Analysis, Random Walks and Groups

Exercise sheet 4

Homework exercises: Return these for marking to Kai Hippi in the tutorial on Week 5. Contact Kai by email if you cannot return these in-person, and you can arrange an alternative way to return your solutions. Remember to be clear in your solutions, if the solution is unclear and difficult to read, you can lose marks. Also, if you do not know how to solve the exercise, attempt something, you can get awarded partial marks.

Kai's comments etc. are in red color.

1. (5pts)

(a) Define the subgroup $\Gamma := \{0, 2\} \subset \mathbb{Z}_4$. Let μ be any probability distribution on $\mathbb{Z}_4$ with support $\mathrm{spt}(\mu) = \Gamma$. Define the uniform measure on Γ by

$$\nu_\Gamma = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2.$$

Prove the following version of the Upper Bound Lemma:

$$d(\mu^{*n}, \nu_\Gamma) \leq \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \Gamma} |\widehat{\mu}(k)|^{2n}}$$

Hint: the proof of the regular upper bound lemma can help

Solution 1.a

By the L^1 identity

$$4d(\mu^{*n}, \nu_\Gamma)^2 = \Big(\sum_{t=0}^{p-1} |\mu^{*n}(t) - \nu_\Gamma(t)|\Big)^2.$$

Since $\lambda(t) = 1/p$ for all $t \in \mathbb{Z}_p$, we have

$$\Big(\sum_{t=0}^{p-1} |\mu^{*n}(t) - \nu_\Gamma(t)|\Big)^2 = p^2\Big(\sum_{t=0}^{p-1} \lambda(t)|\mu^{*n}(t) - \nu_\Gamma(t)|\Big)^2.$$

Using the definition of the inner product for the functions

$$f(t) := \lambda(t), \quad \text{and} \quad g(t) := |\mu^{*n}(t) - \nu_\Gamma(t)|, \quad t \in \mathbb{Z}_p,$$

and Cauchy-Schwartz Inequality we obtain

$$\Big(\sum_{t=0}^{p-1} \lambda(t)|\mu^{*n}(t) - \nu_\Gamma(t)|\Big)^2 = |\langle f, g\rangle|^2 \leq \|f\|_2^2 \|g\|_2^2.$$

The L^2 norms here are

$$\|f\|_2^2 = \sum_{t \in \mathbb{Z}_p} \lambda(t)^2 = \sum_{t \in \mathbb{Z}_p} p^{-2} = p^{-1}$$

and by definition of g:

$$\|g\|_2^2 = \sum_{t \in \mathbb{Z}_p} |\mu^{*n}(t) - \nu_\Gamma(t)|^2.$$

Hence we have proved

$$4d(\mu^{*n}, \lambda)^2 \leq p \sum_{t \in \mathbb{Z}_p} |\mu^{*n}(t) - \nu_\Gamma(t)|^2 = p\|\mu^{*n} - \nu_\Gamma\|_2^2$$

By Plancherel's Theorem, we have that

$$p\|\mu^{*n} - \nu_\Gamma\|_2^2 = \|\widehat{\mu^{*n} - \nu_\Gamma}\|_2^2 = \|\widehat{\mu^{*n}} - \widehat{\nu_\Gamma}\|_2^2 = \sum_{k=0}^{p-1} |\widehat{\mu^{*n}}(k) - \widehat{\nu_\Gamma}(k)|^2.$$

Computing Fourier transform of ν_Γ we see that

$$\widehat{\nu_\Gamma}(k) = \frac{1}{2}(1 + e^{-\pi i k}) = \begin{cases} 1, & k \in \Gamma; \\ 0, & k \notin \Gamma. \end{cases}$$

