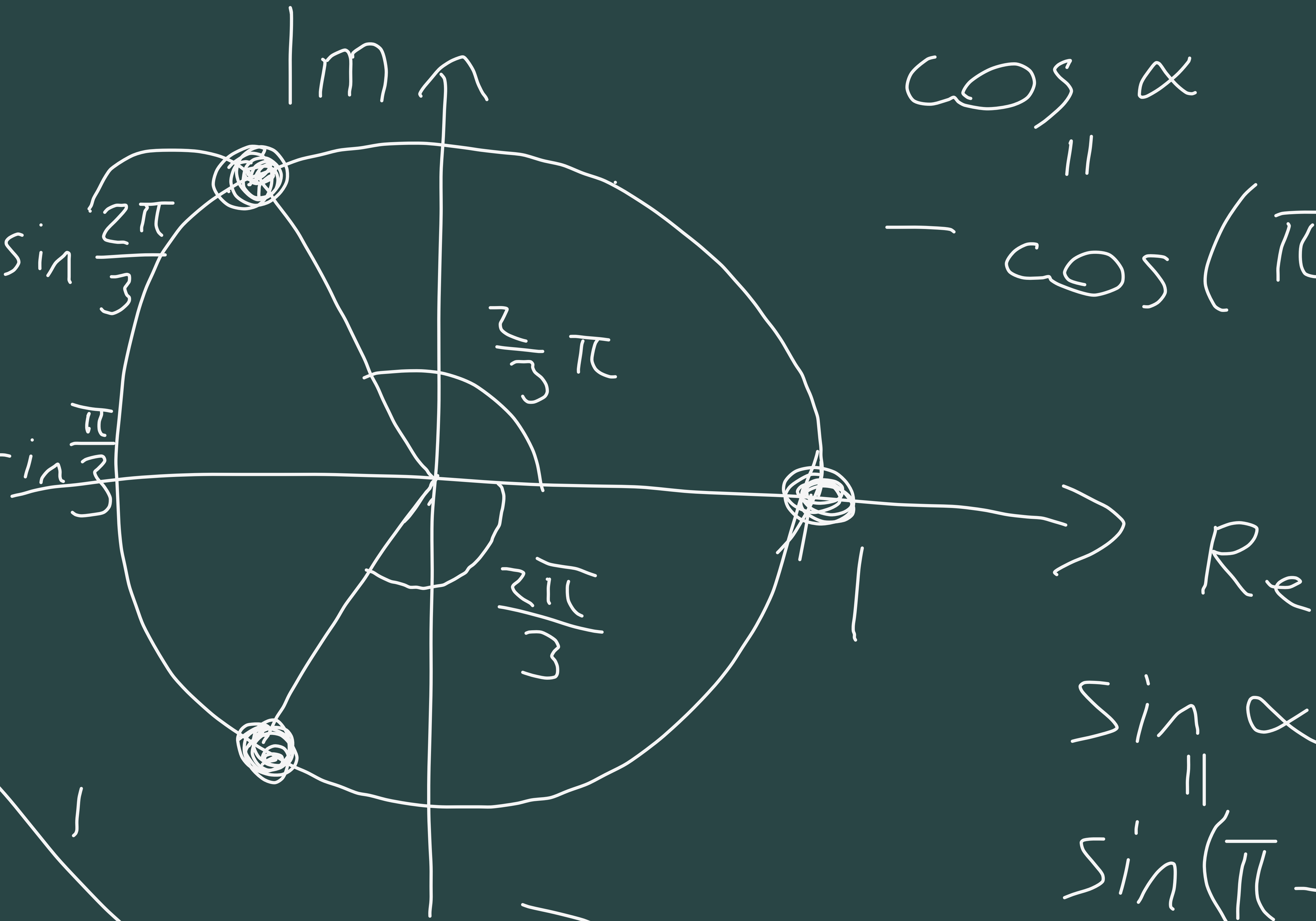
$$\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

$$\parallel$$

$$-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$



$$\cos\alpha$$

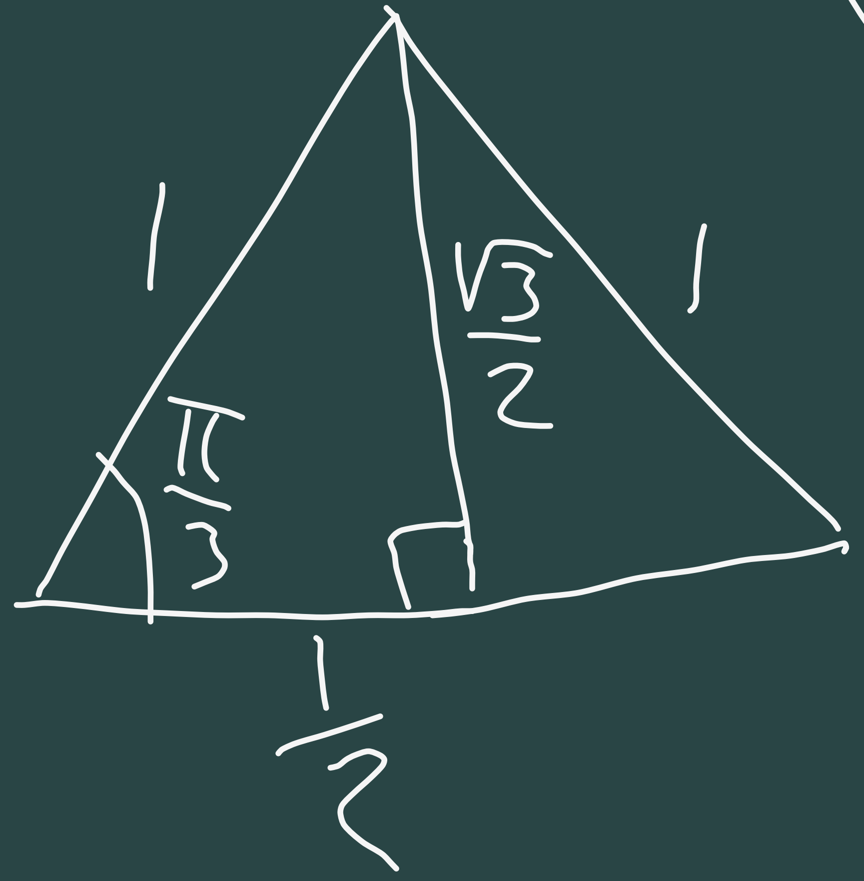
$$\parallel$$

$$-\cos(\pi-\alpha)$$

$$\sin\alpha$$

$$\parallel$$

$$\sin(\pi-\alpha)$$



$$\cos\frac{\pi}{3}=\frac{1}{2},\quad \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

$$P(x) = a_n X^n + a_{n-1} X^{n-1} + \dots + a_0$$

$$n \geq 1, \quad a_n \neq 0$$

$$\Longrightarrow$$

on olemassa $z \in \mathbb{C}$

s.e. $P(z) = 0$

$P(z)$, α juuri $(P(\alpha)=0)$

$\Longrightarrow \dfrac{P(z)}{(z-\alpha)}$ on myös Polynomi

Juuret $1, i, -i$

$$P(x) = (x-1)(x-i)(x+i)$$

$$= (x-1)(x^2+1)$$

$$= x^3 - x^2 + x - 1$$

Lause:

Jos: $P(x)$ on polynomi

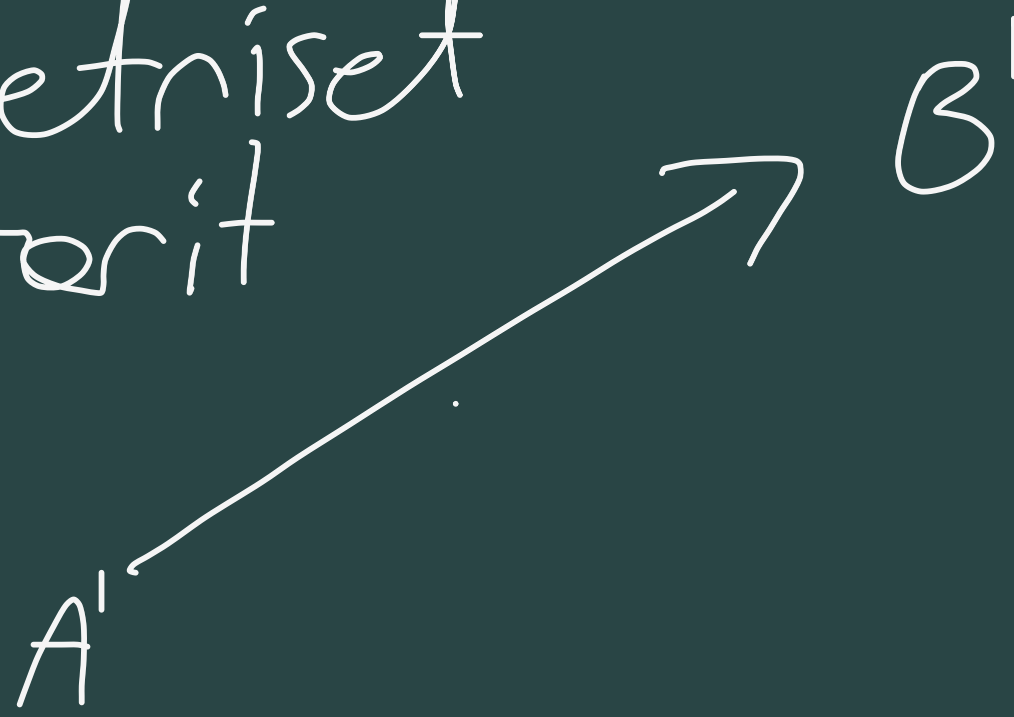
$$\begin{aligned} &\parallel \\ &a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \end{aligned}$$

$a_i \in \mathbb{R}$, kaikki i

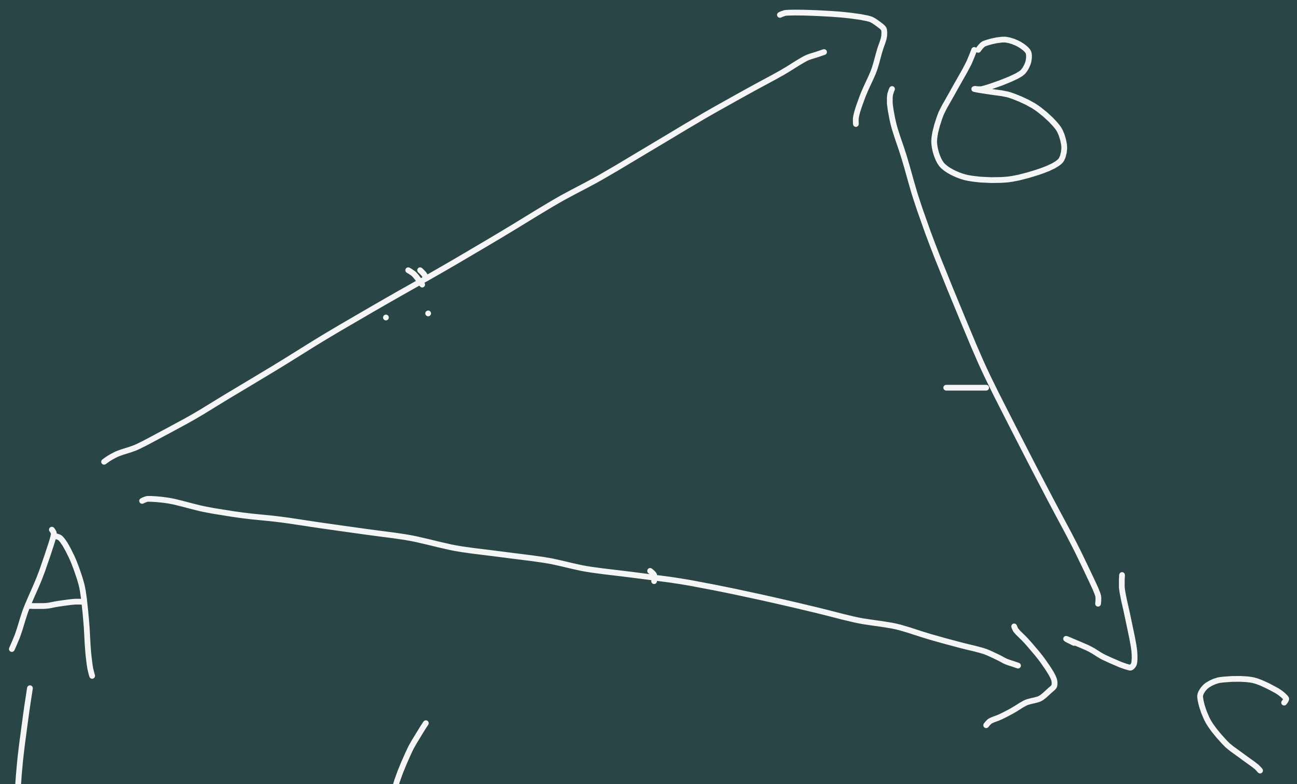
Niin: $P(z) = 0 \iff P(\overline{z}) = 0$

Geometriset vektorit

$$\overrightarrow{AB} = \overrightarrow{A'B'}$$

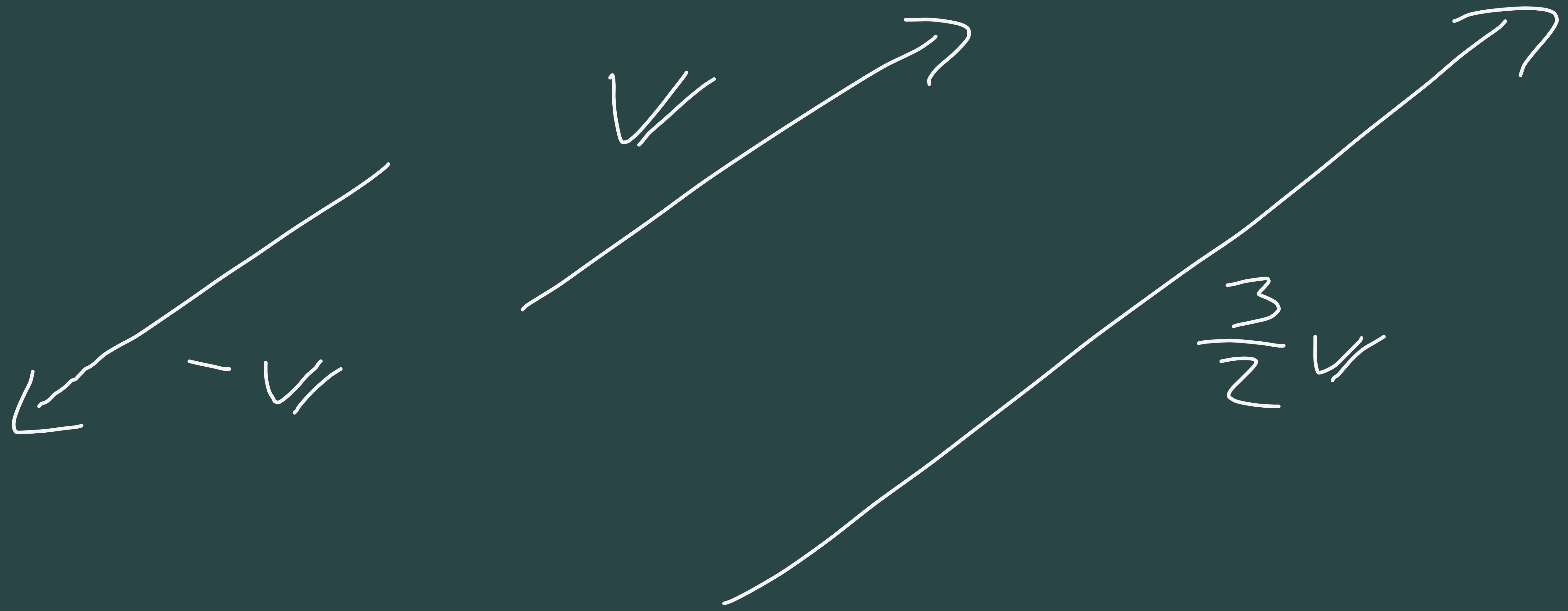


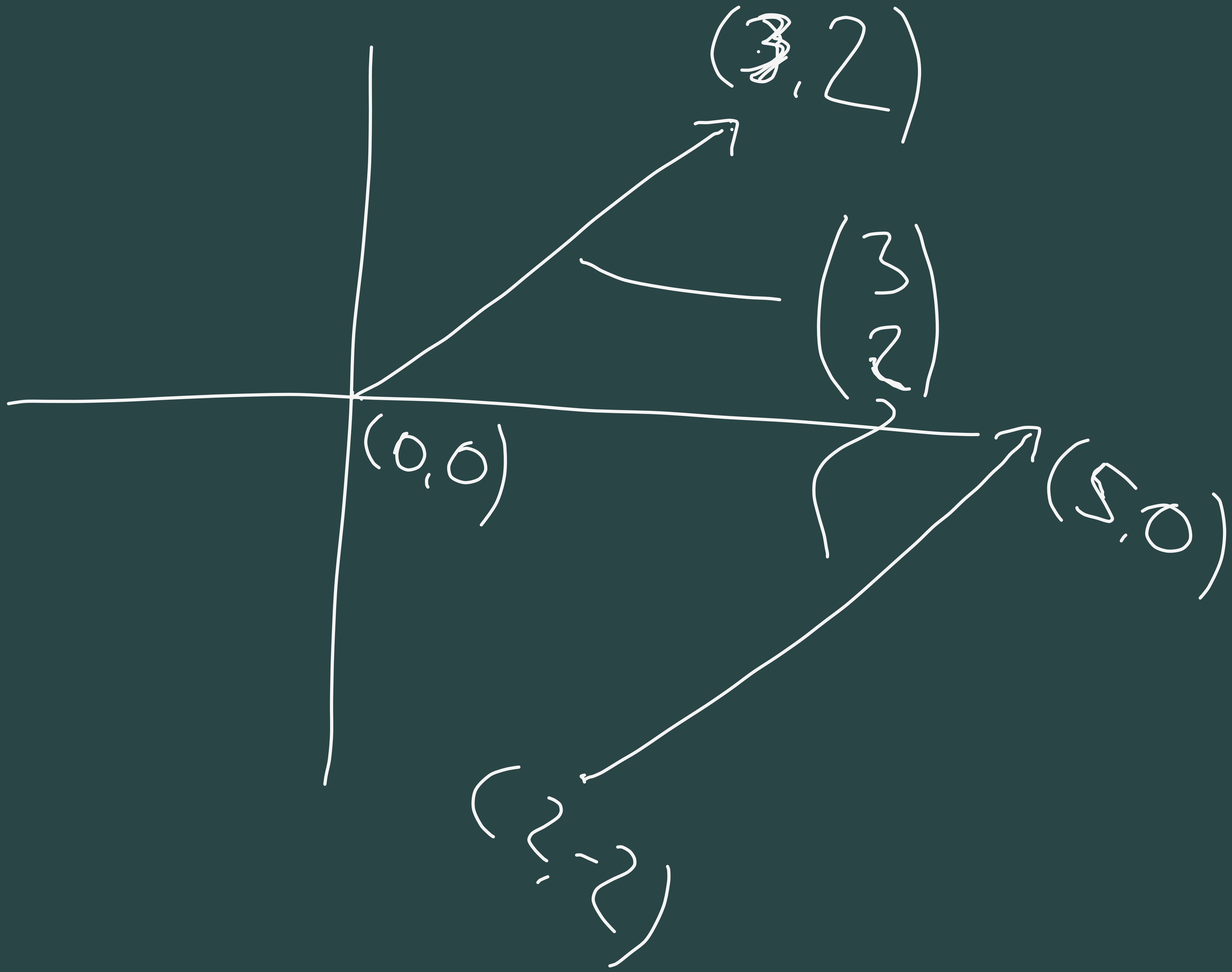
eli



$$\overrightarrow{AB} + \overrightarrow{BC}$$
$$\shortparallel$$
$$\overrightarrow{AC}$$

