Note for lecture 11

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I. WIEDEMANN-FRANZ LAW FOR TUNNELING

There is an important connection between heat and particle transport in terms of conductances. Specifically for NIN tunneling we have

$$\dot{Q}_L = \frac{1}{e^2 R_T}\int d\epsilon(\epsilon - eV)[f_L(\epsilon - eV) - f_R(\epsilon)]. \tag{1}$$

At $eV = 0$ we have

$$\dot{Q}_L = \frac{1}{e^2 R_T}\int d\epsilon\ \epsilon[f_L(\epsilon) - f_R(\epsilon)]. \tag{2}$$

The thermal conductance is then given by

$$\begin{aligned} G_{\rm th} &= \frac{d\dot{Q}_L}{dT} \\ &= -\frac{1}{k_{\rm B}T^2}\frac{1}{e^2 R_T}\int d\epsilon\ \epsilon\frac{df_L(\epsilon)}{d\beta}, \end{aligned} \tag{3}$$

where we used the fact that $\beta = \frac{1}{k_{\rm B}T}$. Knowing that $f(\epsilon) = \frac{1}{1+e^{\beta\epsilon}}$ we have

$$\begin{aligned} \frac{df_L(\epsilon)}{d\beta} &= \frac{-\epsilon e^{\beta\epsilon}}{(1+e^{\beta\epsilon})^2} \\ &= -\epsilon f(\epsilon)[1-f(\epsilon)]. \end{aligned} \tag{4}$$

Substituting above equation into Eq. (3) we have

$$\begin{aligned} G_{\rm th} &= \frac{1}{k_{\rm B}T^2}\frac{1}{e^2 R_T}\int d\epsilon\ \epsilon^2 f(\epsilon)[1-f(\epsilon)] \\ &= \frac{k_{\rm B}^2 T}{e^2 R_T}\int_{-\infty}^{\infty} dx\ x^2 \frac{1}{1+e^x}\frac{1}{1+e^{-x}} \\ &= (\frac{\pi^2 k_{\rm B}^2}{3e^2})T\frac{1}{R_T}, \end{aligned} \tag{5}$$

i.e.

$$G_{\rm th} = \mathcal{L}_0 T G_T, \tag{6}$$

which is equivalent to **Wiedemann-Franz law**, with $\mathcal{L}_0 = \frac{\pi^2 k_{\rm B}^2}{3e^2}$ the Lorenz number.

II. HAMILTONIAN OF THE QUANTUM CIRCUIT

We start with an LC-oscillator, shown in Fig. 2(a), and write its Hamiltonian. The kinetic and potential energies, K and V, respectively, read

$$K = \frac{1}{2} C \dot{\Phi}^2 \tag{7}$$

$$V = \frac{\Phi^2}{2L}, \tag{8}$$

where $\Phi = \int_0^t dt' \; v(t')$ with v as voltage. The Lagrangian of the circuit is then

$$\mathcal{L} = K - V = \frac{1}{2} C \dot{\Phi}^2 - \frac{\Phi^2}{2L}, \tag{9}$$

From the Lagrangian we can obtain the charge, Q, which is the conjugate momentum of node flux Φ by the Legendre transformation

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C \dot{\Phi}. \tag{10}$$

Here Q is the charge on the capacitor. Then we have

$$H = K + V = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}, \tag{11}$$

i.e. a harmonic oscillator. Introducing the creation and annihilation operators such that