On the other hand, by the Convolution Theorem we have

$$\widehat{\mu^{*n}}(k) = \widehat{\mu}(k)^n.$$

As $\operatorname{spt} \mu = \{0, 2\}$ we know that there exists $0 < \alpha < 1$ such that $\mu = \alpha\delta_0 + (1 - \alpha)\delta_2$. Thus

$$\widehat{\mu}(k) = \alpha + (1 - \alpha)e^{-\pi i k}.$$

Thus

$$\widehat{\mu}(0) = 1 \quad \text{and} \quad \widehat{\mu}(2) = 1.$$

Hence the difference

$$\widehat{\mu^{*n}}(k) - \widehat{\nu_\Gamma}(k) = \begin{cases} 0, & k \in \Gamma; \\ \widehat{\mu^{*n}}(k), & k \notin \Gamma. \end{cases}$$

This gives

$$\sum_{k=0}^{p-1} |\widehat{\mu^{*n}}(k) - \widehat{\nu_\Gamma}(k)|^2 = \sum_{k \in \mathbb{Z}_p \setminus \Gamma} |\widehat{\mu}(k)|^{2n}.$$

Dividing by 4 and taking square roots from both sides gives the claim.

(b) In the previous part (a), after how many convolutions is the total variation distance

$$d(\mu^{*n}, \nu_\Gamma) < \frac{1}{100}?$$

Solution 1.b

We have that $\mu = \alpha\delta_0 + (1 - \alpha)\delta_2$ for some $0 < \alpha < 1$ since $\operatorname{spt} \mu = \Gamma = \{0, 2\}$. Hence

$$\widehat{\mu}(k) = \alpha + (1 - \alpha)e^{-\pi i k}$$

Thus

$$\widehat{\mu}(1) = \widehat{\mu}(3) = 2\alpha - 1$$

so by the previous exercise

$$d(\mu^{*n}, \nu_\Gamma) \leq \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \Gamma} |\widehat{\mu}(k)|^{2n}} \leq \frac{1}{2}\sqrt{2|2\alpha - 1|^{2n}} = \frac{\sqrt{2}}{2}|2\alpha - 1|^n.$$

Thus after taking logarithms we know that (when $\alpha \neq 1/2$) that

$$\frac{\sqrt{2}}{2}|2\alpha - 1|^n < \frac{1}{100}$$

if and only if

$$n > \frac{\log(1/(50\sqrt{2}))}{\log|2\alpha - 1|}$$

so if $\alpha \neq 1/2$ and

$$n \geq \left\lceil \frac{\log(1/(50\sqrt{2}))}{\log|2\alpha - 1|} \right\rceil$$

then

$$d(\mu^{*n}, \nu_\Gamma) < \frac{1}{100}.$$

When $\alpha = 1/2$, then $\mu = \nu_\Gamma$, so as Γ is a subgroup we have for all $n \in \mathbb{N}$ that

$$\mu^{*n} = \nu_\Gamma^{*n} = \nu_\Gamma,$$

which implies

$$d(\mu^{*n}, \nu_\Gamma) = 0.$$

Thus $n \geq 1$ is enough (recall that $\mu^{*1} = \mu$).

2. (5pts)

Prove the upper bound lemma in $\mathbb{Z}_2^d$.
Hint: the proof of the regular upper bound lemma can help

Solution 2.

Let $\mu : \mathbb{Z}_2^d \to [0,1]$ be a probability distribution and $\lambda(t) = 1/2^d$, $t \in \mathbb{Z}_2^d$, the uniform distribution on $\mathbb{Z}_2^d$. Fix $n \in \mathbb{N}$. We claim that

$$d(\mu^{*n}, \lambda) \leq \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_2^d \setminus \{0\}} |\widehat{\mu}(k)|^{2n}}.$$

By the L^1 identity, we have

$$4d(\mu^{*n}, \lambda)^2 = \Big(\sum_{t \in \mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|\Big)^2.$$

Since $\lambda(t) = 1/2^d$ for all $t \in \mathbb{Z}_2^d$, we have

$$\Big(\sum_{t \in \mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|\Big)^2 = 2^{2d}\Big(\sum_{t \in \mathbb{Z}_2^d} \lambda(t)|\mu^{*n}(t) - \lambda(t)|\Big)^2.$$