Paikkavektorit

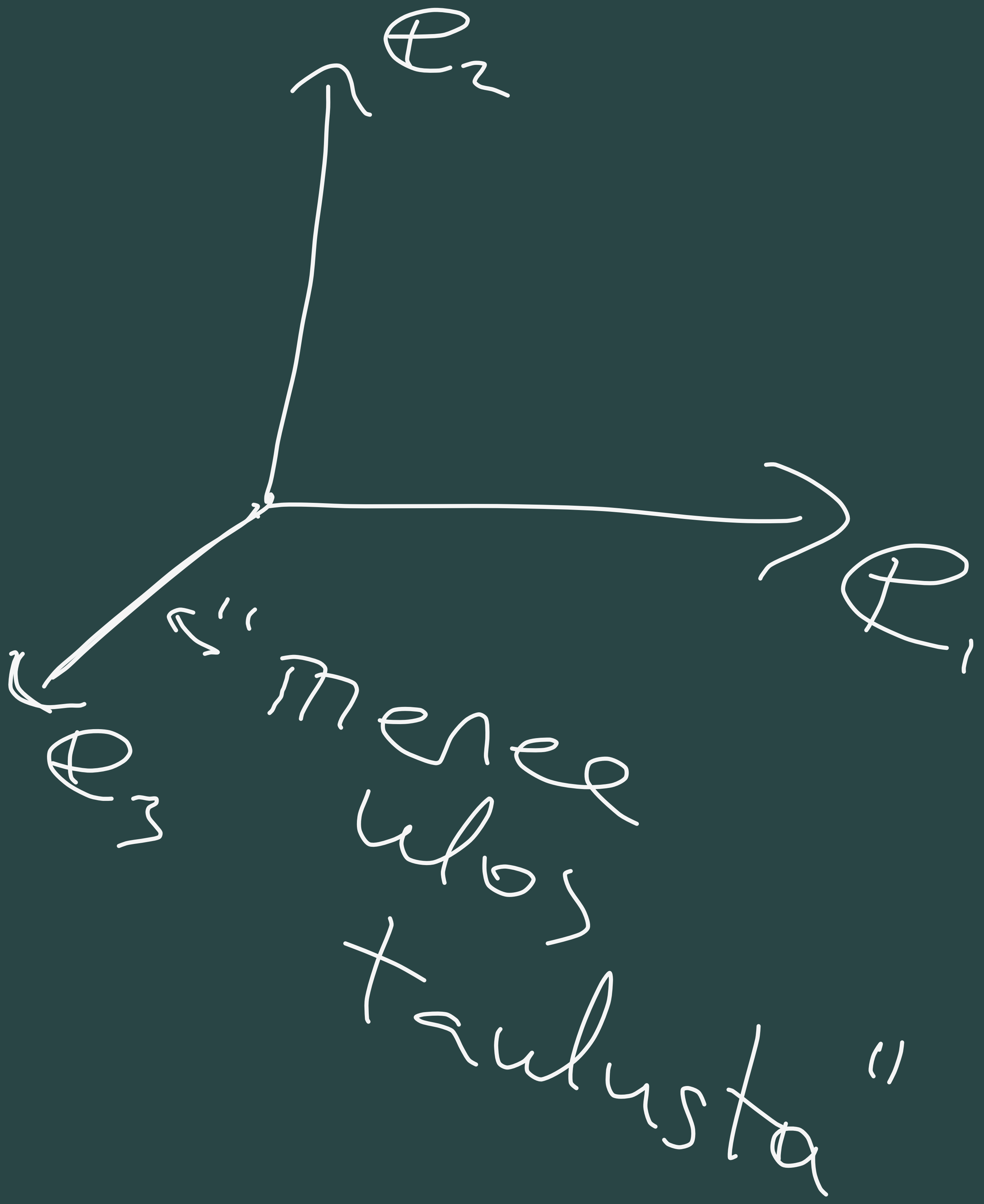

$-v$
v
$\frac{3}{2}v$


(3,2)
$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
(0,0)
(5,0)
(2,-2)

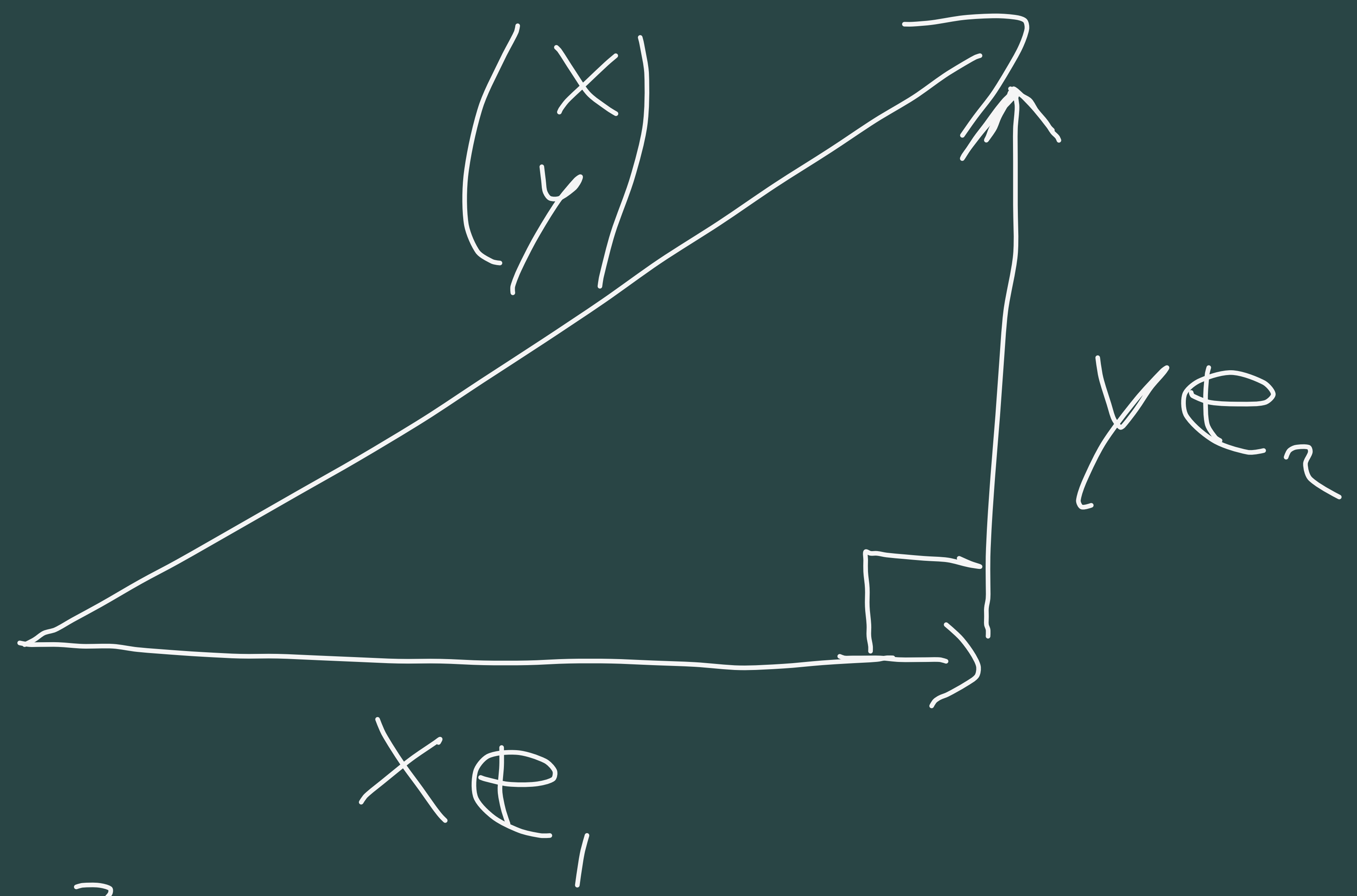
$$x + y = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\pi\\2\pi\end{pmatrix} = \begin{pmatrix}1+\pi\\2\pi\end{pmatrix}$$

$$\frac{1}{\pi}y - x = \begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$$

$$x + 2z = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-6\\2\\8\end{pmatrix}$$ EI MÄÄRITELTY



"menee ulos taulusta"



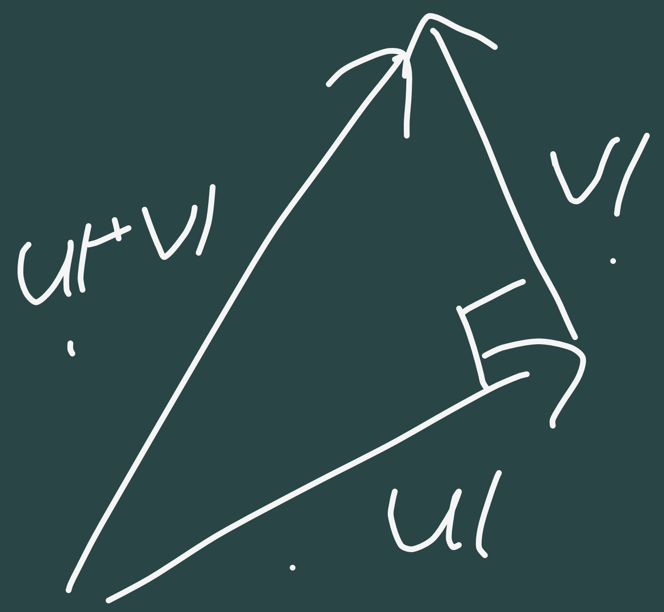
$$\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^2 = x^2+y^2$$

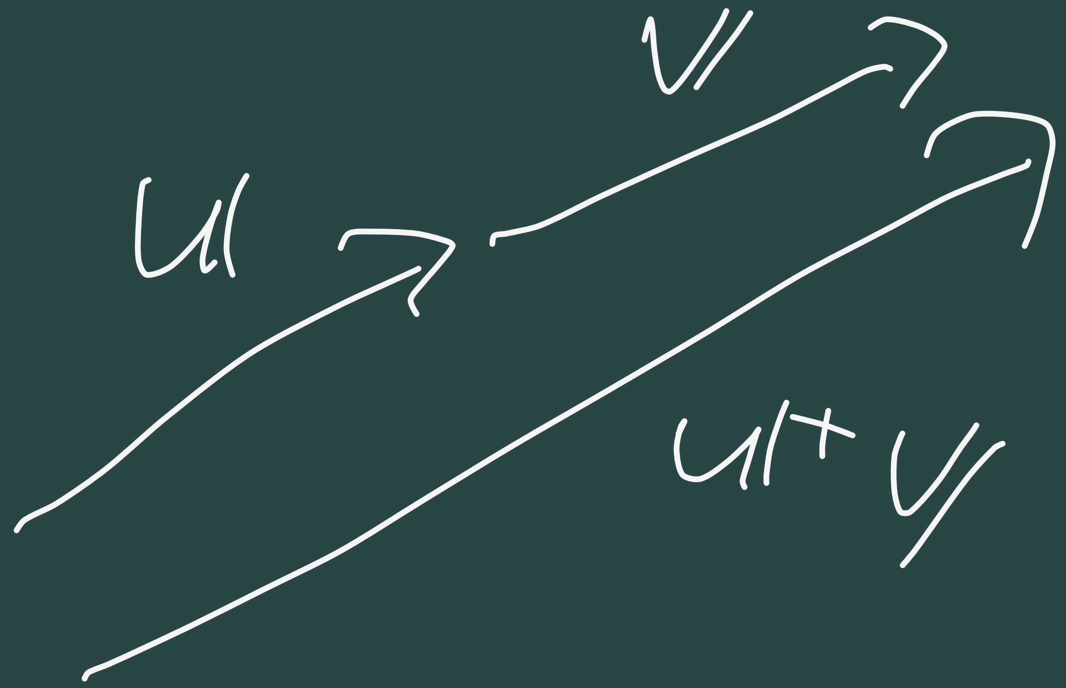
$$\| a \| = \| 2e_1 + 3e_2 + e_3 \|$$

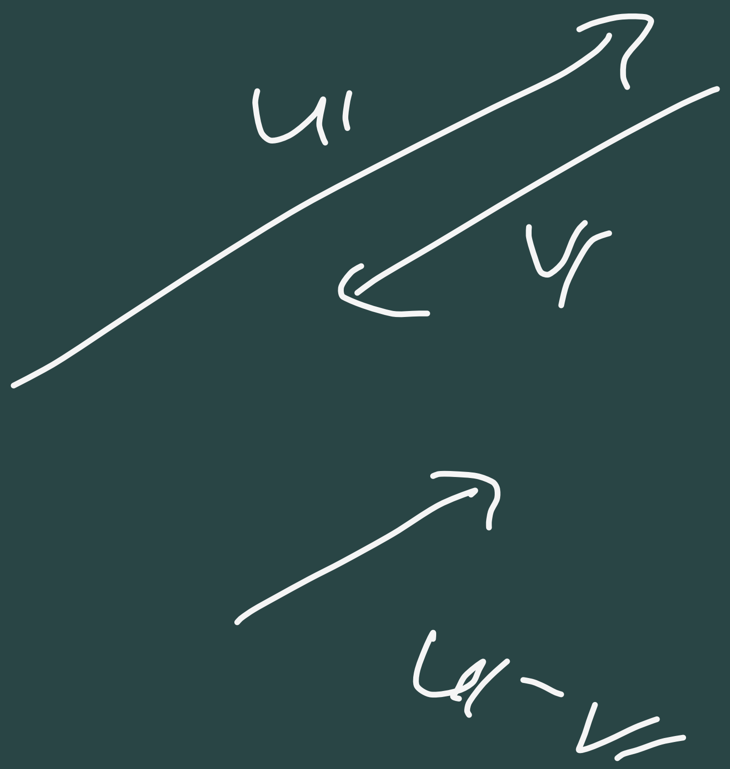
$$= \left\| \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\| = \sqrt{2^2 + 3^2 + 1^2}$$

$$= \sqrt{4 + 9 + 1}$$

$$= \sqrt{14}$$


u+v
v
u

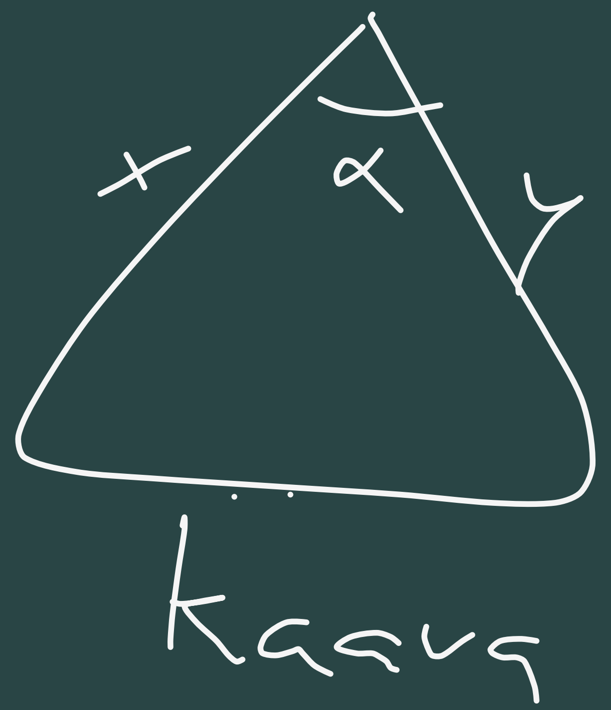

u
v
u+v

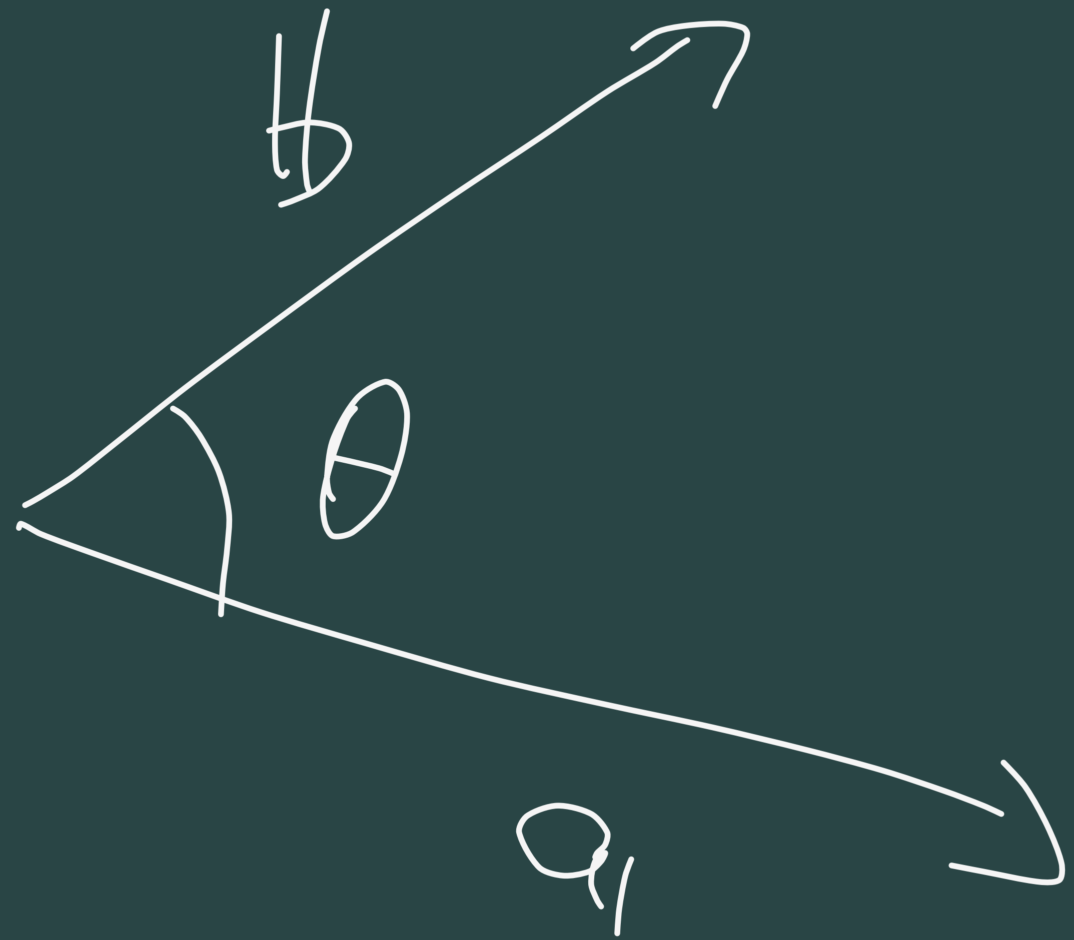

u
v
u−v

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$$

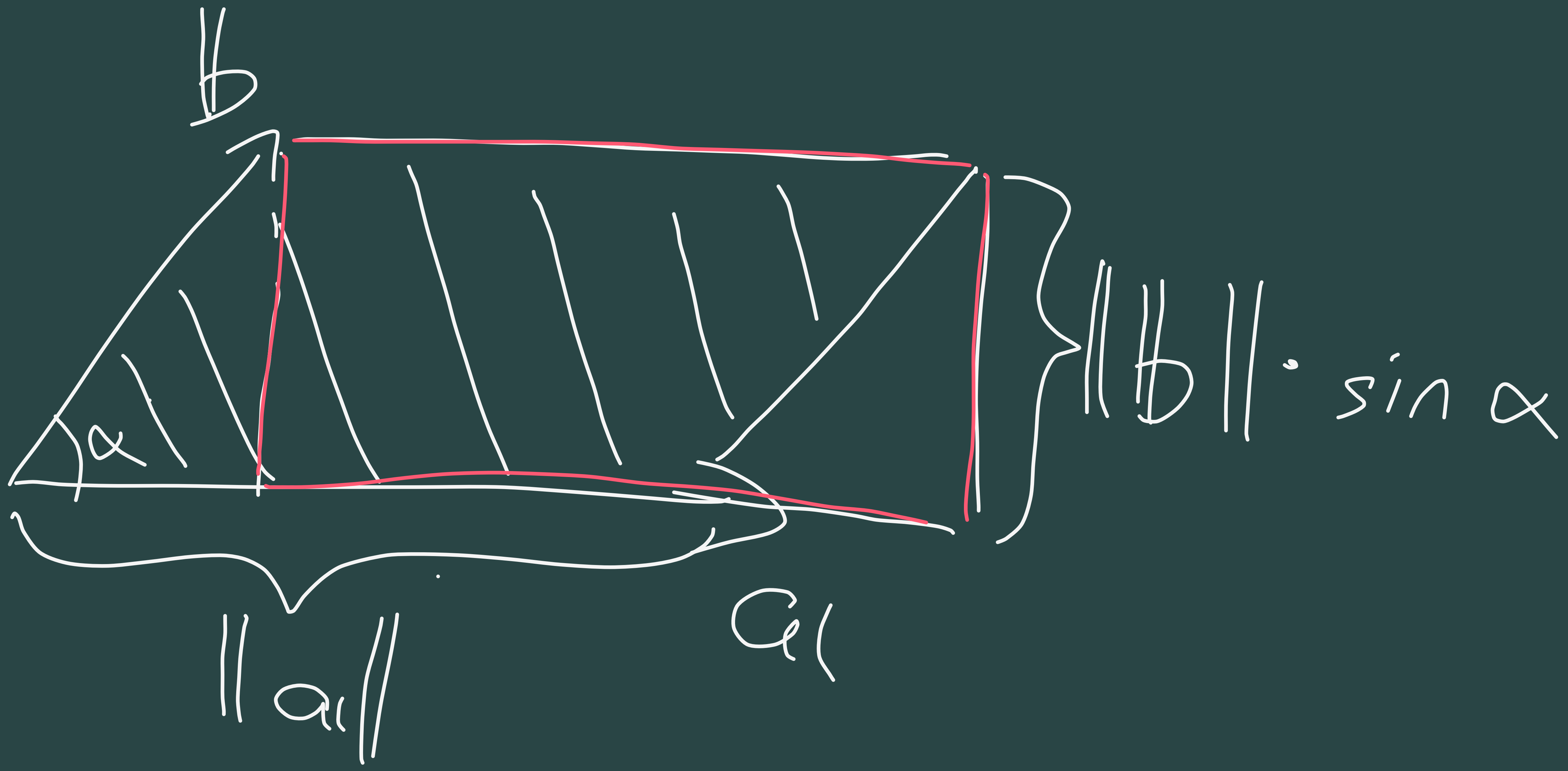
$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$





$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cdot \cos\theta$$

$$\cancel{\begin{pmatrix}a\\b\\c\end{pmatrix} * \begin{pmatrix}d\\e\\f\end{pmatrix} = \begin{pmatrix}ad\\be\\cf\end{pmatrix}}$$


b
α
||b|| · sin α
||a||
a

"

$$i \times j = k$$

$$j \times k = i$$

$$k \times i = j$$

"

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$$

$$\mathbf{a} \times \mathbf{b} = - \mathbf{b} \times \mathbf{a}$$

$$\mathbf{x} \times \mathbf{x} = \mathbf{0}$$

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \qquad c = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$a_1 \times b = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \qquad b \times c = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$a_1 \times c = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Lin. riippuvuus:

Joku vektoreista on esitettävissä muiden vektorien lin. kombinaationa.

