Shortcuts to Fermi-Dirac and Bose-Einstein distribution

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I. FERMI-DIRAC DISTRIBUTION

Assume the single-particle state ϵ is a grand canonical ensemble. For this the grand partition function based on Pauli exclusion principle reads

$$\begin{aligned}\mathcal{Z} &= e^{-\beta\times 0} + e^{-\beta(\epsilon-\mu)} \\ &= 1 + e^{-\beta(\epsilon-\mu)},\end{aligned} \tag{1}$$

where $\beta \equiv 1/(k_{\rm B}T)$ and μ is the chemical potential of the system. The grand partition function here is $\mathcal{Z} = e^{-\beta\Phi}$, where Φ is called the grand potential of the system. In general, the grand potential of a system is given by

$$\Phi = \langle E\rangle - TS - \mu\langle N\rangle, \tag{2}$$

where $\langle N\rangle$ is the average occupation in the state ϵ,

$$\langle N\rangle = -\frac{\partial\Phi}{\partial\mu} = \frac{1}{\beta\mathcal{Z}}\frac{d\mathcal{Z}}{d\mu} = \frac{1}{1+e^{\beta(\epsilon-\mu)}}. \tag{3}$$

Here we identify the population $\langle N\rangle$ with the distribution

$$f(\epsilon) = \frac{1}{1+e^{\beta(\epsilon-\mu)}} \tag{4}$$

i.e. the Fermi-Dirac distribution function.

II. BOSE-EINSTEIN DISTRIBUTION

With the same procedure as in the previous subsection, we again consider a single-particle state as a grand canonical system. Unlike in the previous case, here for bosons multiple oocupations are allowed whereby the partition function reads

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{-N\beta(\epsilon-\mu)} = \frac{1}{1-e^{-\beta(\epsilon-\mu)}}, \tag{5}$$

where we summed the geometric series in the second step. Then as before we find the average occupation $\langle N\rangle = \frac{1}{\beta\mathcal{Z}}\frac{d\mathcal{Z}}{d\mu} = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$, and we again identify the population $\langle N\rangle$ with the distribution

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1} \tag{6}$$

i.e. the Bose-Einstein distribution function.