Using the definition of the inner product for the functions

$$f(t) := \lambda(t), \quad \text{and} \quad g(t) := |\mu^{*n}(t) - \lambda(t)|, \quad t \in \mathbb{Z}_2^d,$$

and Cauchy-Schwartz Inequality we obtain

$$\Big(\sum_{t \in \mathbb{Z}_2^d} \lambda(t)|\mu^{*n}(t) - \lambda(t)|\Big)^2 = |\langle f, g\rangle|^2 \leq \|f\|_2^2 \|g\|_2^2.$$

The L^2 norms here are

$$\|f\|_2^2 = \sum_{t\in\mathbb{Z}_2^d} \lambda(t)^2 = \sum_{t\in\mathbb{Z}_2^d} 2^{-2d} = 2^{-d}$$

and by definition of g:

$$\|g\|_2^2 = \sum_{t\in\mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|^2.$$

Hence we have proved

$$4d(\mu^{*n}, \lambda)^2 \leq 2^d \sum_{t\in\mathbb{Z}_2^d} |\mu^{*n}(t) - \lambda(t)|^2 = 2^d \|\mu^{*n} - \lambda\|_2^2$$

By Plancherel's Theorem, we have that

$$2^d\|\mu^{*n} - \lambda\|_2^2 = \|\widehat{\mu^{*n} - \lambda}\|_2^2 = \|\widehat{\mu^{*n}} - \widehat{\lambda}\|_2^2 = \sum_{k\in\mathbb{Z}_2^d} |\widehat{\mu^{*n}}(k) - \widehat{\lambda}(k)|^2.$$

In $\mathbb{Z}_2^d$ we have that

$$\widehat{\lambda}(k) = \begin{cases} 1, & k = 0; \\ 0, & k \neq 0. \end{cases}$$

On the other hand, as μ^{*n} is a probability distribution, the Fourier transform

$$\widehat{\mu^{*n}}(0) = \sum_{t\in\mathbb{Z}_2^d} \mu^{*n}(t) = 1.$$

Hence the difference

$$\widehat{\mu^{*n}}(k) - \widehat{\lambda}(k) = \begin{cases} 0, & k = 0; \\ \widehat{\mu^{*n}}(k), & k \neq 0. \end{cases}$$

Moreover, by the Convolution Theorem we have

$$\widehat{\mu^{*n}}(k) = \widehat{\mu}(k)^n.$$

Thus

$$\sum_{k\in\mathbb{Z}_2^d} |\widehat{\mu^{*n}}(k) - \widehat{\lambda}(k)|^2 = \sum_{k\in\mathbb{Z}_2^d\setminus\{0\}} |\widehat{\mu}(k)|^{2n}.$$

Dividing by 4 and taking square roots from both sides gives the claim.

Further exercises: Attempt these before the tutorial, they are not marked and will be discussed in the tutorial. If you cannot attend the tutorial, but want to do the attendance marks, you can return your attempts to these before the tutorial to Kai. Here Kai will not mark the further exercises, but will look if an attempt has been made and awards the attendance mark for that week's tutorial.

3.

Let $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ in $\mathbb{Z}_4$. Find an upper bound for the mixing time $n_{\text{mix}}(1/100)$ for μ, that is, after how many convolutions μ^{*n} is the total variation distance

$$d(\mu^{*n}, \lambda) < \frac{1}{100}?$$

Solution 3.

By the Upper Bound Lemma, we have

$$d(\mu^{*n}, \lambda) \leq \frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \{0\}} |\widehat{\mu}(k)|^{2n}}.$$

In the Week 7 exercise 2 (the measure ν there), we computed that

$$|\widehat{\mu}(1)| = \frac{\sqrt{2}}{2}, \quad |\widehat{\mu}(2)| = 0, \quad |\widehat{\mu}(3)| = \frac{\sqrt{2}}{2}.$$