Jos $v_1 \ldots v_m \in \mathbb{R}^n$

lin. riippumattomat

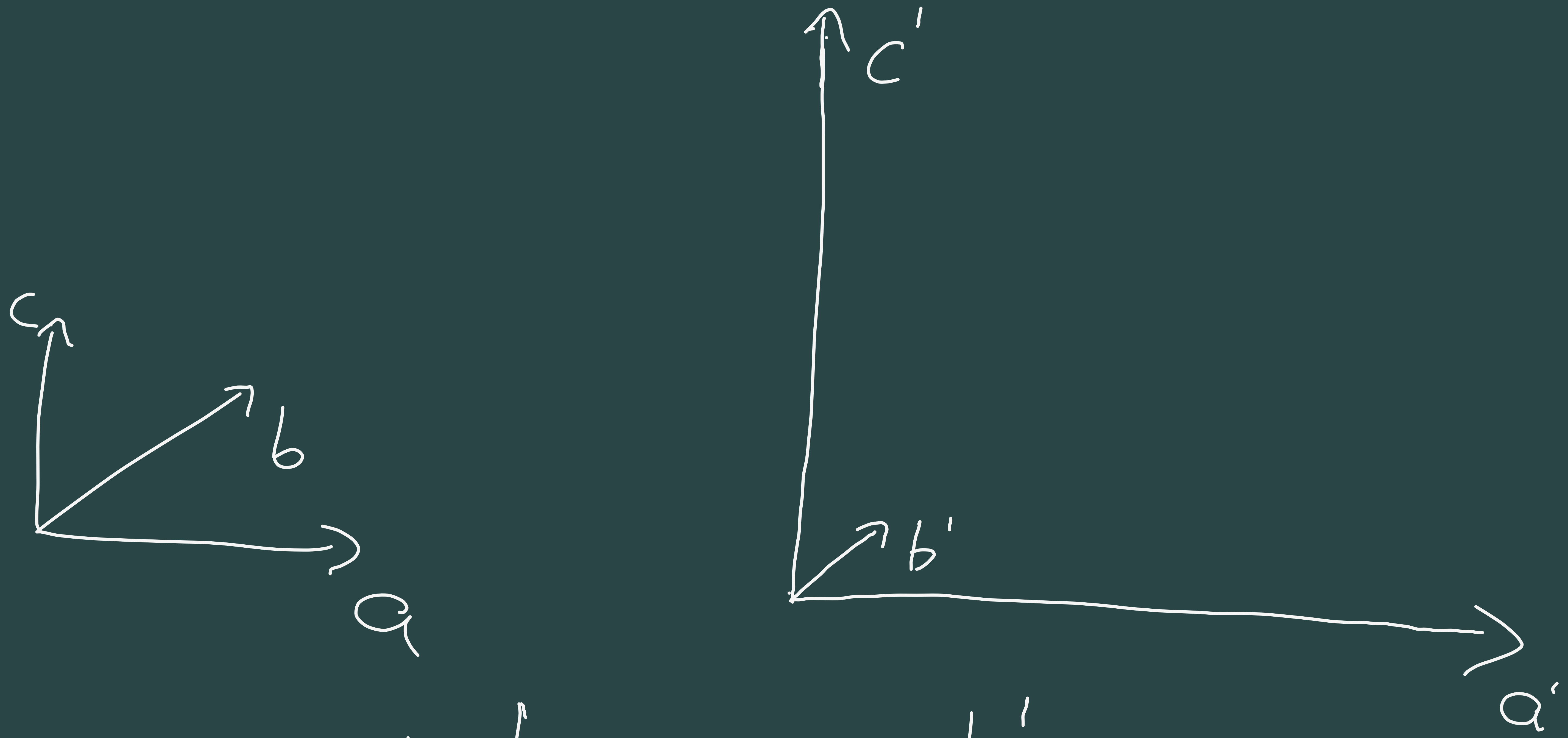
Niin $\langle v_1 \ldots v_m \rangle = \{a_1 v_1 + \ldots + a_m v_m\}$

on $(n-m)$ yhtälön määrittämä

avaruus.

Siispä:

- $\mathbb{R}$^:ssä on korkeintaan n riippumatonta vektoria
- n satunnaisesti valittua vektoria $\mathbb{R}$^:ssä ovat "melkein varmasti" riippumattomat



$$a \times b = a' \times b'$$

$$b \times c = b' \times c'$$

$$a \times c \neq a' \times c'$$

Esim

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x+y \qquad \text{lin.}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y \\ y+z \end{pmatrix} \qquad \text{lin.}$$

Vastaesimerkkejä:

$$x \xmapsto{f} 3x+4$$

EI LIN.

VAAN

AFFIINI.

$$f(x+y)=3(x+y)+4$$

$$\neq$$

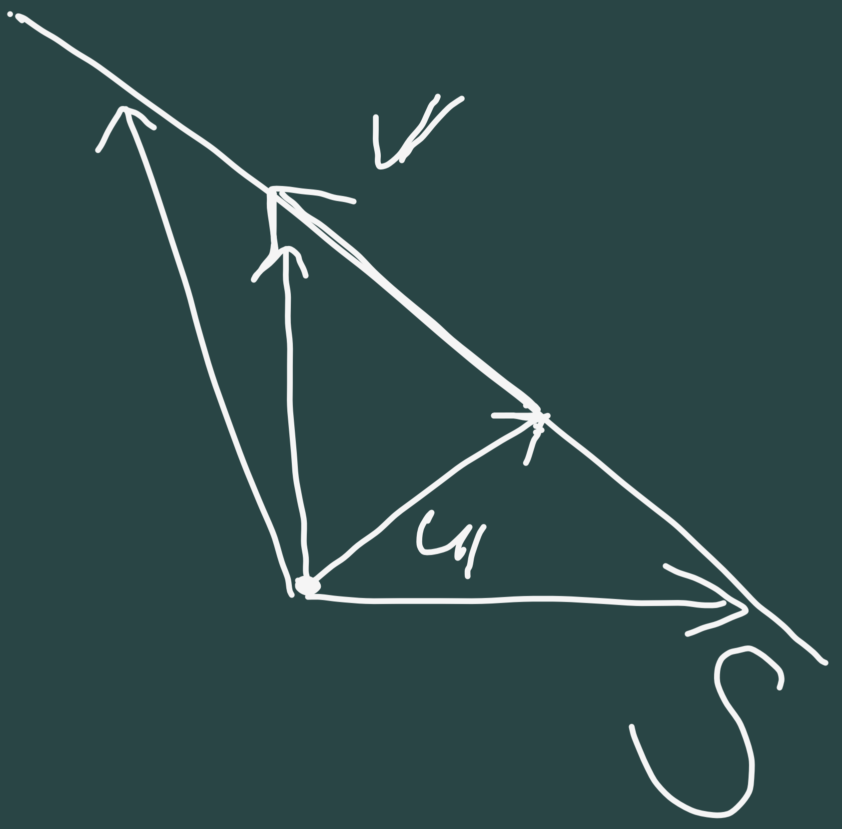
$$f(x)+f(y)=3x+4+3y+4$$

$f(x) = x^2$ EI LIN

$$(x+y)^2 \neq x^2 + y^2$$

$f(x) = \sin x$ $\underline{\underline{\text{ei}}}$ lin.

$$\sin(x+y) \neq \sin x + \sin y$$



$$f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

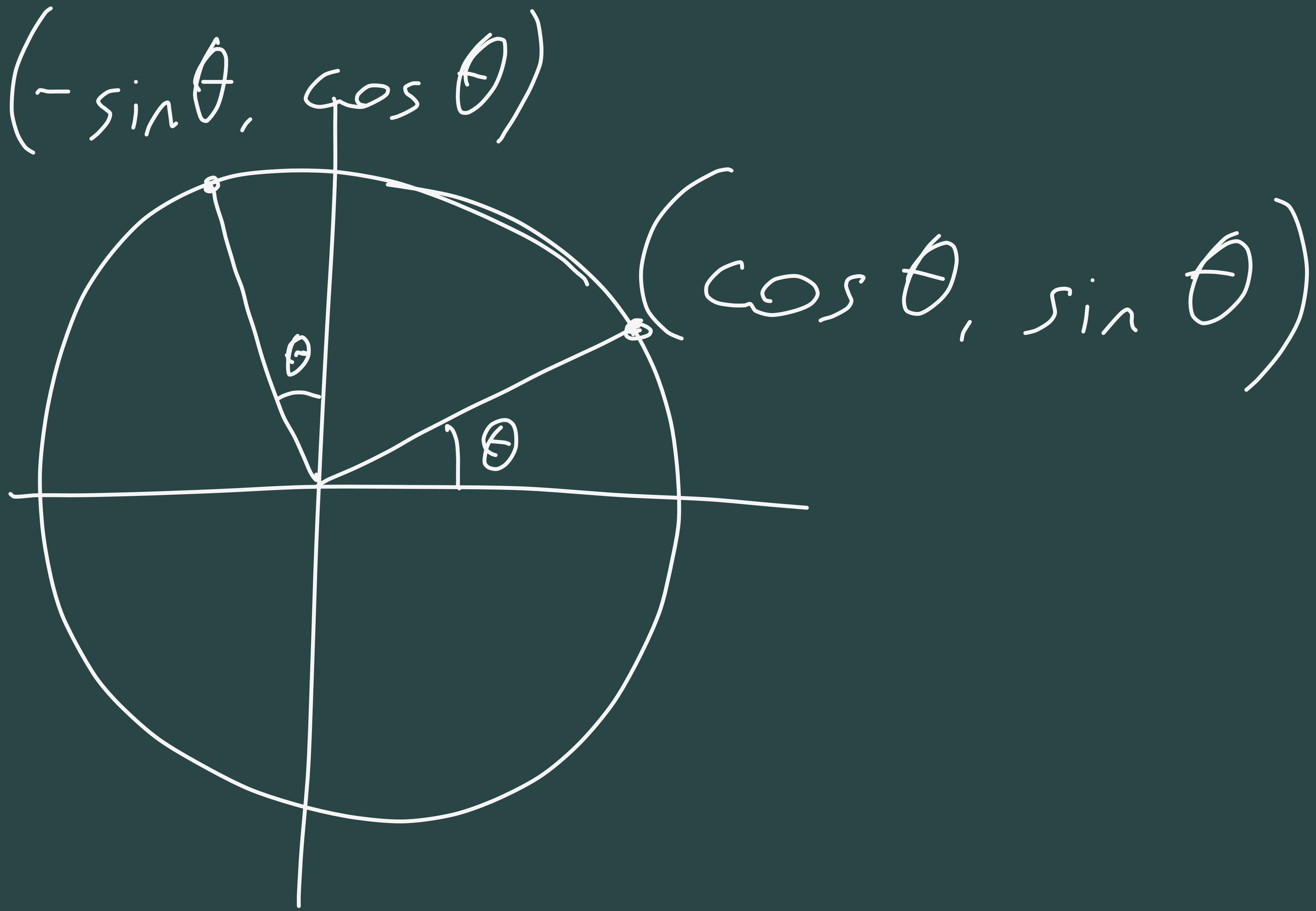
Suora $S=\{u+tv : t\in\mathbb{R}\}$

$$f(S)=\{f(u+tv)\}=\{f(u)+tf(v)\}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 3x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} -x \\ y \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : \quad r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \longmapsto r\begin{pmatrix} \cos\left(\frac{\pi}{4}+\theta\right) \\ \sin\left(\frac{\pi}{4}+\theta\right) \end{pmatrix}$$


(−sin θ, cos θ)
(cos θ, sin θ)
θ
θ

$$f(x) = \sin x$$

$$g(x) = x^2$$

$$f \circ g(x) = \sin(x^2)$$

$$g \circ f(x) = \sin(x)^2$$

HARJ

Jos $f : \mathbb{R}^{n} \longrightarrow \mathbb{R}^{m}$

$g : \mathbb{R}^{m} \longrightarrow \mathbb{R}^{k}$

ovat lineaarisia,

niin $g \circ f : \mathbb{R}^{n} \longrightarrow \mathbb{R}^{k}$

on myös lineaarinen

$$g\circ f(cx+y) \overset{!}{=} c\,g\circ f(x) + g\circ f(y)$$

$$\|$$

määr.

f lin.

$$g(f(cx+y))$$

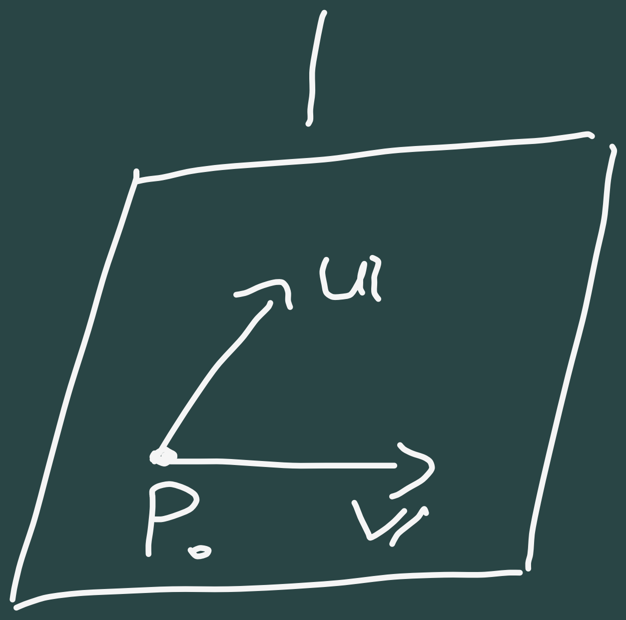
$$\|$$

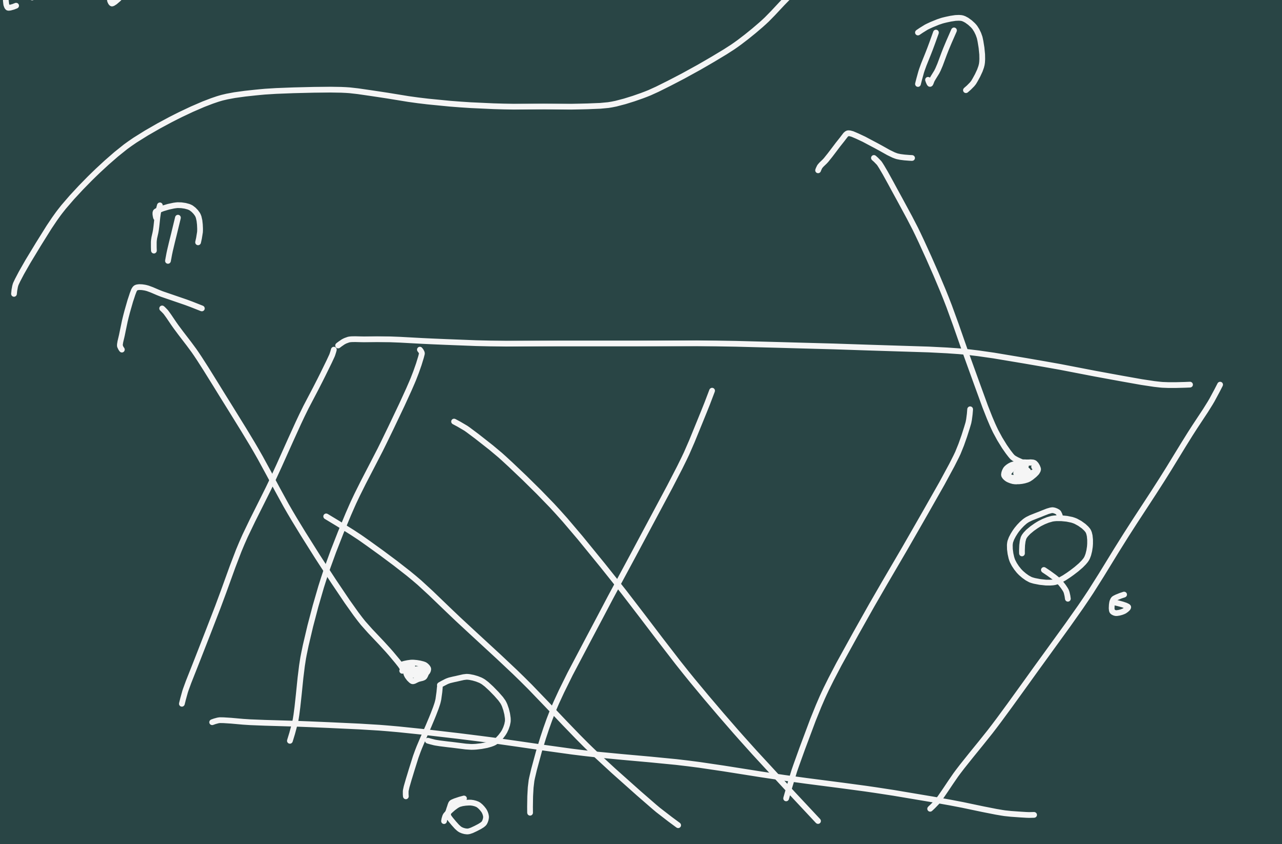
$$g(cf(x)+f(y)) = cg(f(x)) + g(f(y))$$

g lin.