$$[c, c^\dagger] = 1, \tag{12}$$

we have

$$\Phi = \sqrt{\frac{\hbar Z_0}{2}} (c + c^\dagger) \tag{13}$$

$$Q = -i \sqrt{\frac{\hbar}{2 Z_0}} (c - c^\dagger) \tag{14}$$

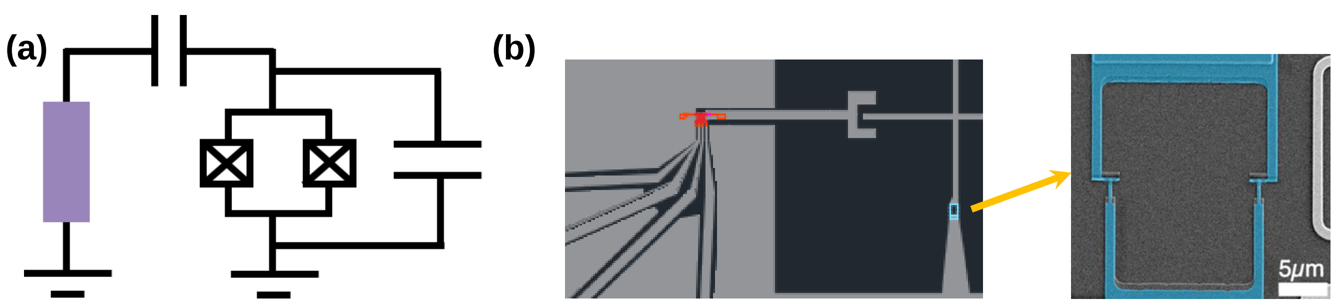

FIG. 1. Exemplary quantum circuit for analysis. (a) Circuit diagram. On the right side a SQUID (superconducting quantum interference device) composed of two Josephson junctions. This element is capacitively coupled to a resistive element on the left. (b) The actual on-chip circuit. The coupling capacitor is the fork-like structure in the middle, and the SQUID is zoomed out in blue on the right. The resistive element is contacted by four NIS tunnel junctions to control and measure temperature, on the left.

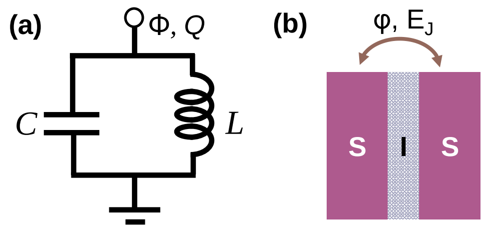

FIG. 2. (a) LC circuit. (b) Josephson junction.

$$H = \frac{\hbar\omega_0}{2}(c^\dagger c + cc^\dagger) = \hbar\omega_0(c^\dagger c + \frac{1}{2}), \tag{15}$$

where $\omega_0 = \sqrt{\frac{1}{LC}}$, and $Z_0 = \sqrt{\frac{L}{C}}$.

For a Josephson tunnel junction, shown in Fig. 2(b), the Josephson relations are

$$\hbar\dot{\phi} = 2ev \tag{16}$$

$$I = I_c \sin\phi, \tag{17}$$

where ϕ is the phase difference across the junction and v is the voltage. The first relation implies that flux and phase are related by

$$\phi = \frac{2e}{\hbar}\Phi. \tag{18}$$

In the second Josephson relation, I is the current through the junction and the r.h.s. applies for a tunnel junction, with critical current I_c. For different types of weak links, sinusoidal dependence does not necessarily hold. We discussed earlier that energy stored in the system (=work done by the source) is

$$E = \int^t Iv(t')dt' = I\Phi. \tag{19}$$

Thus for a current biased case $I = \frac{\partial E}{\partial \Phi}$ and

$$E = \int^\Phi I\, d\Phi \quad = \frac{\hbar I_c}{2e}\int^\phi \sin\phi' d\phi' \tag{20}$$

$$= -\frac{\hbar I_c}{2e}\cos\phi \equiv -E_J\cos\phi. \tag{21}$$

We call this "Josephson energy" and in quantum mechanics "Josephson Hamiltonian" $\hat{H}_J$. Now we may expand this energy for small values of ϕ, as

$$E \simeq -E_J(1 - \frac{\phi^2}{2}) = \frac{E_J}{2}\phi^2 = (\frac{2e}{\hbar})^2\Phi^2 \equiv \frac{\Phi^2}{2L_J}. \tag{22}$$

Here $L_J = (\frac{\hbar}{2e})^2\frac{1}{E_J} = \frac{\hbar}{2eI_c}$ is the Josephson inductance. Therefore, in the "linear regime":

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L_J}, \tag{23}$$

i.e. Josephson junction behaves approximately as a harmonic oscillator.

III. DENSITY MATRIX ρ

Liouville - von Neumann equation follows mechanically from the Schrödinger equation as follows. First, in the Schrödinger picture we have

$$i\hbar\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \tag{24}$$

for a state $|\psi(t)\rangle$ and Hamiltonian H. We can write the solution of it formally as

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle, \tag{25}$$

where $U(t,t_0)$ is the time evolution operator between the initial time t_0 and t. It then obeys

$$i\hbar\partial_t U(t,t_0) = HU(t,t_0). \tag{26}$$

The density matrix of the whole system can be written as

$$\rho_S(t) = \sum_\lambda p_\lambda |\psi_\lambda(t)\rangle\langle\psi_\lambda(t)|, \tag{27}$$

where p_λ are the (positive) weights for the states $|\psi_\lambda(t)\rangle$ that obey the Schrödinger equation. We then find

$$\rho_S(t) = U(t,t_0)\rho_S(t_0)U^\dagger(t,t_0). \tag{28}$$

Using the chain rule in differentiating, we then find easily that

$$\dot{\rho}_S(t) = \frac{i}{\hbar}[\rho_S(t), H], \tag{29}$$

which is the Liouville - von Neumann equation.