Hence

$$\frac{1}{2}\sqrt{\sum_{k \in \mathbb{Z}_4 \setminus \{0\}} |\widehat{\mu}(k)|^{2n}} \leq \frac{1}{2}\sqrt{\left(\frac{\sqrt{2}}{2}\right)^{2n} + \left(\frac{\sqrt{2}}{2}\right)^{2n}} = \left(\frac{\sqrt{2}}{2}\right)^{n+1}.$$

Now, to have

$$\left(\frac{\sqrt{2}}{2}\right)^{n+1} < \frac{1}{100}$$

after taking logarithms, we have

$$(n+1)\log\left(\frac{\sqrt{2}}{2}\right) < \log\left(\frac{1}{100}\right)$$

and as $\frac{\sqrt{2}}{2} < 1$, the logarithm is negative, so

$$n > \frac{\log 100}{\log 2 - \log\sqrt{2}} - 1 \approx 12.2877.$$

Hence $n_{\mathrm{mix}}(1/100) \leq 13$.

4.

Prove the lower bound lemma (Theorem 5.2. in the lecture notes):
Let $\mu : \mathbb{Z}_p \to [0,1]$ be a probability distribution. Then for all $n \in \mathbb{N}$ we have

$$d(\mu^{*n}, \lambda) \geq \frac{1}{2}\sqrt{\frac{1}{p}\sum_{k \in \mathbb{Z}_p \setminus \{0\}} |\widehat{\mu}(k)|^{2n}}.$$

Hint: the ideas of the upper bound lemma are useful here. Instead of using the Cauchy-Schwarz inequality, try to use some function to get from the L1-identity form into the inner product form and then use Plancherel's theorem.

Solution 4.

So $\mu : \mathbb{Z}_p \to [0,1]$ is a probability distribution on $\mathbb{Z}_p$. Let $n \in \mathbb{N}$. Consider:

$$\begin{aligned}
& 4d(\mu^{*n},\lambda)^2 \\
= \quad & [L1-identity] \quad \left(\sum_{k=0}^{p-1} |\mu^{*n}(k)-\lambda(k)|\right)^2 \\
\geq \quad & [Square\ of\ sums\ \geq sum\ of\ squares] \quad \left(\sum_{k=0}^{p-1} |\mu^{*n}(k)-\lambda(k)|^2\right) \\
= \quad & \left(\sum_{k=0}^{p-1} (\mu^{*n}(k)-\lambda(k))^2\right) \\
= \quad & [inner\ product] \quad \langle \mu^{*n}-\lambda, \mu^{*n}-\lambda\rangle \\
= \quad & [Plancherel's\ theorem] \quad \frac{1}{p}\langle \widehat{\mu^{*n}-\lambda}, \widehat{\mu^{*n}-\lambda}\rangle \\
= \quad & \frac{1}{p}\langle \widehat{\mu^{*n}}-\widehat{\lambda}, \widehat{\mu^{*n}}-\widehat{\lambda}\rangle \\
= \quad & \frac{1}{p}\sum_{l=0}^{p-1} (\widehat{\mu^{*n}}(l)-\widehat{\lambda}(l))^2 \\
= \quad & [Convolution\ theorem] \quad \frac{1}{p}\sum_{l=0}^{p-1} (\widehat{\mu}(l)^n-\widehat{\lambda}(l))^2 \\
= \quad & [\widehat{\lambda} = \chi_{\{0\}}, \widehat{\mu}(0) = 1] \quad \frac{1}{p}\sum_{l=1}^{p-1} (\widehat{\mu}(l)^n)^2 \\
= \quad & \frac{1}{p}\sum_{l=1}^{p-1} |\widehat{\mu}(l)|^{2n}
\end{aligned}$$

Hence we have:

$$\begin{aligned}
& 4d(\mu^{*n},\lambda)^2 \geq \frac{1}{p}\sum_{l=1}^{p-1} |\widehat{\mu}(l)|^{2n} \\
\iff \quad & 2d(\mu^{*n},\lambda) \geq \sqrt{\frac{1}{p}\sum_{l=1}^{p-1} |\widehat{\mu}(l)|^{2n}} \\
\iff \quad & d(\mu^{*n},\lambda) \geq \frac{1}{2}\sqrt{\frac{1}{p}\sum_{l=1}^{p-1} |\widehat{\mu}(l)|^{2n}}
\end{aligned}$$

□

5.