$$k\left\{\overbrace{\begin{pmatrix} & & \\ & B & \\ & & \end{pmatrix}}^{m}\right.\overbrace{\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix}}^{n}\left.\vphantom{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}\right\}m = \overbrace{\begin{pmatrix} & & \\ & BA & \\ & & \end{pmatrix}}^{n}\left.\vphantom{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}\right\}k$$

$$\{P_0 + t\mathbf{v} + s\mathbf{u} : s,t \in \mathbb{R}\}$$





Tason yhtälö

$$ax + by + cz = d$$

Jos $d = 0 \implies$ sisältää origon

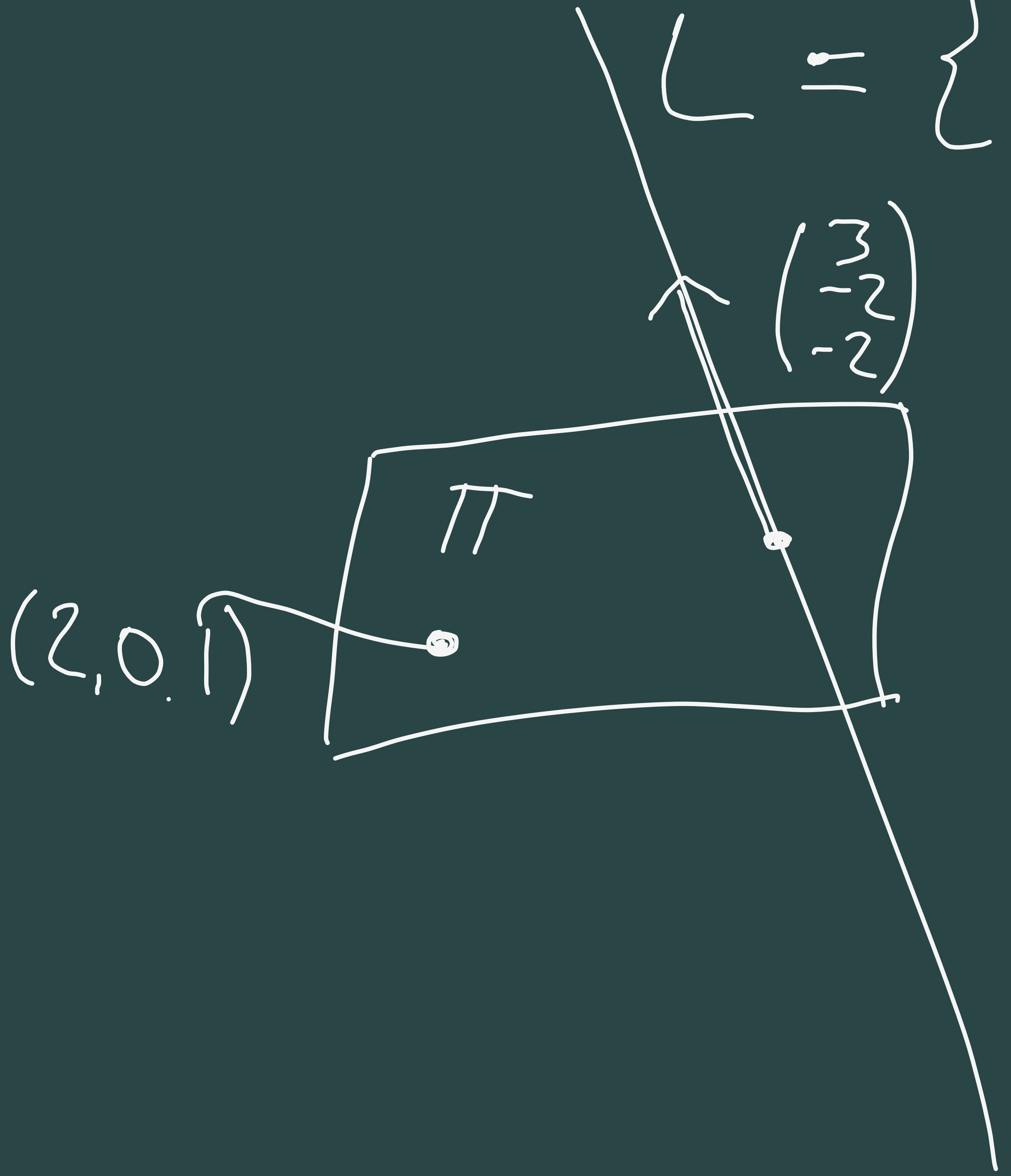
Muuten : $\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z = 1$

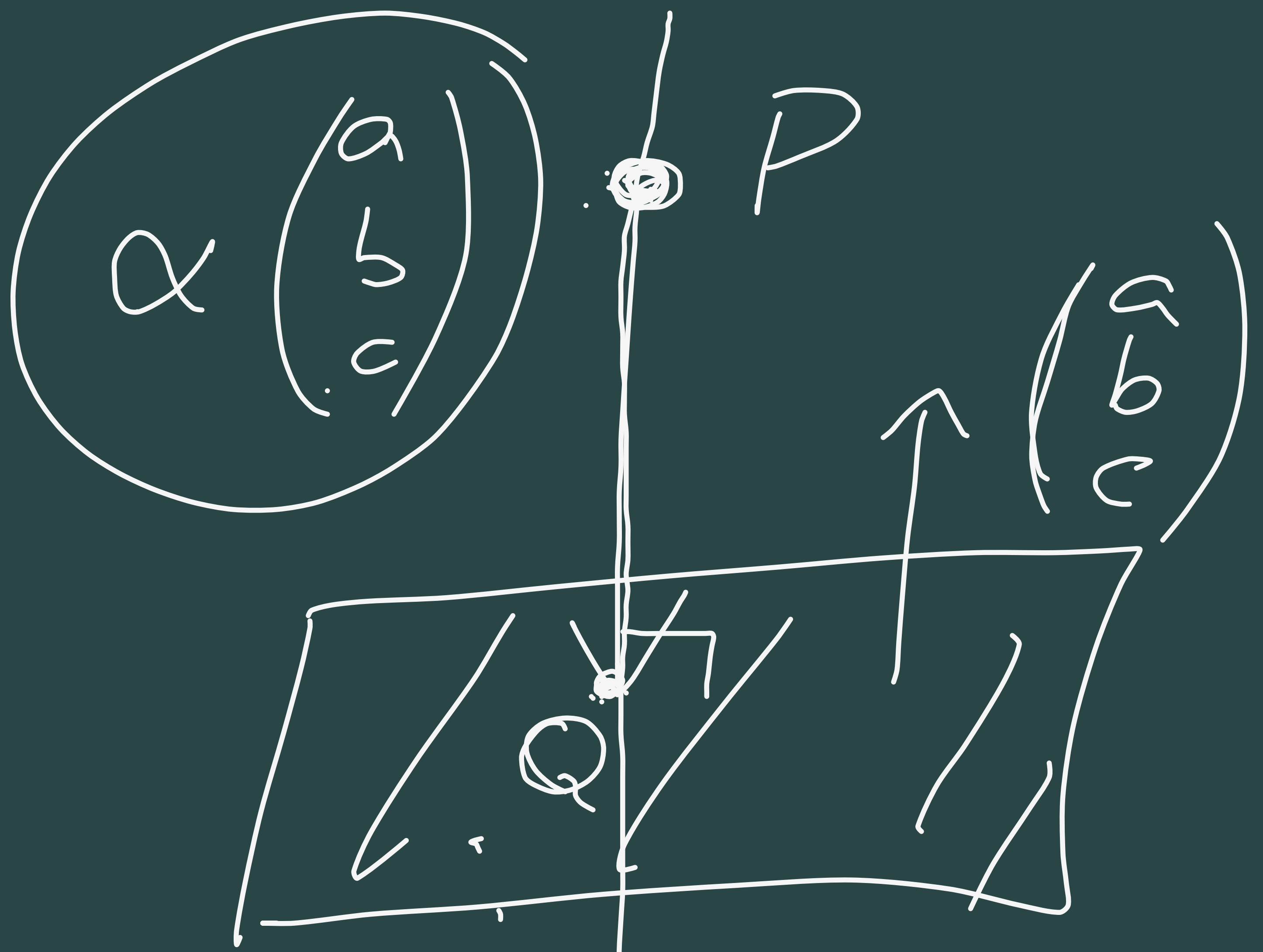
Suora 3-avaruudessa

$$\{P_0 + t\mathbf{v} : t \in \mathbb{R}\}$$

$$\begin{cases} x - x_0 = ta \\ y - y_0 = tb \\ z - z_0 = tc \end{cases} \Longrightarrow \left\{ \begin{array}{c} \dfrac{x - x_0}{a} \\ \shortparallel \\ \dfrac{y - y_0}{b} \\ \shortparallel \\ \dfrac{z - z_0}{c} \end{array} \right.$$

$$L = \{(1+3t,\ 1-2t,\ -2t)\}$$



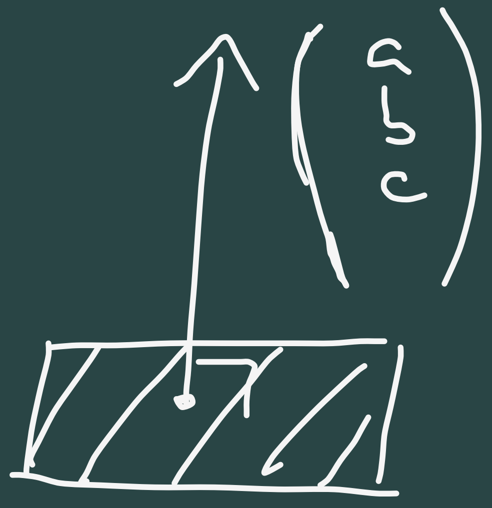


$$\boxed{\alpha\begin{pmatrix}a\\b\\c\end{pmatrix}}$$

$$\Pi : ax+by+cz=d$$

$$ax_0+by_0+cz_0 = d-\alpha\left(a^2+b^2+c^2\right)$$

Tason yhtälö: $\underbrace{ax+by+cz}=d$

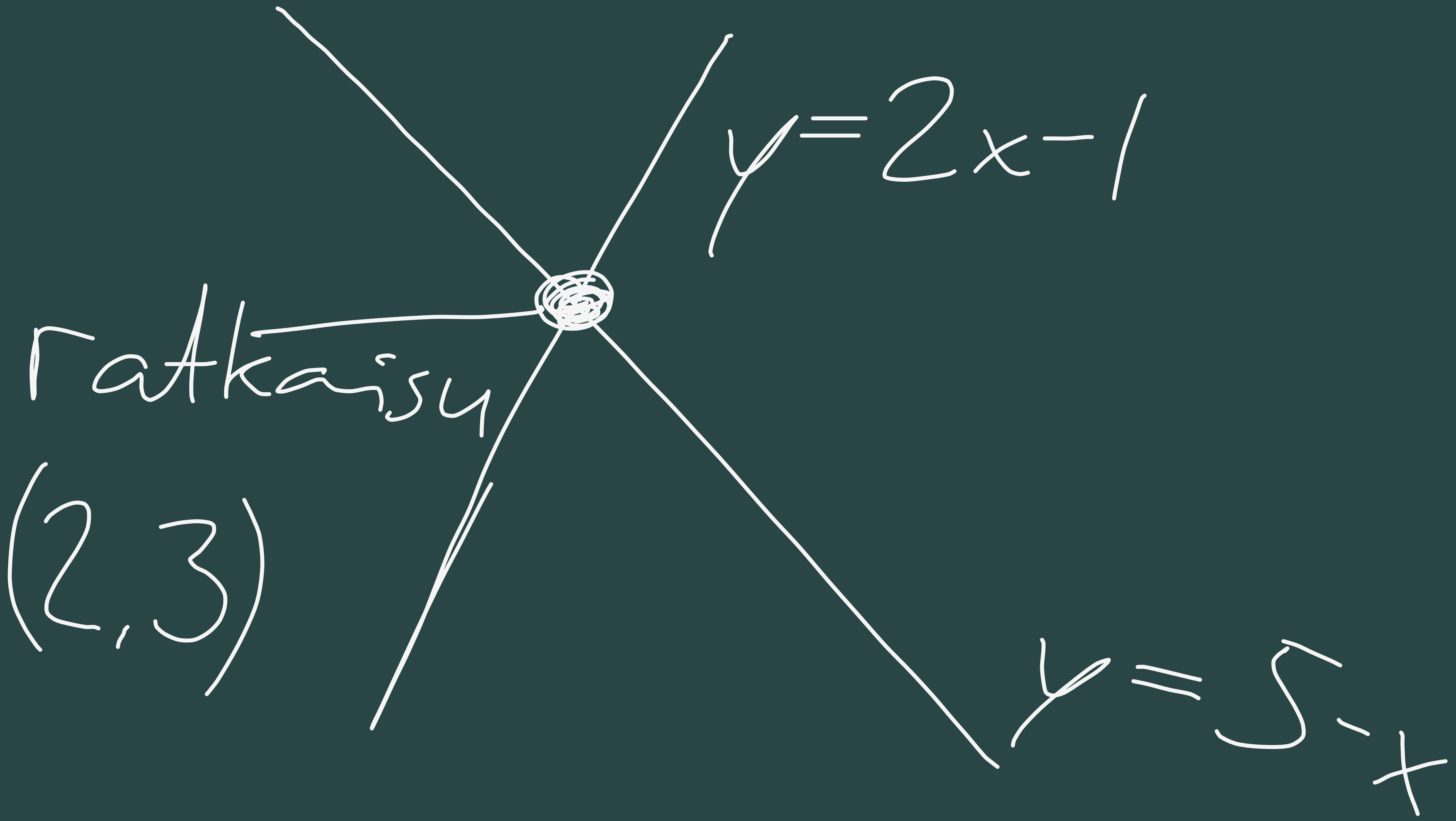


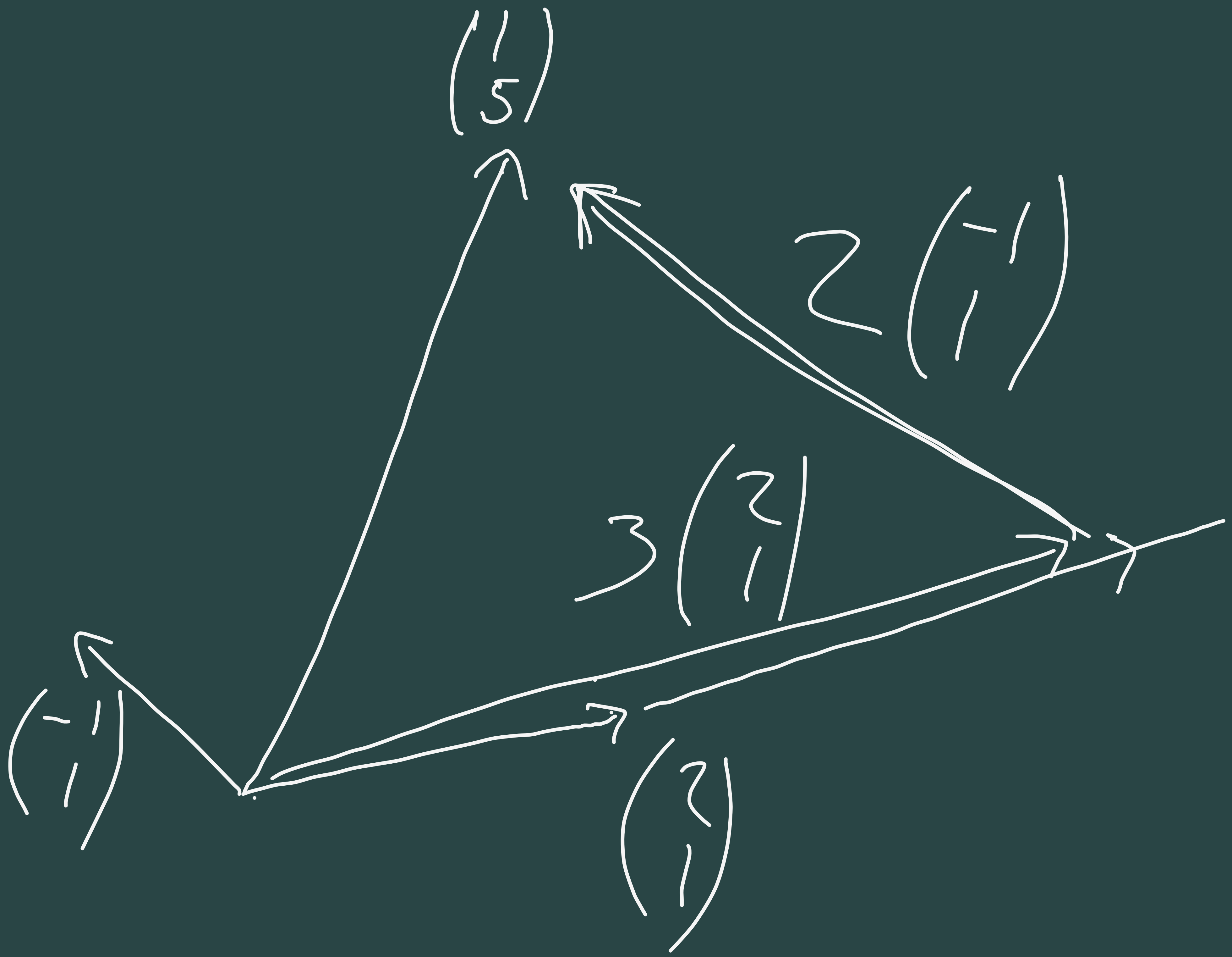
$$\alpha = \frac{d-ax_0-by_0-cz_0}{\left\|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\|^2}$$

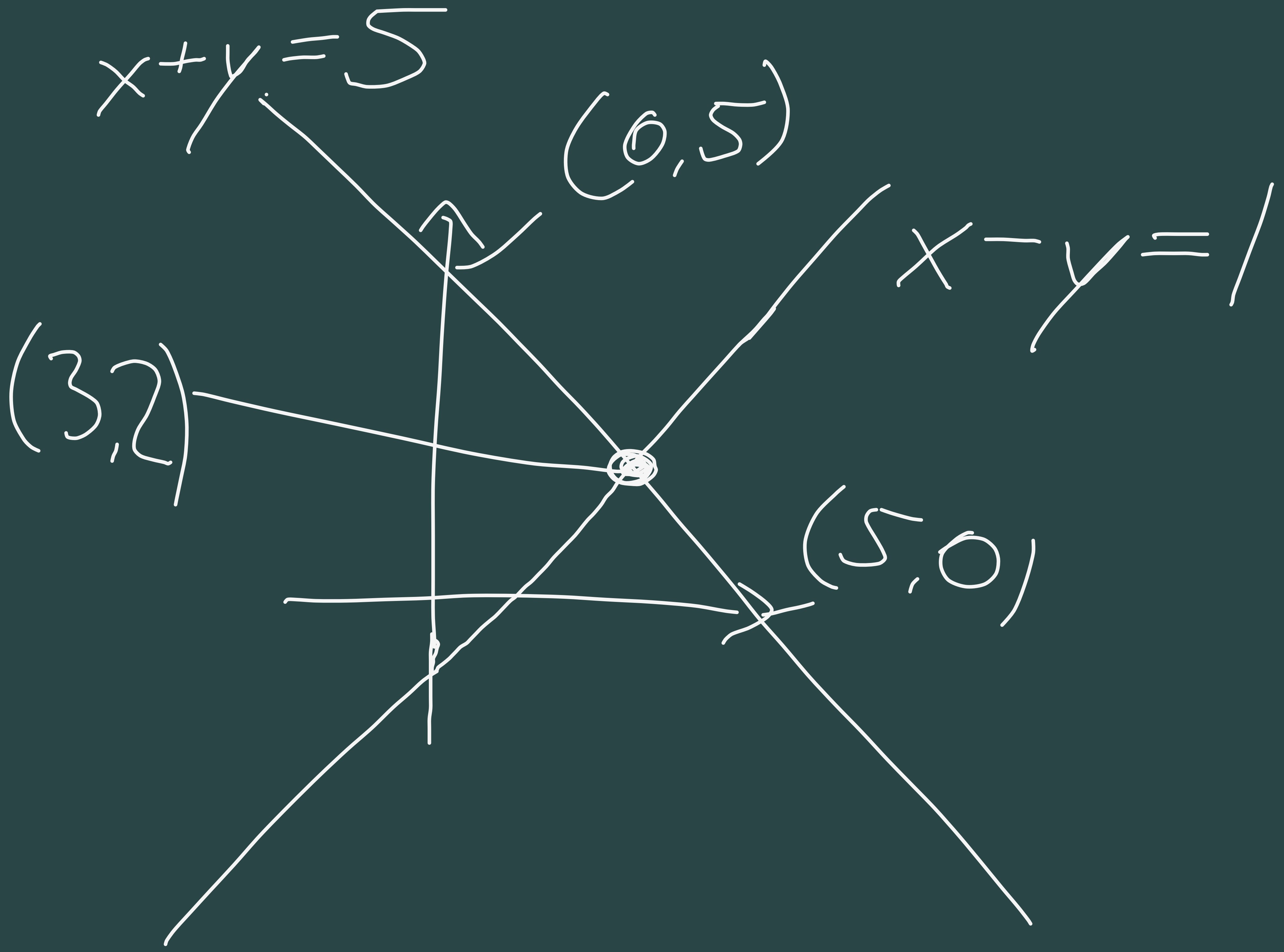
$$\boxed{\|n\|} = \alpha\left\|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right\|$$

