MS-C1081 – Abstract Algebra **Problem set 5**
2022–2023 (Period III)
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During the exercise sessions you may work on the Warm-up problems, and the TAs will present solutions to those. Submit your solutions to Homework 1,2,3 and to the Fill-in-the-blanks problem on MyCourses before the deadline. Remember that the Homeworks and the Fill-in-the-blanks go to separate return boxes, and each return box accepts a single pdf file.

Warm-ups

Warm-up 1. Consider the ring of functions $F := \{\text{functions } \mathbb{R} \to \mathbb{R}\}$ with pointwise addition and multiplication, that is, for all $x \in \mathbb{R}$, define

$$(f+g)(x) := f(x) +_{\mathbb{R}} g(x) \qquad \text{and} \qquad (f \cdot g)(x) := f(x) \cdot_{\mathbb{R}} g(x).$$

For each of the following subsets of F, determine whether it is a subring, and if yes, determine if it is an ideal:

1. $C := \{f_c \mid c \in \mathbb{R}\}$, where for all $c, x \in \mathbb{R}$ we define $f_c(x) := c$,
2. $P := \{\text{polynomial functions } \mathbb{R} \to \mathbb{R}\}$,
3. $N := \{\text{functions } \mathbb{R} \to \mathbb{R} \text{ whose values are all non-negative}\}$,
4. $D := \{\text{differentiable functions } \mathbb{R} \to \mathbb{R}\}$,
5. $S_f := \{f \cdot g \mid g \in F\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a fixed continuous function.

Warm-up 2. In the following, consider $\mathbb{Z}$, $\mathbb{R}$, etc, with their usual sum and multiplication. Determine which of the following maps are ring homomorphisms, and for those that are ring homomorphisms determine their kernel:

$$\begin{aligned} f_1 \colon \mathbb{Z} &\longrightarrow \mathbb{Z} \\ n &\longmapsto 3n \end{aligned} \qquad \begin{aligned} f_4 \colon \mathbb{R} &\longrightarrow \mathbb{C} \\ x &\longmapsto x + ix \end{aligned} \qquad \begin{aligned} f_7 \colon \mathbb{Z}_6 &\longrightarrow \mathbb{Z}_6 \\ [x]_6 &\longmapsto [3x]_6 \end{aligned} \qquad \begin{aligned} f_{10} \colon M_{2,2}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ A &\longmapsto \det(A) \end{aligned}$$

$$\begin{aligned} f_2 \colon \mathbb{Z} &\longrightarrow \mathbb{R} \\ n &\longmapsto 2^n \end{aligned} \qquad \begin{aligned} f_5 \colon \mathbb{R} &\longrightarrow \mathbb{C} \\ x &\longmapsto ix \end{aligned} \qquad \begin{aligned} f_8 \colon \mathbb{Z}_6 &\longrightarrow \mathbb{Z}_6 \\ [x]_6 &\longmapsto [2x]_6 \end{aligned} \qquad \begin{aligned} f_{11} \colon M_{2,2}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ A &\longmapsto A_{11} \end{aligned}$$

$$\begin{aligned} f_3 \colon \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \overline{z} \end{aligned} \qquad \begin{aligned} f_6 \colon \mathbb{R} &\longrightarrow \mathbb{C} \\ x &\longmapsto x \end{aligned} \qquad \begin{aligned} f_9 \colon \mathbb{Z}_2 &\longrightarrow \mathbb{Z}_6 \\ [x]_2 &\longmapsto [3x]_6 \end{aligned} \qquad \begin{aligned} f_{12} \colon M_{2,2}(\mathbb{R}) &\longrightarrow M_{2,2}(\mathbb{R}) \\ A &\longmapsto A^T, \end{aligned}$$

where $\overline{z}$ is the complex conjugate of z, A_{11} is the top-left entry of the matrix A, and A^T is the transpose of A.

Warm-up 3. The union of ideals is in general not an ideal. However, show that if you have a chain of inclusions of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \subseteq I_j \subseteq I_{j+1} \subseteq \ldots,$$

then $\bigcup_{j \in \mathbb{N}} I_j$ is an ideal.

Homework

Homework 1. Let R be a commutative unital ring.

1. Let $I_1, I_2, \dots, I_k \subseteq R$ be ideals. Recall that $I_1 + I_2 + \cdots + I_k$ denotes $(I_1 \cup I_2 \cup \cdots \cup I_k)$, that is, the ideal generated by the union. Show that

   $$I_1 + I_2 + \cdots + I_k = \{b_1 + b_2 + \cdots + b_k \mid b_j \in I_j\}.$$

   Hint. Show that the right-hand side is an ideal, and it is contained in every ideal containing $I_1 \cup I_2 \cup \cdots \cup I_k$. [3 points]

2. Let $a_1, a_2, \dots, a_k \in R$. Recall that we write $(a_1, a_2, \dots, a_k) := (\{a_1, a_2, \dots, a_k\})$. Show that

   $$(a_1, a_2, \dots, a_k) = \{r_1 a_1 + r_2 a_2 + \cdots + r_k a_k \mid r_j \in R\}.$$

   [3 points]

Homework 2. Let $(R, +_R, \cdot_R)$ and $(T, +_T, \cdot_T)$ be two rings. Define the ring $(R \times T, +, \cdot)$, where for all $(a, b), (a', b') \in R \times T$ we set

$$(a,b) + (a',b') := (a +_R a',\, b +_T b') \qquad \text{and} \qquad (a,b) \cdot (a',b') := (a \cdot_R a',\, b \cdot_T b').$$

The additive identity of $R \times T$ is $(0_R, 0_T)$. Assume $R \supsetneq \{0_R\}$.

1. Show that if $T \supsetneq \{0_T\}$, then $(R \times T, +, \cdot)$ is not a domain. [3 points]
2. Show that if R is a domain and $T = \{0_T\}$, then $(R \times T, +, \cdot)$ is a domain. [3 points]

Homework 3. Let R and R' be commutative unital rings, and let $f\colon R \to R'$ be a ring homomorphism satisfying $f(1_R) = 1_{R'}$.

1. If J is a prime ideal of R', show that the pre-image $I := f^{-1}(J)$ is a prime ideal of R.

   [2 points]
2. If R' is an integral domain, show that $\ker(f)$ is a prime ideal of R. [2 points]
3. If f is bijective (in addition to being a ring homomorphism, so that f is a ring isomorphism), show that the inverse function $f^{-1}\colon R' \to R$ is a ring isomorphism.

   [2 points]

Fill-in-the-blanks. Complete the proof of the following claim:

Claim. Let R be a commutative ring with unity 1. If $\{0\}$ is a maximal ideal of R, then R is a field.

Proof. First of all we need to check that R is not the zero ring: by the definition of a maximal ideal, $\{0\}$ is in particular a ____________ ideal, so that $R \supsetneq \{0\}$.

Next, let $a \in R \setminus \{0\}$. We need to check that ____________________________________. Since $\{0\}$ is maximal and $a \notin \{0\}$, the principal ideal ______ has to be equal to R. Then in particular, since $1 \in R$, there exists some __________ satisfying _____ $\cdot a = 1$. Hence a is invertible. ☐

[3 points]