Let $\sigma_1,\ \sigma_1\sigma_2,\ \sigma_1\sigma_2\sigma_3,\ \ldots$ be the random walk on S_{52} driven by the probability distribution μ describing the weak Borel shuffle (" μ 'chooses' permutation σ_i at the i^{th} step randomly and attachs it to the end of the walk: $\sigma_i \ldots \sigma_{i-1} \to \sigma_1 \ldots \sigma_i$). Write down the formula for this measure μ. Then, let $e \in S_{52}$ be the identity permutation. Apply the right convolution $\mu *_R \mu$ in the group S_{52} to compute the probability

$$\mathbb{P}(\sigma_1\sigma_2 = e).$$

Hint: recall the first exercise sheet of the course for the weak Borel shuffle; right convolution can be found also from the lecture notes.

Solution 5.

Define a permutation

$$\sigma_k(j) = \begin{cases} k & \text{if } j = 0; \\ j-1 & \text{if } 1 \le j \le k; \\ j & \text{if } k < j \le 51; \end{cases}$$

Then define the random permutation $\sigma \in S_{52}$ by choosing $k \in \{0, 1, \ldots, 52\ 51\}$ uniformly with probability $1/52$ and setting $\sigma = \sigma_k$. Hence the probability distribution on S_{52} can be defined formally for all $\sigma \in S_{52}$ as

$$\mu(\sigma) = \begin{cases} \frac{1}{52}, & \text{if } \sigma = \sigma_k \text{ for some } k = 0, 1, 2, \ldots, 51; \\ 0, & \text{otherwise.} \end{cases}$$

We have that the probability

$$\mathbb{P}(\sigma_1 \sigma_2 = e) = \mu * \mu(e)$$

By the definition of the right convolution we have

$$\mu * \mu(e) = \sum_{\sigma \in S_{52}} \mu(e\sigma^{-1})\mu(\sigma)$$

We note that $e\sigma^{-1} = \sigma^{-1}$ since e is the neutral element of the group S_{52}. Since $\mu(\sigma) = 0$ for all $\sigma \neq \sigma_k$ for some $k = 0, 1, 2, \ldots, 51$, we have

$$\sum_{\sigma \in S_{52}} \mu(e\sigma^{-1})\mu(\sigma) = \sum_{k=0}^{52} \mu(\sigma_k^{-1})\mu(\sigma_k).$$

When $k = 0$ we see that $\sigma_0^{-1} = \sigma_0$ so

$$\mu(\sigma_0^{-1}) = \mu(\sigma_0) = \frac{1}{52}.$$

When $k = 1$ we see that $\sigma_1^{-1} = \sigma_1$ so

$$\mu(\sigma_1^{-1}) = \mu(\sigma_1) = \frac{1}{52}.$$

However, if $k \ge 2$, then σ_k^{-1} can never be any of the permutations σ_ℓ, $\ell = 0, 1, 2, \ldots, 51$ because $\sigma_k^{-1}(k) = 0$ and we have $\sigma_\ell(1) = 0$ when $\ell \ge 1$, which would force $k = 1$. Thus

$$\mu(\sigma_k^{-1}) = 0, \quad \text{for all } k = 2, 3, \ldots, 51.$$

This implies that

$$\sum_{k=0}^{52} \mu(\sigma_k^{-1})\mu(\sigma_k) = \mu(\sigma_0^{-1})\mu(\sigma_0) + \mu(\sigma_1^{-1})\mu(\sigma_1) = \frac{1}{52^2} + \frac{1}{52^2} = \frac{2}{52^2}.$$

Thus we have

$$\mathbb{P}(\sigma_1 \sigma_2 = e) = \mu * \mu(e) = \frac{2}{52^2}.$$