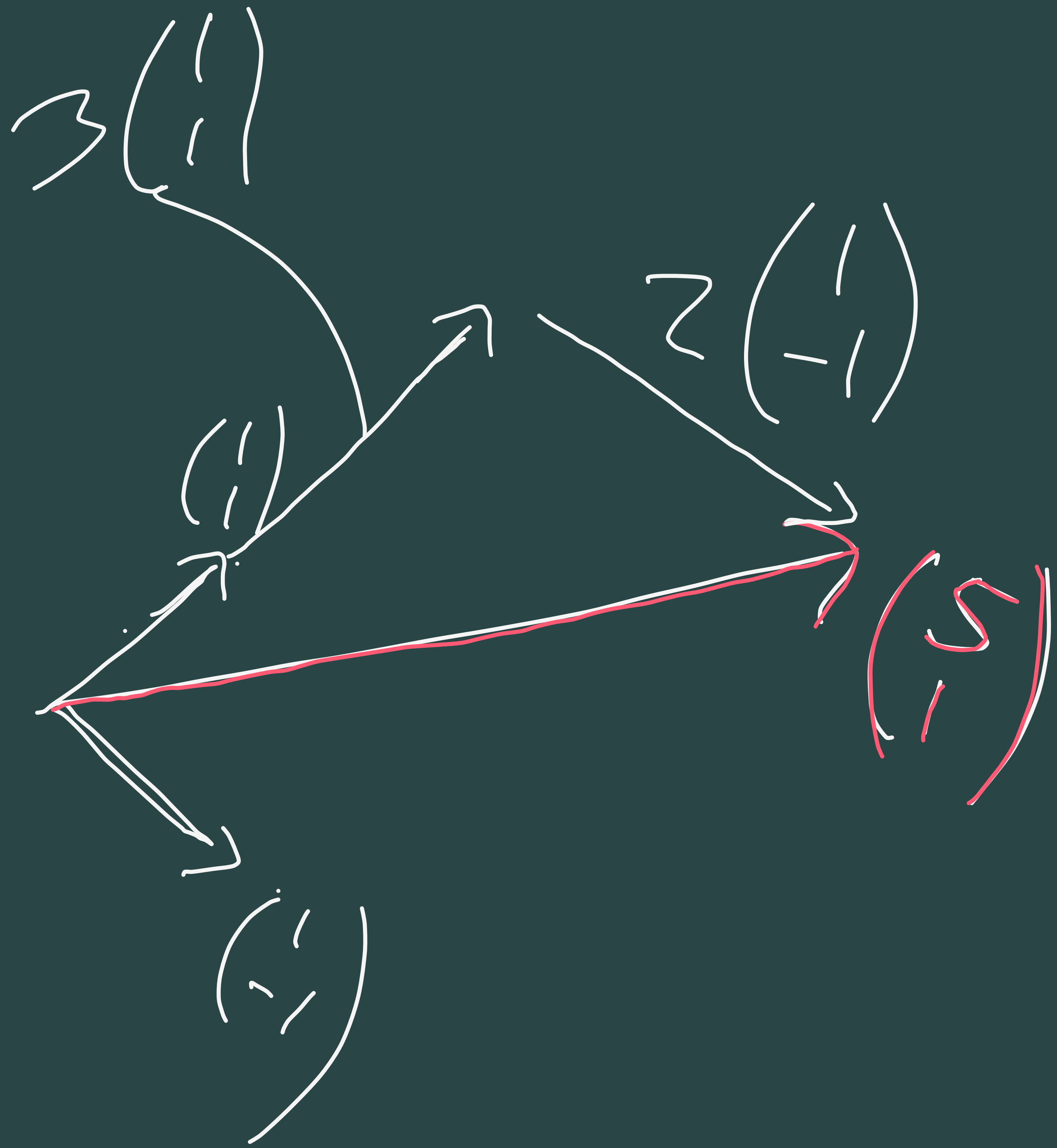
$$= \frac{|d-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}}$$

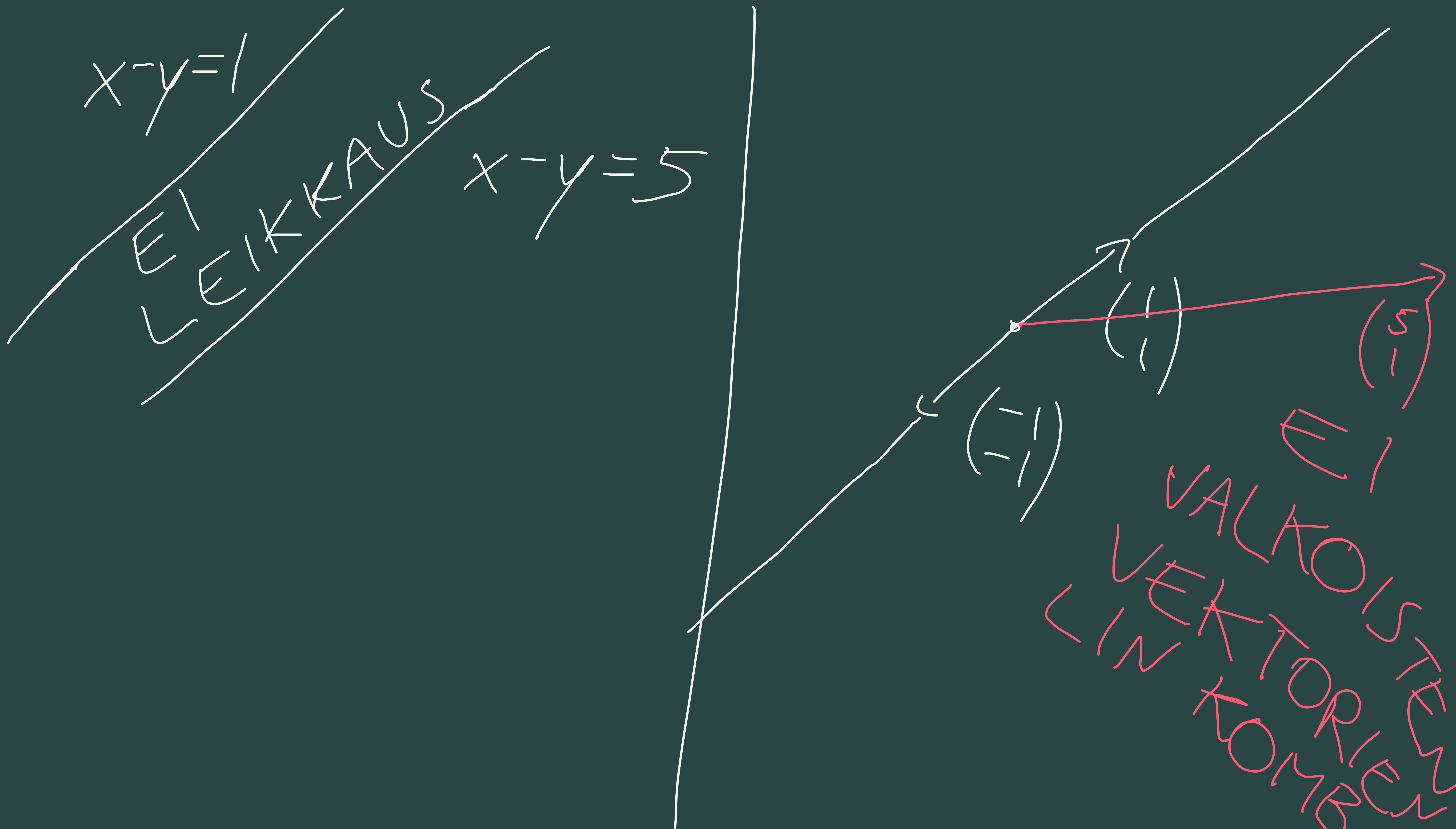
$$\{\overbrace{\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2x-y\\ x+y\end{pmatrix}$$


y=2x-1
ratkaisu
(2,3)
y=5-x


$\begin{pmatrix}1\\5\end{pmatrix}$
$2\begin{pmatrix}-1\\1\end{pmatrix}$
$3\begin{pmatrix}2\\1\end{pmatrix}$
$\begin{pmatrix}-1\\1\end{pmatrix}$
$\begin{pmatrix}2\\1\end{pmatrix}$


x+y=5
(0,5)
x-y=1
(3,2)
(5,0)


3(1 1)
(1 1)
2(1 -1)
(5 1)
(1 -1)


x-y=1
EI LEIKKAUS
x-y=5
(1, 1)
(-1, -1)
(5, 1)
EI VALKOISTEN VEKTORIEN LIN KOMBEN

Jos $c \neq 0$

$$a = b$$

$$\Downarrow$$

$$ca = cb$$

$$\Downarrow$$

$$a = \frac{1}{c}ca = \frac{1}{c}cb = b$$

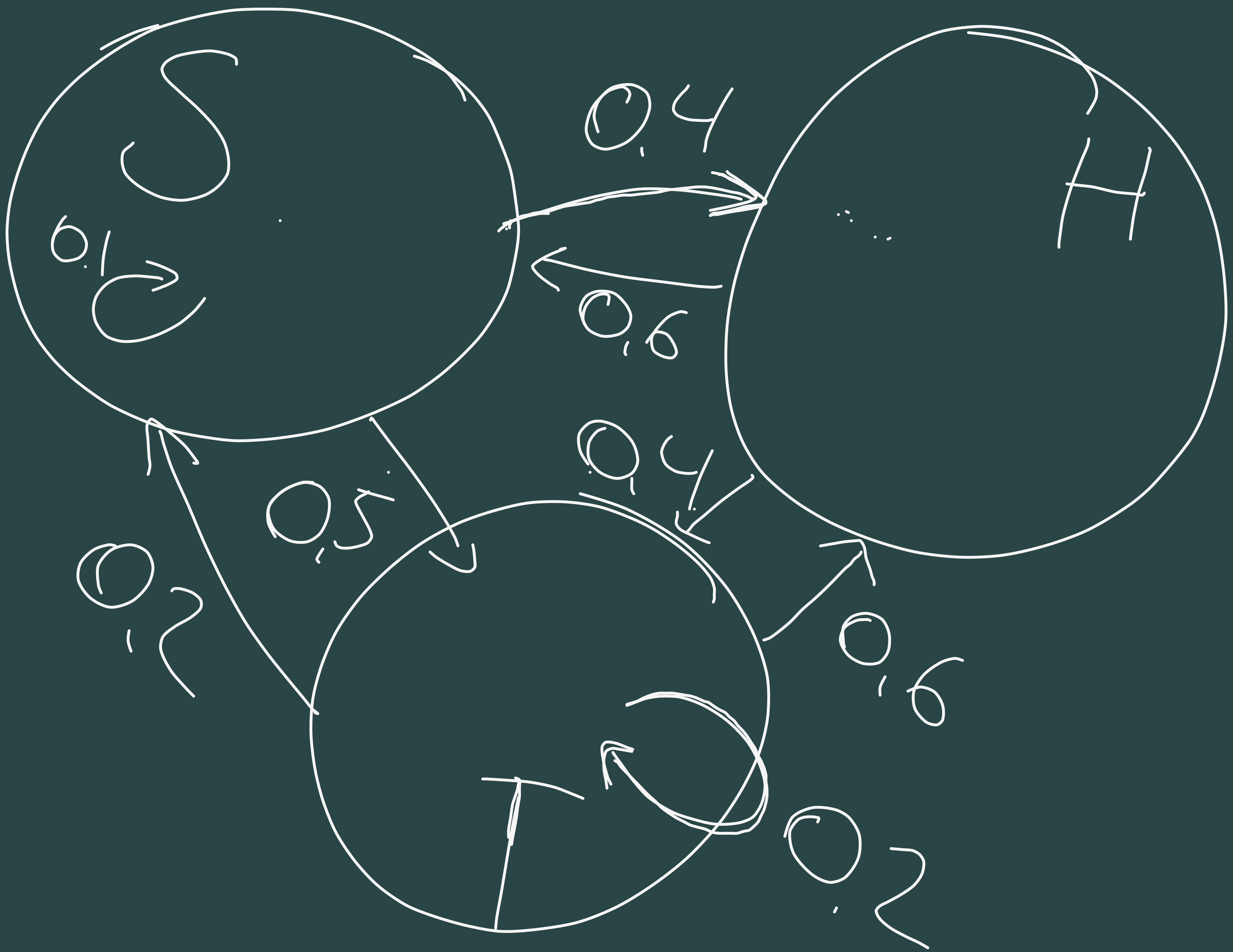
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -2 \end{pmatrix}$$

$$\begin{matrix} \\ -1 \\ 4 \end{matrix}\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \begin{matrix} 1 \\ \\ \\ \end{matrix}$$

Vapaa

$$\begin{cases} x_1 = x_4 = t \\ x_2 = -x_4 = -t \\ x_3 = \cancel{x_4} = t \\ x_4 = t \end{cases}$$


S
H
T
0.1
0.4
0.6
0.4
0.5
0.2
0.6
0.2

$$\left(\begin{array}{ccc|c} 1 & -0{,}4 & -0{,}6 & 0 \\ -0{,}6 & 0{,}9 & -0{,}2 & 0 \\ -0{,}4 & -0{,}5 & 0{,}8 & 0 \end{array}\right) \begin{array}{l} \\ 0{,}6 \\ 0{,}4 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -0{,}4 & -0{,}6 & 0 \\ 0 & 0{,}66 & -0{,}56 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \frac{40}{66} \\ \\ \\ \end{array}$$

$$\begin{cases} P_h = \\ P_s = \dfrac{0{,}56\alpha}{0{,}66} \\ P_t = \alpha \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \overset{\beta}{\boxed{-0{,}6-\frac{40}{66}\cdot 0.6}} & 0 \\ 0 & 1 & -\frac{56}{66} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$P_h = -\beta\alpha$$

$$P_s = \frac{56}{66}\alpha$$

$$P_t = \alpha$$

$$x\,CO_2 + y\,H_2O \longrightarrow z\,O + w\,C_6H_{12}O_6$$

$$\begin{cases} x = 6w \\ 2x + y = z + 6w \\ 2y = 12w \end{cases}$$

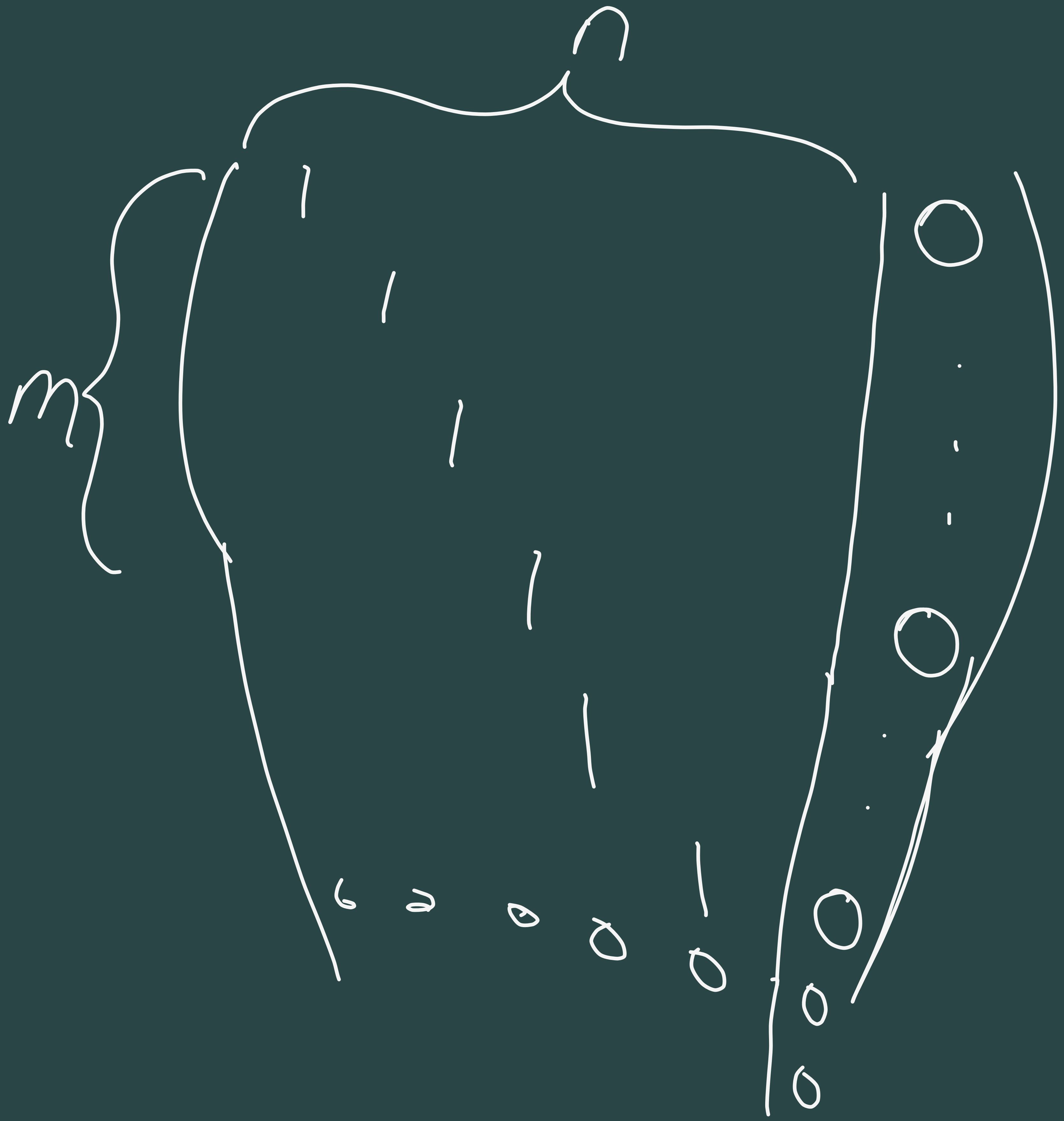
$$\begin{array}{c} \\ -2 \curvearrowright \\ \tfrac{1}{2} \curvearrowright \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -6 & 0 \\ 2 & 1 & -1 & 6 & 0 \\ 0 & 2 & 0 & -12 & 0 \end{array}\right)$$

