MS-C1081 – Abstract Algebra **Problem set 6**
2022–2023 (Period III)
Milo Orlich – Rahinatou Njah **Deadline: Tue 21.2.2023 at 10am**

During the exercise sessions you may work on the Warm-up problems, and the TAs will present solutions to those. Submit your solutions to Homework 1,2,3 on MyCourses before the deadline. Remember that the return box accepts a single pdf file.

Definitions. Let R be a ring.

- An element $a \in R \setminus \{0\}$ is called a ***nilpotent*** if there exists $n \in \mathbb{Z}_{>0}$ such that $a^n = 0$.
- If R has no nilpotents, R is called a ***reduced ring***.
- Given an ideal $I \subseteq R$, the set $\sqrt{I} := \{r \in R \mid \exists_{n \in \mathbb{Z}_{>0}} r^n \in I\}$ is called the ***radical of*** I.
- An ideal $I \subseteq R$ such that $I = \sqrt{I}$ is called a ***radical ideal***. (In commutative rings.)

Warm-ups

Read the definitions above before solving the following problem:

Warm-up 1.

1. Give examples of rings that are reduced and rings that are not.
2. Check that $I \subseteq \sqrt{I}$ for any ideal I.
3. In $\mathbb{Z}$, verify that the ideal (6) is a radical ideal, but (9) is not.
4. Let R be a commutative ring and let $I \subseteq R$ be an ideal. Show that $\sqrt{I}$ is an ideal of R.
   Hint: In order to check that the sum of two elements of $\sqrt{I}$ is still in $\sqrt{I}$, recall the binomial formula $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

Warm-up 2 (Field extensions).

1. Is $\sqrt{2}+i$ algebraic over $\mathbb{Q}$? If yes, what is the minimal polynomial of $\sqrt{2}+i$ over $\mathbb{Q}$?
2. We know the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$ from the lectures. What is the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}[\sqrt{2}]$? And over $\mathbb{Q}[\sqrt{3}]$?

Homework

Read the definitions above the warm-ups before solving the following problem:

Homework 1. Let R be a commutative ring. Let $I \subseteq R$ be an ideal.

1. If I is a radical ideal, show that R/I is a reduced ring. [3 points]
2. If R/I is a reduced ring, show that I is a radical ideal. [3 points]

Hint: If you need inspiration, look at Proposition 4.74 of the lecture notes.

Homework 2. Let R be a commutative unital ring. Recall that every ring has at least the two *trivial* ideals (except for the zero ring, in which case these two ideals coincide).

1. Show that if R is a field, then R has exactly two ideals. [3 points]

2. Show that if R has exactly two ideals, then R is a field.

   Hint: Use Proposition 4.74. [3 points]

Homework 3.

1. Let $z \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that the complex conjugate $\overline{z}$ is algebraic over $\mathbb{Q}$.

   Hint. For any two given complex numbers z_1 and z_2, recall that $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$ and $\overline{z_1}\,\overline{z_2} = \overline{(z_1 z_2)}$. [3 points]

2. Let $\gamma = \sqrt{1+\sqrt{2}}$. Show that γ is algebraic over $\mathbb{Q}$, and that $\sqrt{2} \in \mathbb{Q}[\gamma]$. [3 points]

Fill-in-the-blanks. No proof this week. If you respond to the feedback survey, you will get three points. Thank you! [3 points]