Analysis, Random Walks and Groups

Exercise sheet 5 (last sheet of the course)

Homework exercises: Return these for marking to Kai Hippi in the tutorial on Week 6. Contact Kai by email if you cannot return these in-person, and you can arrange an alternative way to return your solutions. Remember to be clear in your solutions, if the solution is unclear and difficult to read, you can lose marks. Also, if you do not know how to solve the exercise, attempt something, you can get awarded partial marks.

1. (5pts) Let G be a finite group and $\varrho_1 : G \to U(V_{\varrho_1})$ and $\varrho_2 : G \to U(V_{\varrho_2})$ be unitary representations and let $\varphi : V_1 \to V_2$ be a morphism between ϱ_1 to ϱ_2. Prove the following version of Schur's lemma:

(a) if ϱ_1 is irreducible, then φ is either injective or zero;
(b) if ϱ_2 is irreducible, then φ is either surjective or zero.

2. (5pts) Fix $p \geq 2$ and let H be a subgroup of $\mathbb{Z}_p$. Prove the **Poisson summation formula**: for any $f : \mathbb{Z}_p \to \mathbb{C}$, we have

$$\frac{1}{|H|} \sum_{h \in H} f(h) = \frac{1}{p} \sum_{k \in H^\perp} \widehat{f}(k),$$

where

$$H^\perp := \{k \in \mathbb{Z}_p : e^{2\pi i k t/p} = 1 \text{ for all } t \in H\}.$$

Hint: There are couple of ways to do this. One way is to apply the inverse Fourier transform to the function $F(s) = \sum_{h \in H} f(sh)$, and then set $s = 1$, or first verifying the Poisson summation formula for Dirac measures δ_t on G and then using linearity to extend for all f.

Further exercises: Attempt these before the tutorial, they are not marked and will be discussed in the tutorial. If you cannot attend the tutorial, but want to do the attendance marks, you can return your attempts to these before the tutorial to Kai. Here Kai will not mark the further exercises, but will look if an attempt has been made and awards the attendance mark for that week's tutorial.

3. Let G be a finite group and $x \in G$, $x \neq 1$. Define a probability distribution in G:

$$\mu_x = \frac{1}{2}\delta_x + \frac{1}{2}\delta_{-x}$$

Give an example of a finite group G such that the Fourier transform

$$\widehat{\mu_x}(\xi)$$

is unitary for all $\xi \in \widehat{G}$.

4. Recall that the **Uncertainty Principle** in a finite abelian group G said: for all $f : G \to \mathbb{C}$ with $f \neq 0$, we have

$$|\mathrm{spt}(f)||\mathrm{spt}(\widehat{f})| \geq |G|.$$

(we did this in the lecture for $G = \mathbb{Z}_p$, but the proof is same for general abelian G)

Now, prove the following structure theorem relates to this: if G is an abelian group and a function $f : G \to \mathbb{C}$ with $0 \in \mathrm{spt}(f)$ satisfies the equality:

$$|\mathrm{spt}(f)||\mathrm{spt}(\widehat{f})| = |G|,$$

then $\mathrm{spt}(f)$ is a subgroup of G.

5. Construct a probability distribution μ on S_3 with entropy $H(\mu) = \log 2$. Then, using the character table of S_3 and finding the dimensions of the irreducible representations of S_3, construct some $n_0 \in \mathbb{N}$ such that the entropy $H(\mu^{*n}) > \log 6 + \frac{1}{1000}$ for all $n \geq n_0$.