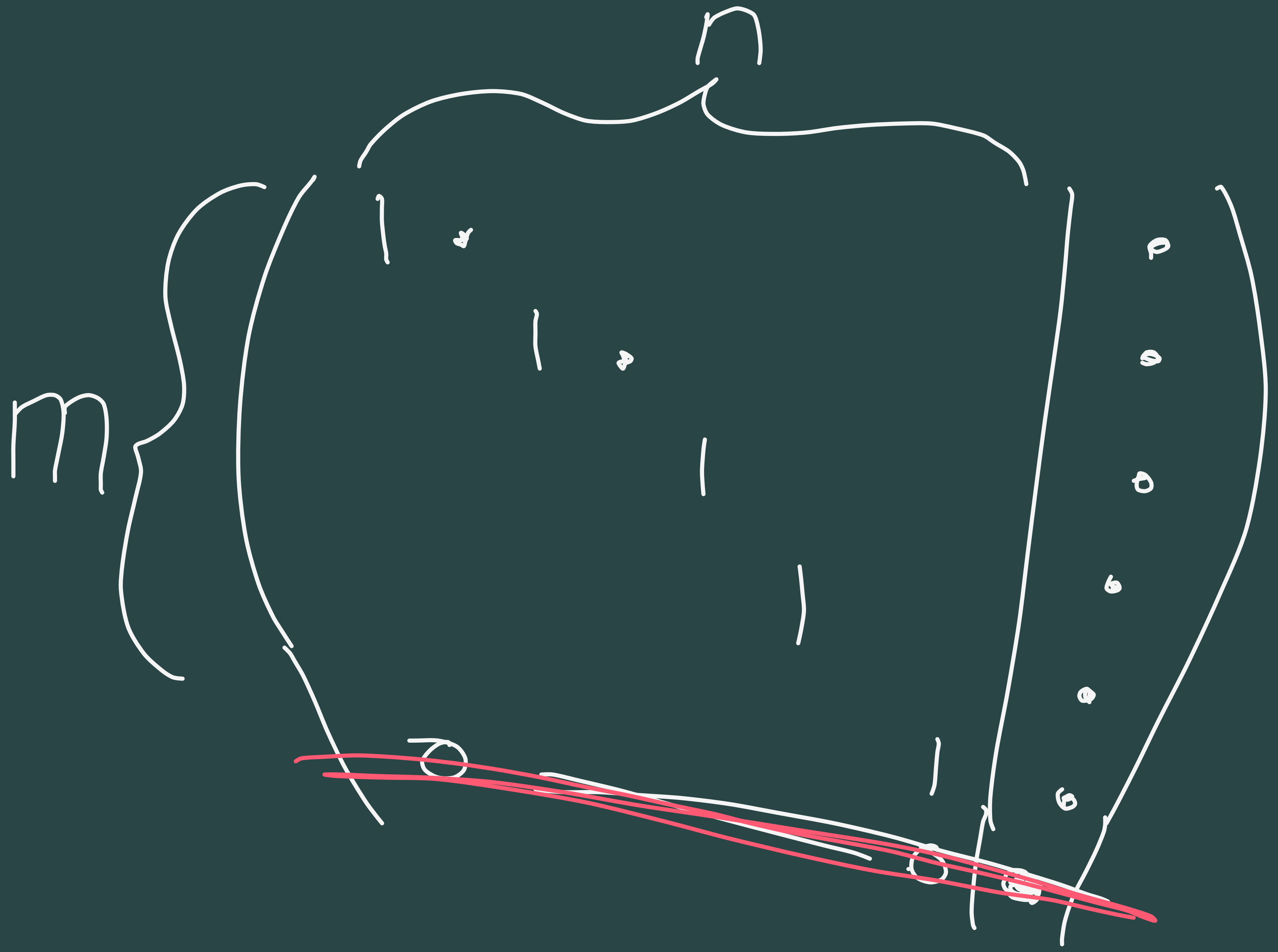
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -6 & 0 \\ 0 & -1 & -1 & 18 & 0 \\ 0 & 1 & 0 & -6 & 0 \end{array}\right) \begin{array}{c} \\ \curvearrowleft \\ 1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -6 & 0 \\ 0 & 1 & 1 & -18 & 0 \\ 0 & 0 & -1 & 12 & 0 \end{array}\right) \begin{array}{l} \\ \curvearrowleft \\ 1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -6 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & -12 & 0 \end{array}\right)$$

$$\begin{cases} x = 6\alpha \\ y = 6\alpha \\ z = 12\alpha \\ w = \alpha \end{cases}$$


n
m


n
m
0
0

$$\begin{array}{ccc}
A\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} = AX & = I_n = & \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix} \\
\| & & \| \\
\begin{pmatrix} \vdots & & \\ Ax_1 & \cdots & Ax_n \\ \vdots & & \end{pmatrix} & & \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}
\end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & 4 & 1 \end{array}\right)$$

$$AA^{-1} = I = A^{-1}A$$

A käännettävä $\Rightarrow$ Yhtälöllä

$$x a_1 + \dots + x_n a_n = 0$$

on vain ratkaisu

$$x_1 = \dots = x_n = 0$$

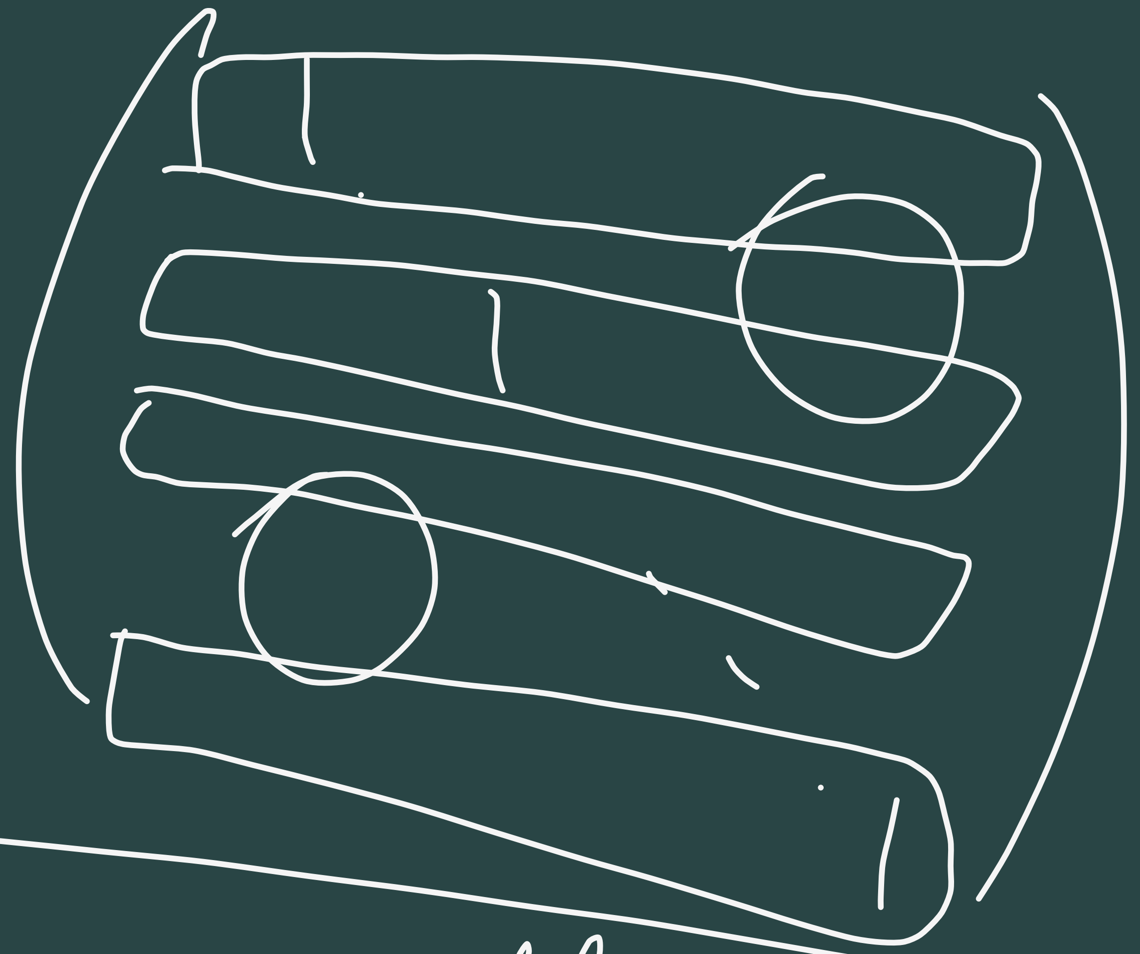
$\Rightarrow$ $a_1 \dots a_n$ lin. riippumatomat

$$(A^T)^{-1}$$

$$\parallel$$

$$(A^{-1})^T$$

$A \sim$



A käännettävä

$\Longleftrightarrow$

A^T —— " ——

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (a, c \neq 0)$$

$$\begin{array}{c} c \circlearrowright \\ a \circlearrowright \end{array} \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \sim \begin{array}{c} \rightarrow \\ -1 \; \hookrightarrow \end{array} \left(\begin{array}{cc|cc} ca & bc & c & 0 \\ ac & ad & 0 & a \end{array}\right)$$

$$\begin{array}{c} \\ -bc \end{array} \begin{pmatrix} ac & bc & \big| & c & 0 \\ 0 & 1 & \big| & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\sim \begin{pmatrix} ac & 0 & \big| & c+\frac{bc^2}{ad-bc} & \frac{-abc}{ad-bc} \\ 0 & 1 & \big| & & \end{pmatrix}$$

Jos $ad-bc = 0$ $\Longrightarrow$ ei käänteistä

"PMMP-sääntö"

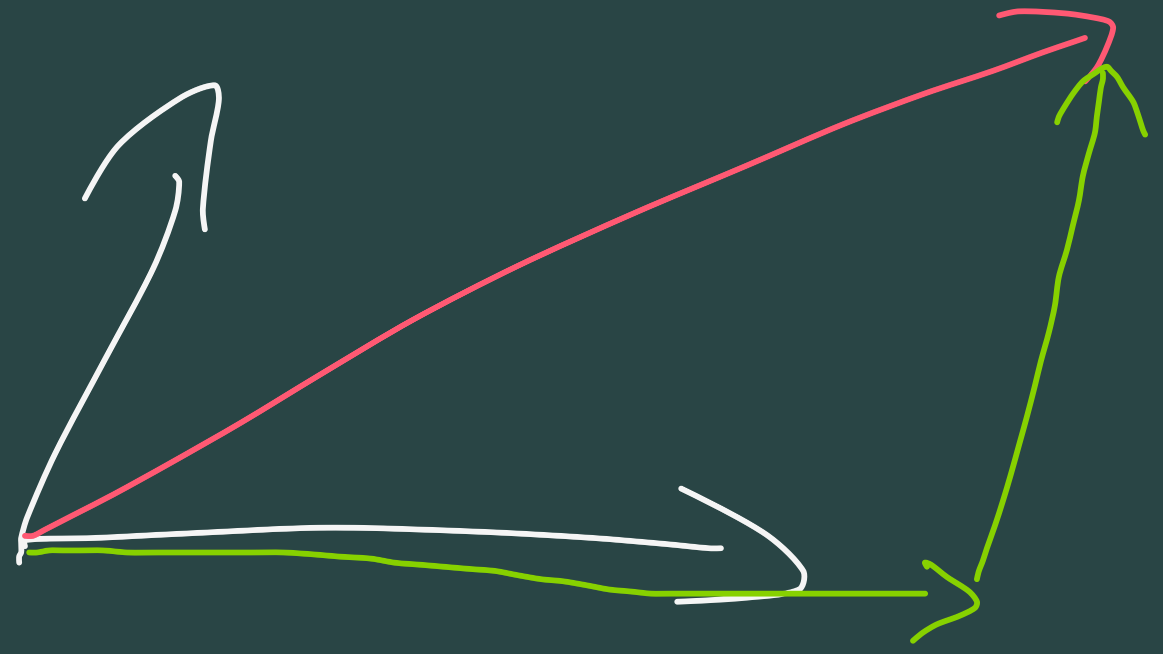
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

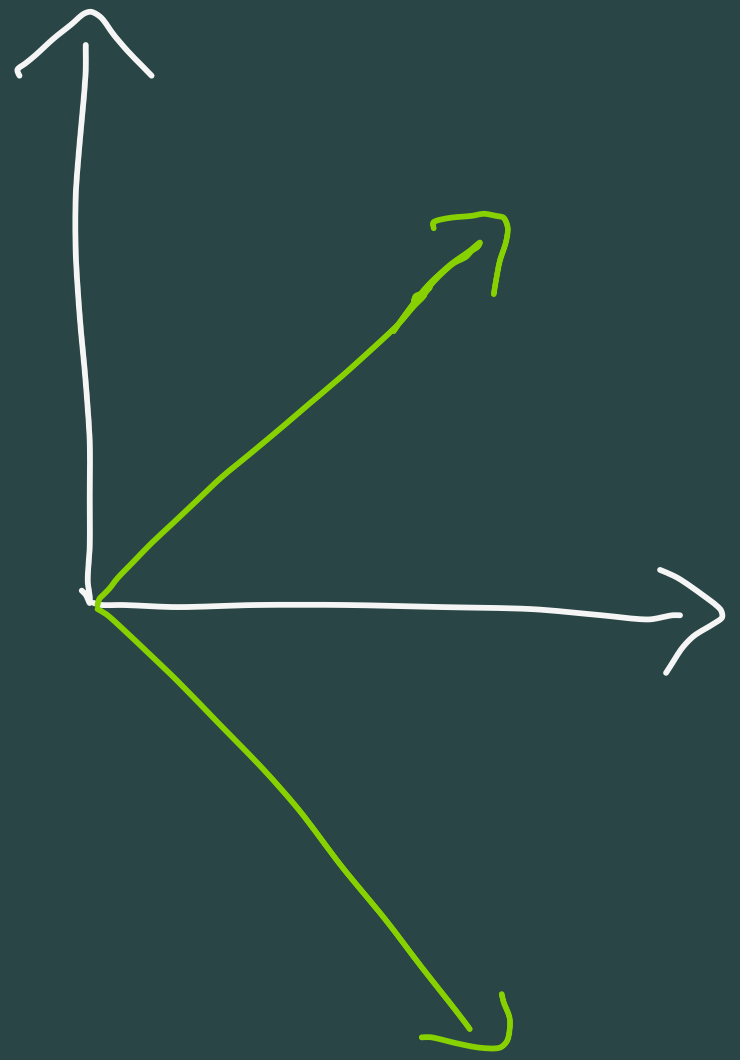
$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & ad-bc \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{cases} \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & \text{jos } ad-bc \neq 0 \\ \text{ei olemassa} & \text{jos } ad-bc = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{array}\right)$$





HALUTAAN TODISTA

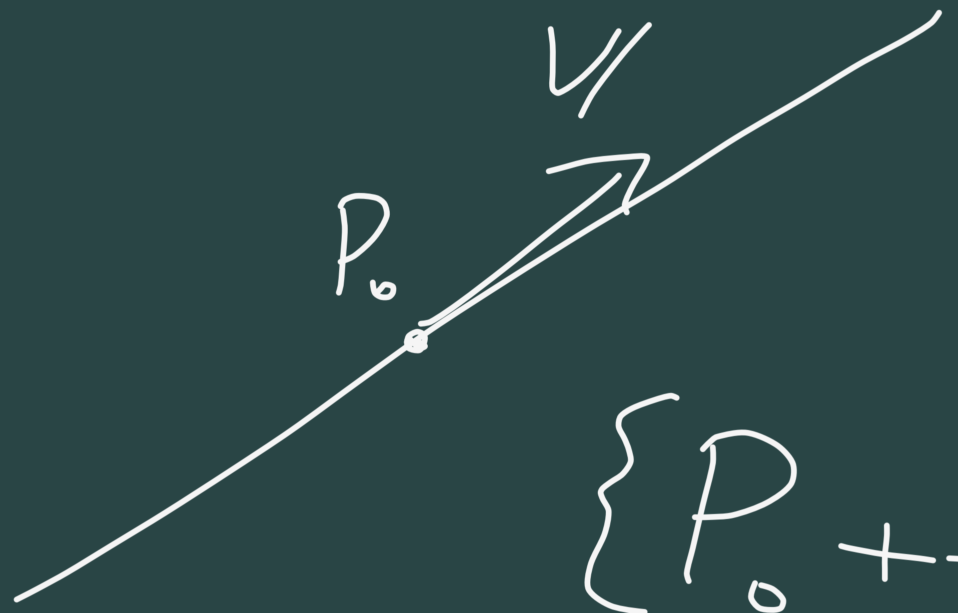
Esitys- $w = x_1 v_1 + \dots + x_n v_n$

on yksikäsitteinen

jos ja vain jos

$v_1, \dots, v_n$ riippumatomat.

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



$$\{P_0 + tv : t \in \mathbb{R}\}$$

$$\{(x,y) : ax+by=c\}$$

$$\{P + tv + su : s,t \in \mathbb{R}\}$$

$$ax+by+cz=d$$

$$A = \begin{pmatrix} \mathbb{v}_1 & \dots & \mathbb{v}_n \end{pmatrix}$$

$$C(A) = \left\langle \mathbb{v}_1 \; . \; . \; . \; \mathbb{v}_n \right\rangle$$

$$= \{ A\mathbb{x} : \mathbb{x} \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

$$\begin{array}{c} r_1 \\ r_2 \end{array}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}\begin{array}{c} r_1 \\ s_2 \end{array}$$

$$\begin{pmatrix} 2 & 3 & 4 & 5 \end{pmatrix} = r_1 + r_2 \quad = r_1 + r_1 + s_2$$

$$\sim \left(\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right) \begin{array}{c} s_1 \\ s_2 \end{array}$$

$$\begin{pmatrix} 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\|$$

$$2 \cdot \begin{pmatrix} 1 & 0 & -1 & -2 \end{pmatrix} + 3 \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}$$

$$= (s_1 + s_2) + (s_1 + s_2) + s_2 =$$

$$(\mathbb{a}_1 \quad \mathbb{a}_2 \quad \mathbb{a}_3) \sim (\mathbb{b}_1 \quad \mathbb{b}_2 \quad \mathbb{b}_3)$$

RIIPPUMATOMAT

RIIPPUMATOMAT

$\Longleftrightarrow$

$\Updownarrow$

$$x_1\mathbb{a}_1 + x_2\mathbb{a}_2 + x_3\mathbb{a}_3 = \mathbb{0}$$

vain triviaali ratkaisu

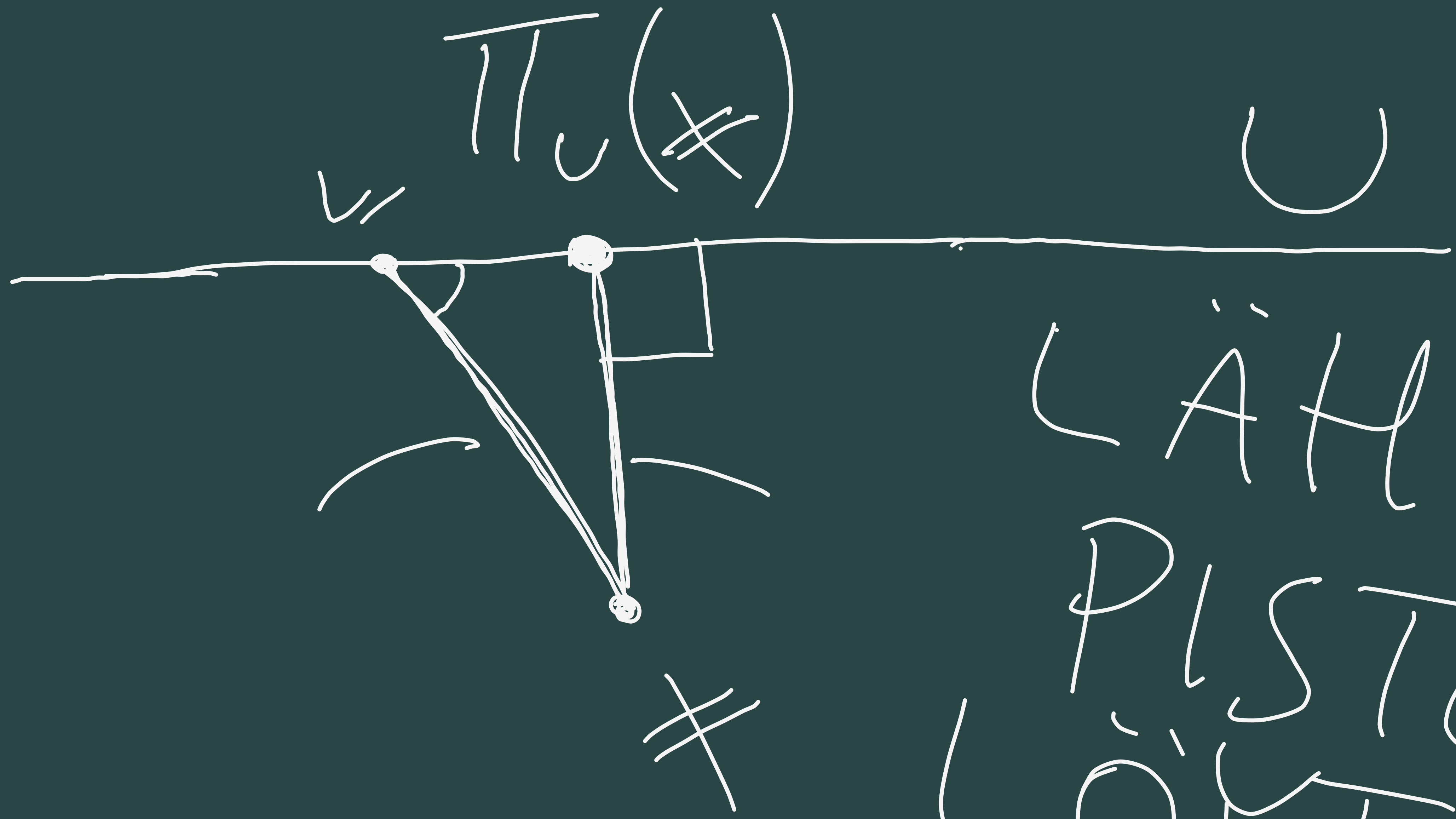
$\Longleftrightarrow$

$$x_1\mathbb{b}_1 + x_2\mathbb{b}_2 + x_3\mathbb{b}_3 = \mathbb{0}$$

vain triv ratkaisu

$$m\left\{\overbrace{\begin{pmatrix} / \\ / \\ / \\ / \\ / \end{pmatrix}}^{k}\right. \; k\left\{\overbrace{\begin{pmatrix} \qquad\qquad\qquad \end{pmatrix}}^{n}\right. = m\left\{\overbrace{\begin{pmatrix} \qquad \text{Aste } k \qquad \end{pmatrix}}^{n}\right.$$

