and the constant function 1, we have

$$\sum_{t\in\mathbb{Z}_p} |f * \mu^{*n}(t)| = \langle |f * \mu^{*n}|, 1\rangle \le \|f * \mu^{*n}\|_2 \|1\|_2$$

Here

$$\|1\|_2 = \sqrt{\sum_{t\in\mathbb{Z}_p} 1^2} = \sqrt{p}$$

and by the Plancherel's theorem and convolution theorem we obtain

$$\|f * \mu^{*n}\|_2 = \frac{1}{\sqrt{p}}\|\widehat{f * \mu^{*n}}\|_2 = \frac{1}{\sqrt{p}}\|\widehat{f}\widehat{\mu^{*n}}\|_2 = \frac{1}{\sqrt{p}}\|\widehat{f}\widehat{\mu}^n\|_2 = \frac{1}{\sqrt{p}}\sqrt{\sum_{k\in\mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}}$$

~~so the claim follows~~. Hence we have:

$$||M_\mu^n f||_1 \le \sqrt{p}||f * \mu^{*n}||_2 = \sqrt{p}\frac{1}{\sqrt{p}}\sqrt{\sum_{k\in\mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}} = \sqrt{\sum_{k\in\mathbb{Z}_p} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^{2n}}.$$

Clearly this proves what we wanted, and it is even better we set out to solve.