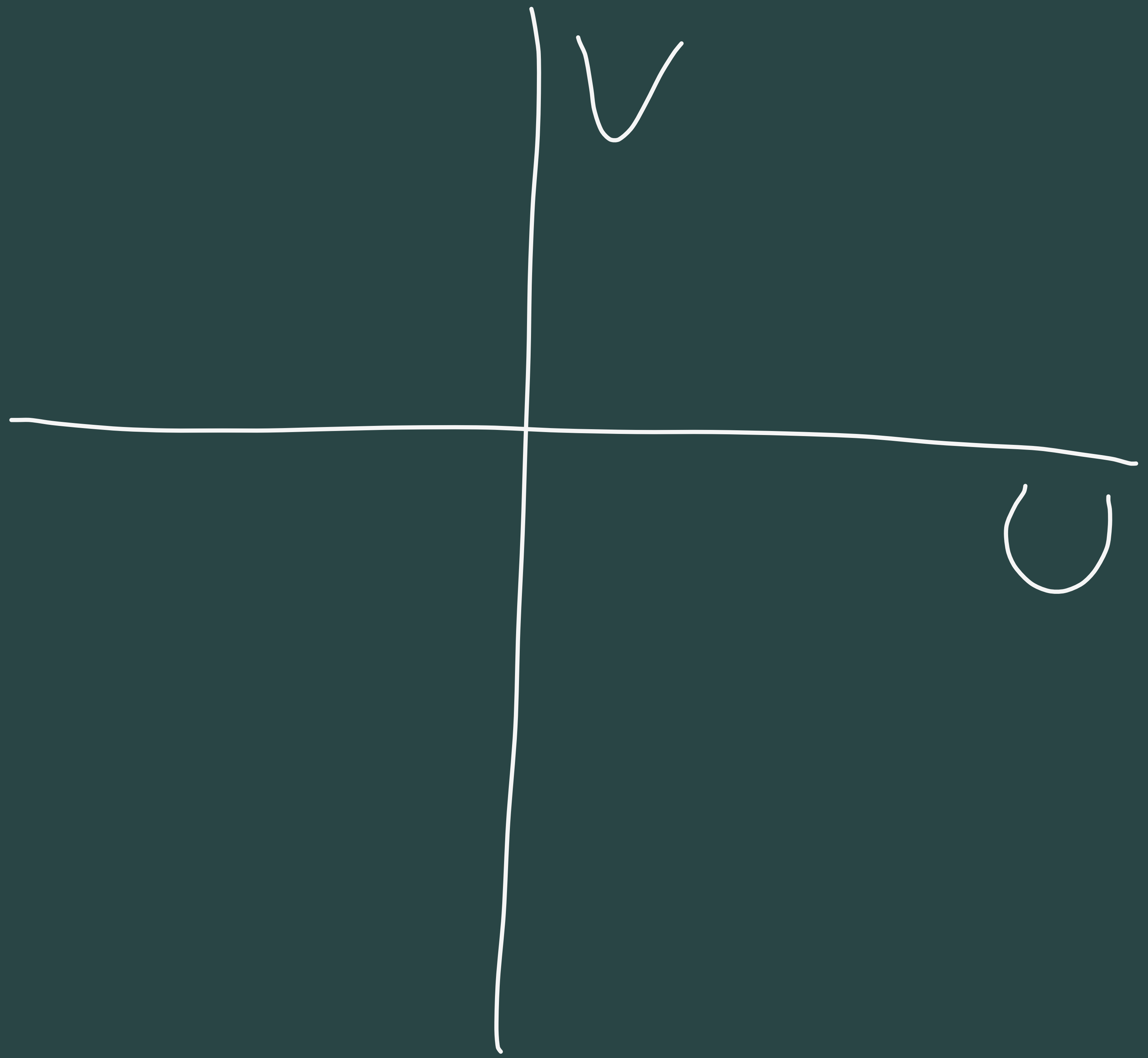
$$\overbrace{\left(\begin{matrix}5\\ \cancel{2\cdot}\\ 3\\ \cancel{2\cdot}\\ 1\\ \cancel{2\cdot}\end{matrix}\quad \text{“MELKEN” 7-ASTE} = \right)}^{100\cdot 10^{6}}\left.\vphantom{\begin{matrix}5\\5\\5\\5\\5\\5\end{matrix}}\right\}\ 100000$$



LÄHIMMÄN PISTEEN LÖYTÄMINEN

KÄYTTÄÄ ORTOGONALISUUTTA

$$\overbrace{\begin{pmatrix} \text{———} \; a_1 \; \text{———} \\ \text{—————————} \\ \text{—————————} \\ \text{—————————} \\ \text{———} \; a_m \; \text{———} \end{pmatrix}}^{n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

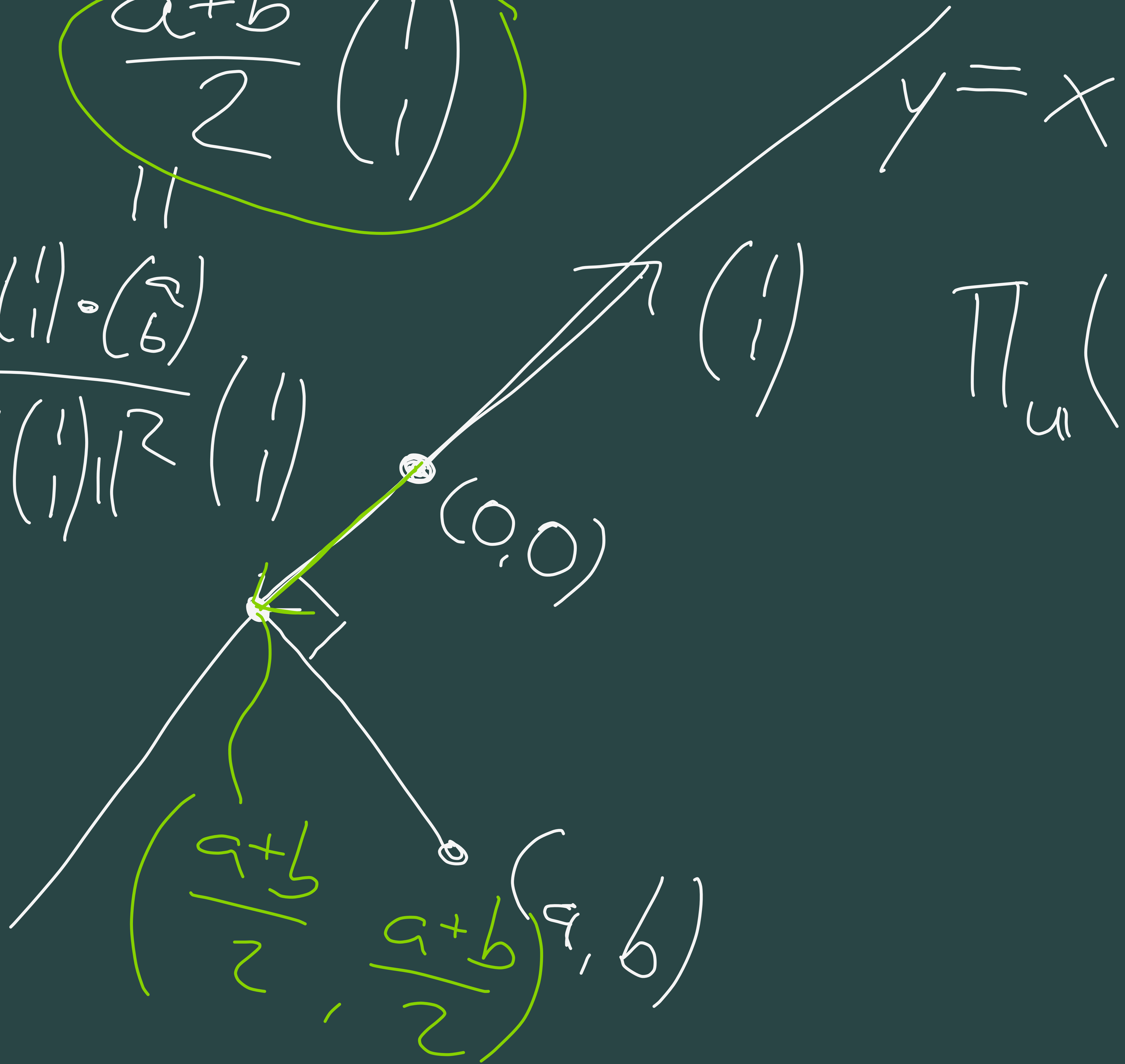


$$V = U^{\perp}$$

$$\Updownarrow$$

$$U = V^{\perp}$$

$$\boxed{\frac{a+b}{2}\begin{pmatrix}1\\1\end{pmatrix}} = \frac{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}a\\b\end{pmatrix}}{\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\|^2}\begin{pmatrix}1\\1\end{pmatrix}$$



$$\Pi_{u}(\mathbf{b}) = \frac{u\cdot \mathbf{b}}{\|u\|^2}u$$

$$\mathcal{N}(A) \perp \mathcal{R}(A)$$

$$\underline{\underline{y}} \in \mathcal{C}(A) \Longleftrightarrow \underline{\underline{y}} = A\underline{\underline{x}}$$

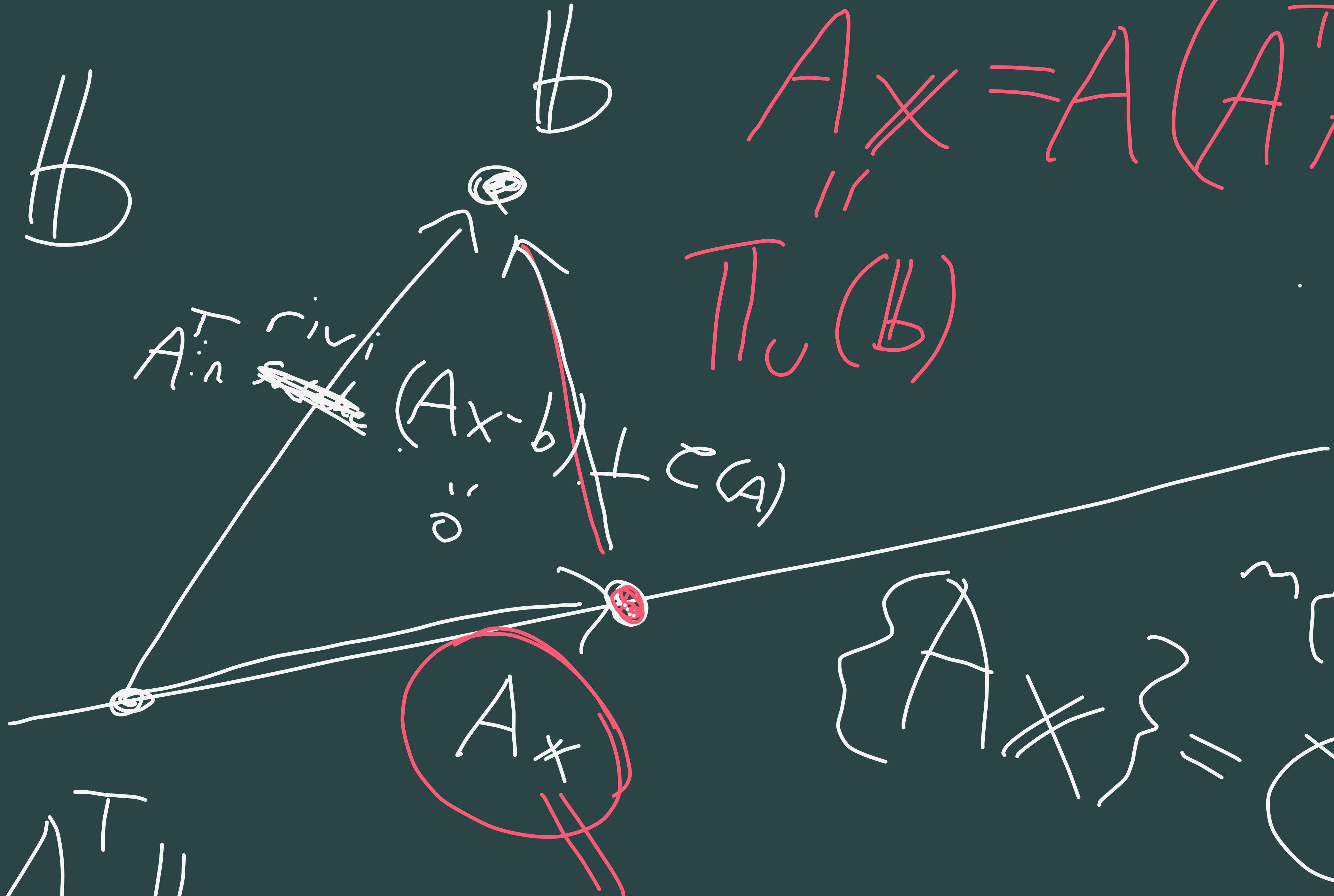
jollekin $\underline{\underline{x}}$.

$$A\mathbb{x} \approx \mathbb{b}$$

$$\Updownarrow$$

$$A^T A\mathbb{x} = A^T\mathbb{b}$$

$$A\mathbb{x} = A(A^TA)^{-1}A^T\mathbb{b}$$



$$A\mathbb{x} = A(A^TA)^{-1}A^T\mathbb{b}$$

$$\parallel$$

$$\Pi_U(\mathbb{b})$$

$$\parallel$$

$$\Pi_U(\mathbb{b})$$

$$\tilde{(A)}$$

$$\{(A^T)$$

$$\{A\mathbb{x}\} = C(A)$$

$$\parallel$$

$$U$$

$$m\left\{\overbrace{\left(\quad A^T \quad\right)}^{n}\right. \overbrace{\left(\quad A \quad\right)}^{m}\left.\vphantom{\begin{pmatrix}A\\A\end{pmatrix}}\right\}n = \overbrace{\left(\quad A^TA \quad\right)}^{m}\left.\vphantom{\begin{pmatrix}A\\A\end{pmatrix}}\right\}m$$

$$U = \mathcal{C}(A)$$

$$A(A^T A)^{-1} A^T \mathbf{b} = \Pi_U(\mathbf{b})$$

$$A^T A = (u_1 \cdots u_n)\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \|u\|^2$$

$$U = \mathcal{C}(u)$$

$$A = (u)$$

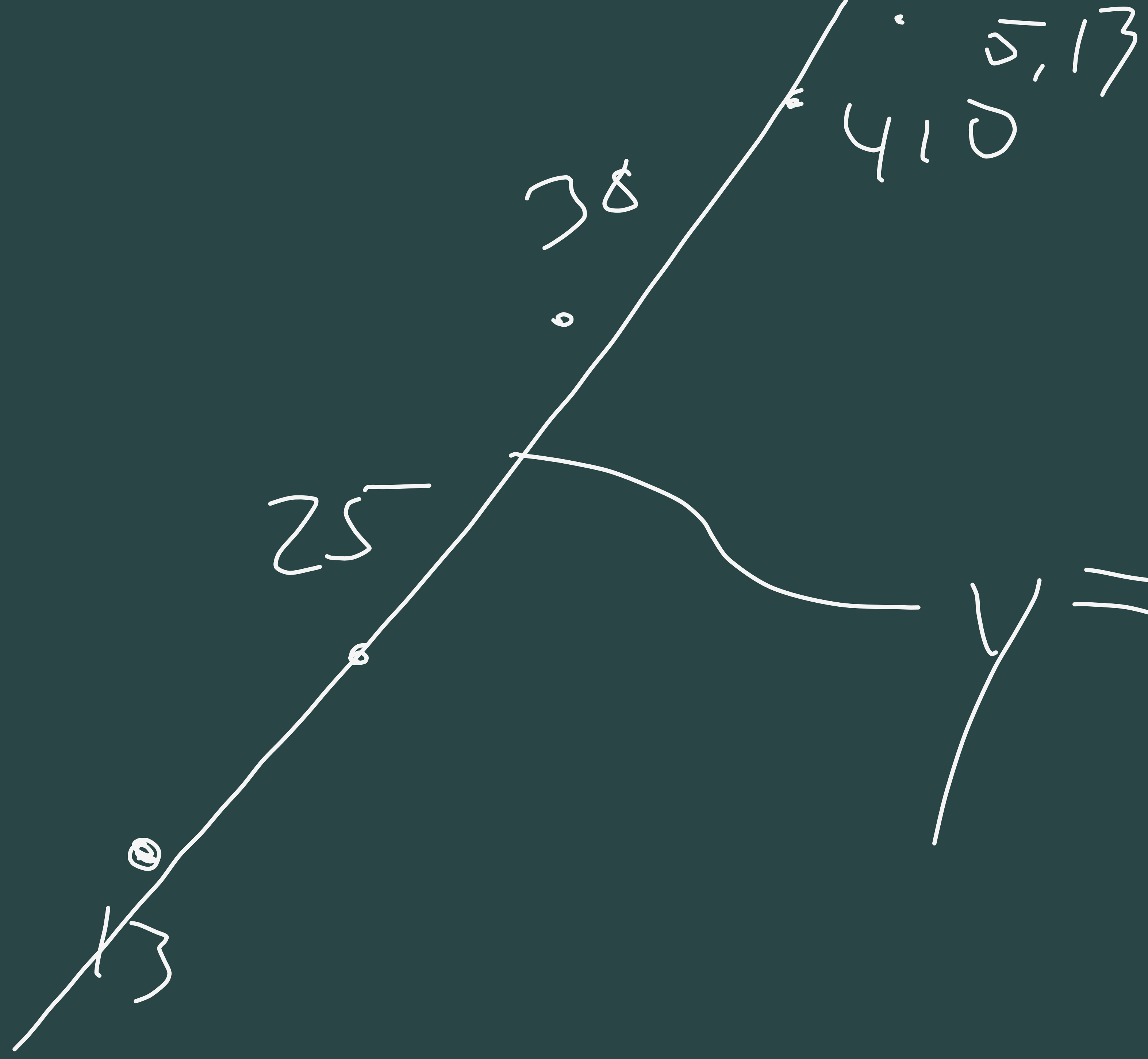
$$A^T$$

$$(A^T A)^{-1} = \frac{1}{\|\mathbf{u}\|^2}$$

$$A (A^T A)^{-1} A^T \mathbf{b} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \frac{1}{\|\mathbf{u}\|^2} (u_1 \cdots u_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \frac{1}{\|\mathbf{u}\|^2} \, \mathbf{u} \cdot \mathbf{b}$$

$$= \frac{\mathbf{u} \cdot \mathbf{b}}{\|\mathbf{u}\|^2} \, \mathbf{u} \, .$$

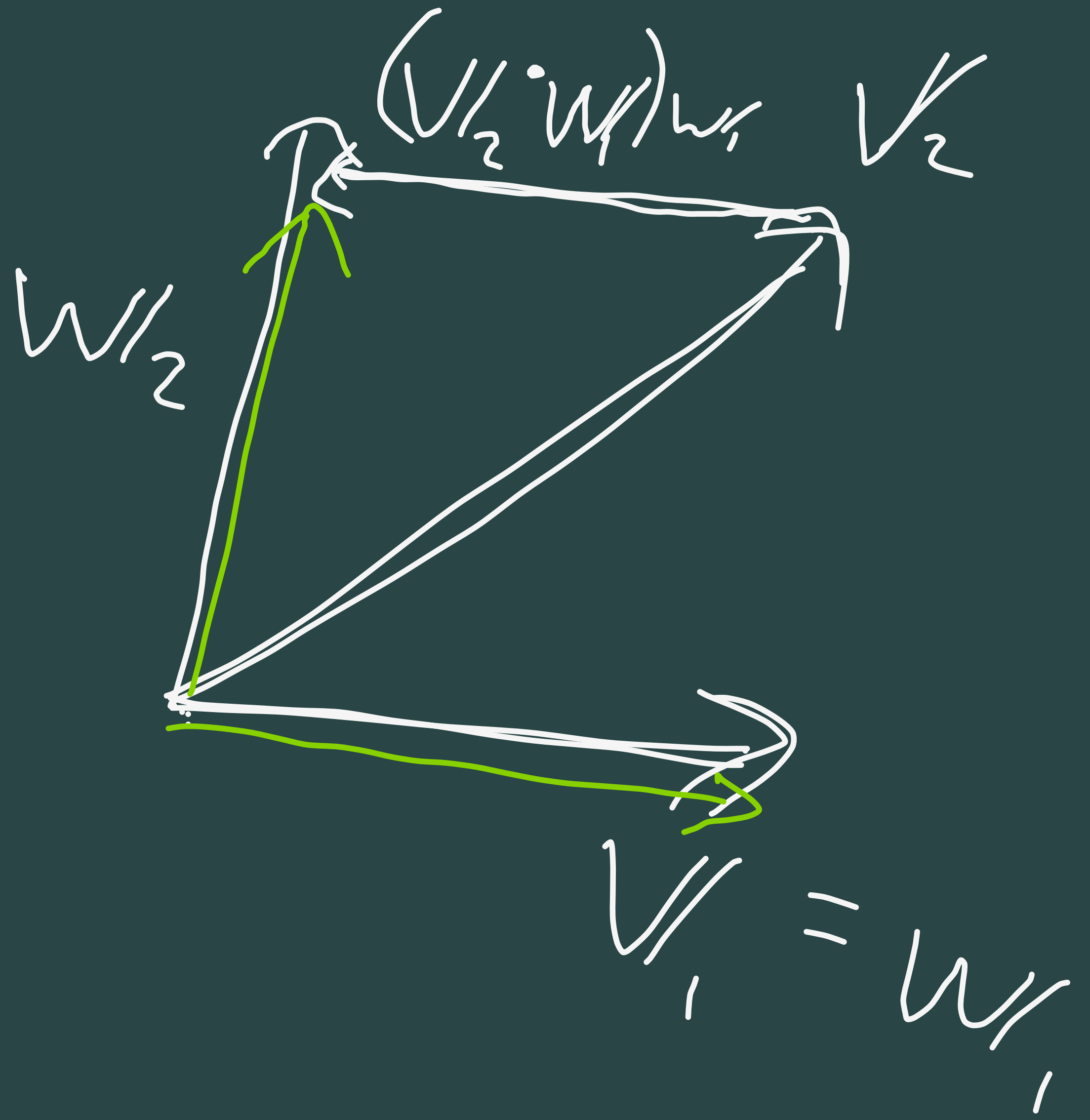


$$y = \frac{5x}{2} + \frac{3}{10}$$

$$v_i = \mathbf{v} \cdot \mathbf{b}_i$$

Muiden kanta-vektorien

$v_i \cdot \mathbf{b}_i \perp$ lin. komb


$(v_2 \cdot w_1)w_1$
v_2
w_2
$v_1 = w_1$

$\begin{pmatrix}1\\1\end{pmatrix} = v_1 = w_1$

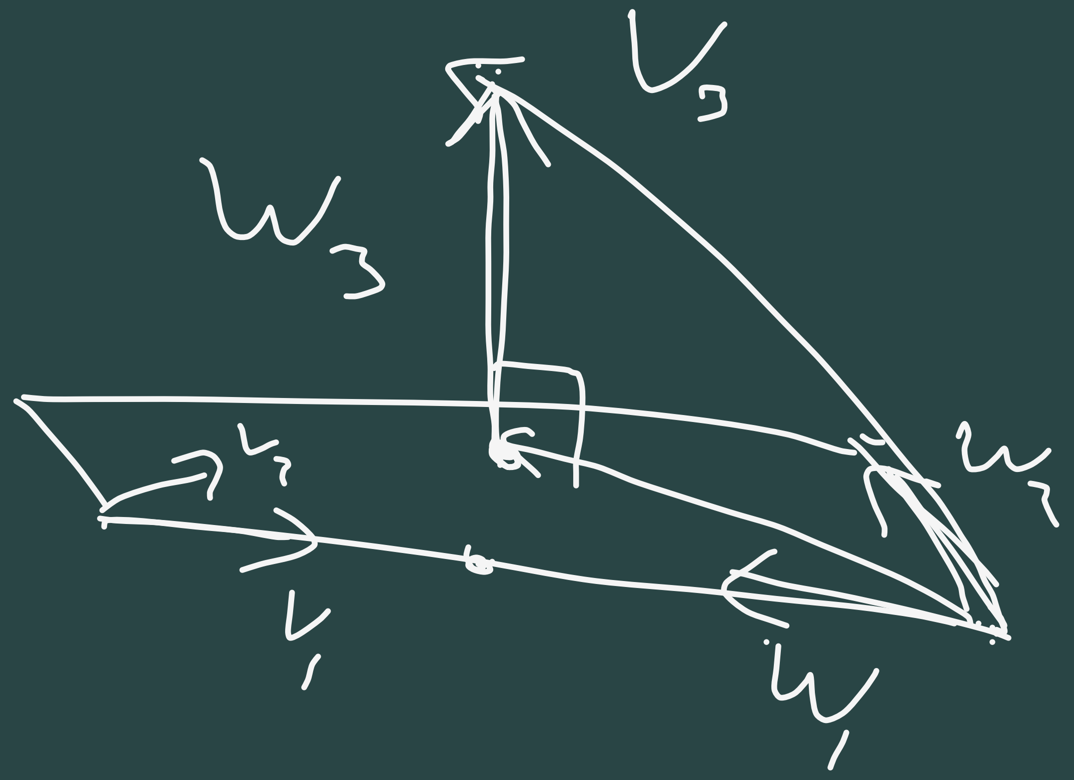
$\begin{pmatrix}1\\0\end{pmatrix} = v_1$

$w_2 = \begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}$

$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\begin{pmatrix}\frac{3}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}_e = \begin{pmatrix}1\\2\end{pmatrix}_u$

$\sqrt{2}\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix} = u_2$


v_3
w_3
v_2
w_2
v_1
w_1

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & -1 & 0 \end{array}\right)$$

vapaa

$$x = y + z + w$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} y \\ y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} w \\ 0 \\ 0 \\ w \end{pmatrix} = y\overset{V_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + z\overset{V_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + w\overset{V_3}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

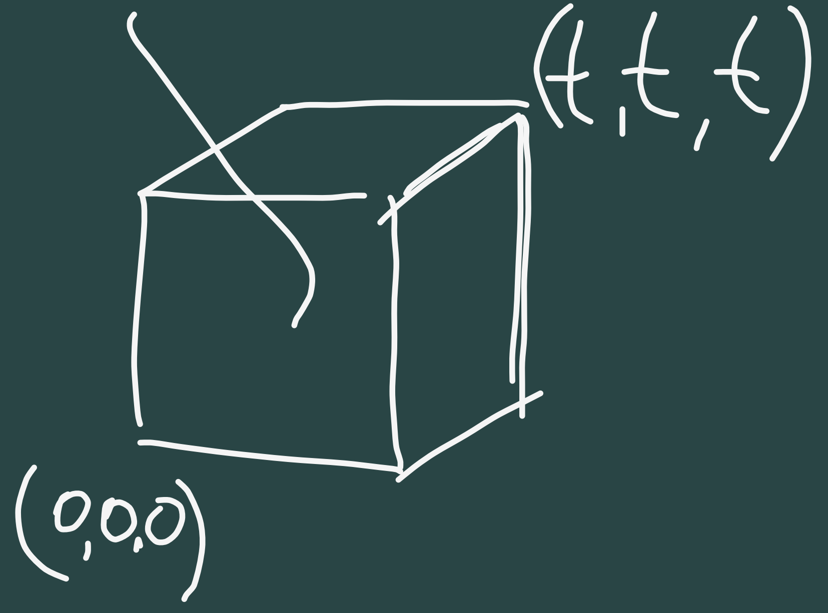
$$w_2 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$w_3 = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix} \qquad u_3 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}$$

Jos ortogonaali

$$\mathbb{v}_B = \begin{pmatrix} \dfrac{\mathbb{v}\cdot\mathbb{b}_1}{\|\mathbb{b}_1\|^2} \\ \vdots \\ \dfrac{\mathbb{v}\cdot\mathbb{b}_n}{\|\mathbb{b}_n\|^2} \end{pmatrix}$$

Tilavuus t^n



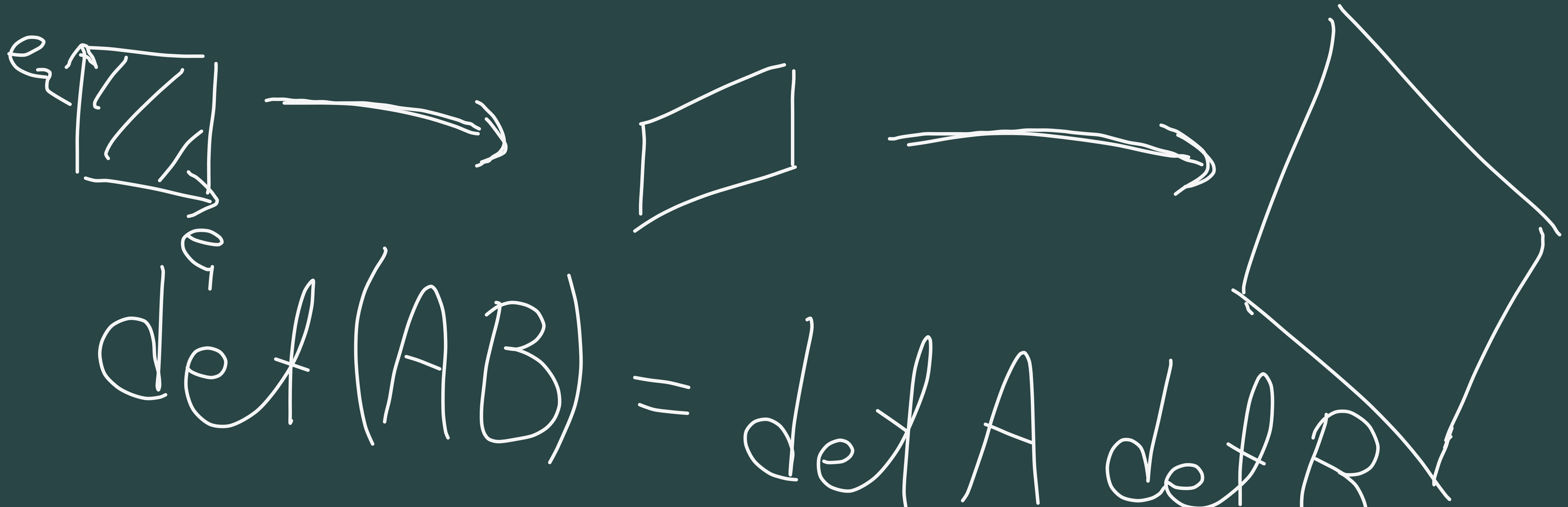
$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$$

$\|A\|$

$\shortparallel$

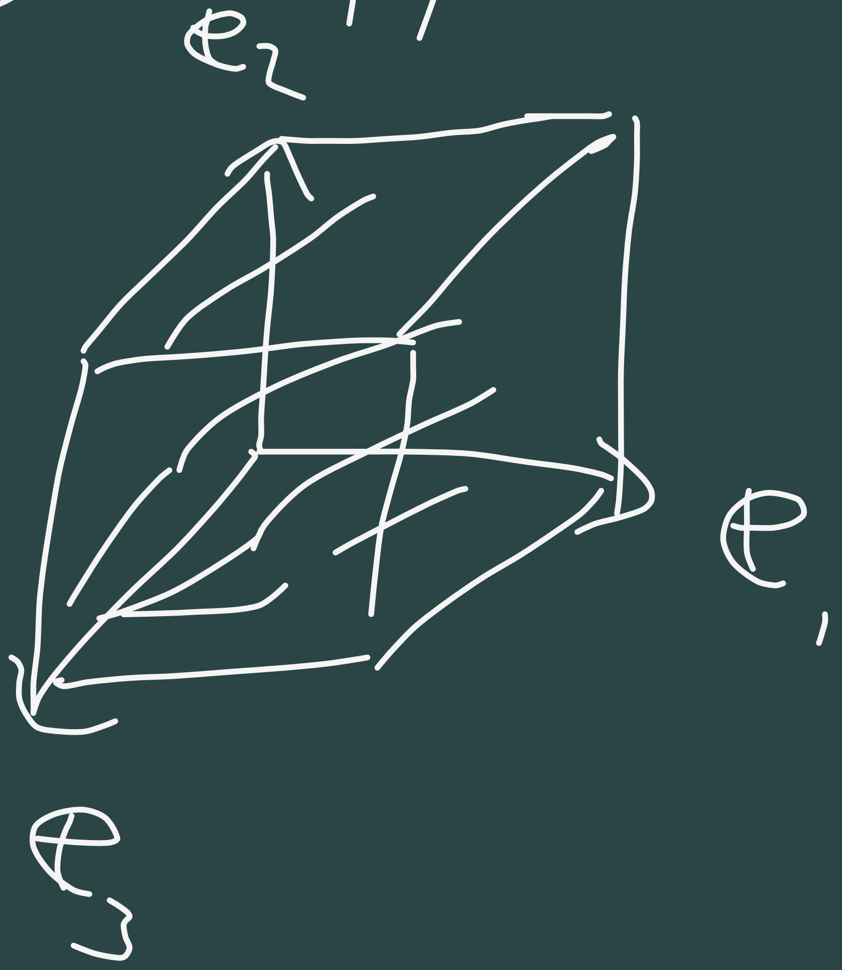
$\mathrm{Vol} = |\det A|$

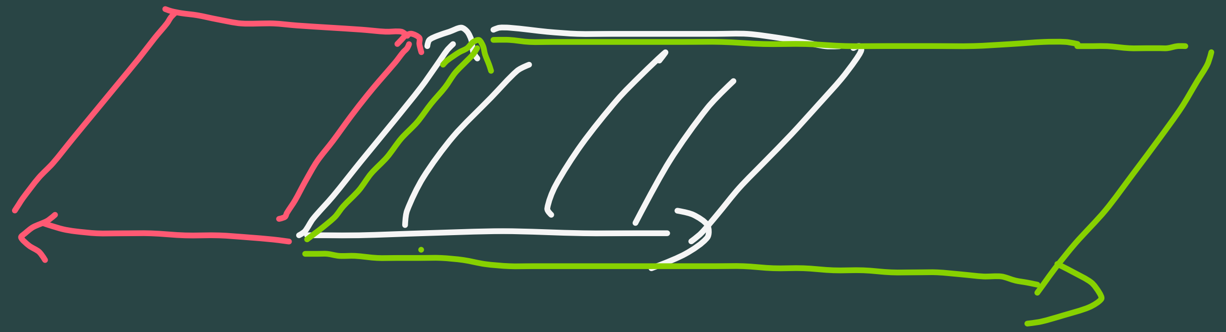
$$\begin{tikzcd}\mathbb{R}^n \arrow[r, bend left, "B"] & \mathbb{R}^n \arrow[r, bend left, "A"] & \mathbb{R}^n\end{tikzcd}$$



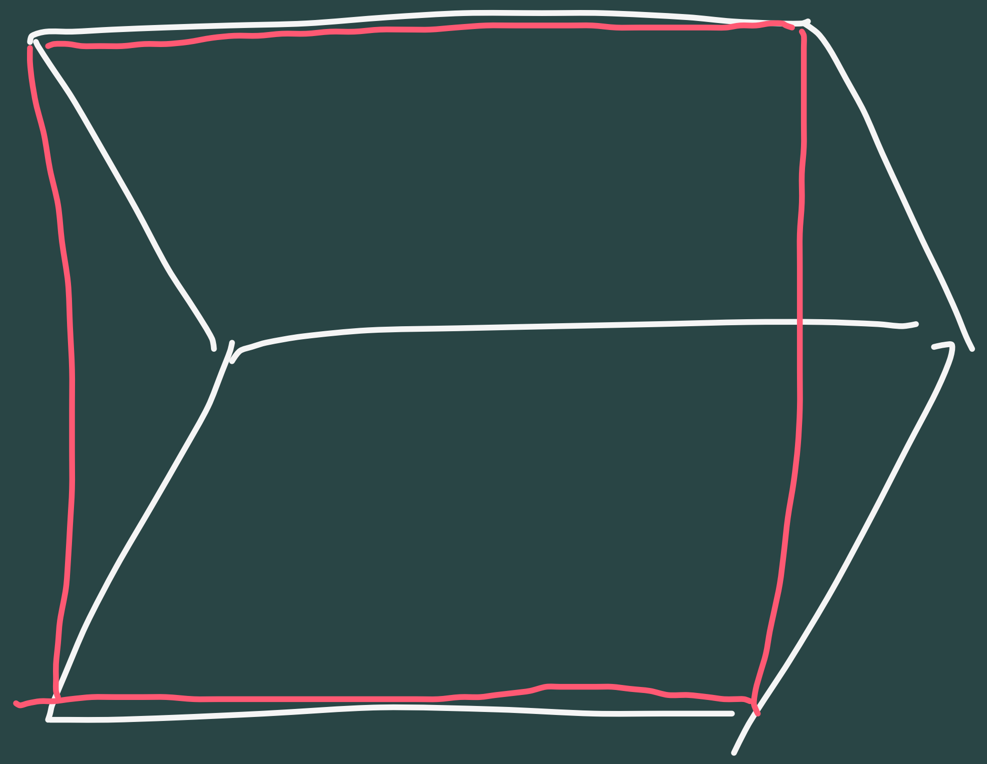
$$\det(AB) = \det A \det B$$

$$\det\begin{pmatrix}1&&&&\\&1&&0&\\&&1&&\\&0&&1&\\&&&&1\end{pmatrix}$$





$$\det \left. \underbrace{\begin{pmatrix} s & & & \\ & s & & 0 \\ & 0 & s & \\ & & & s \end{pmatrix}}_{n} \right\} n = s^{n}$$



$$\det\begin{pmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{pmatrix}+\det\begin{pmatrix}\mathbf{b} & \mathbf{a} & \mathbf{c}\end{pmatrix}$$

$$=\det\begin{pmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{pmatrix}+\det\begin{pmatrix}\mathbf{a} & \mathbf{a} & \mathbf{c}\end{pmatrix}+\det\begin{pmatrix}\mathbf{b} & \mathbf{b} & \mathbf{c}\end{pmatrix}$$

$$+\det\begin{pmatrix}\mathbf{b} & \mathbf{a} & \mathbf{c}\end{pmatrix}$$

$$=\det\begin{pmatrix}a+b & a+b & c\end{pmatrix}=0$$

$$0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\iff ad = bc \iff \lambda \begin{pmatrix} a \\ c \end{pmatrix} = \mu \begin{pmatrix} b \\ d \end{pmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

$$\Longleftrightarrow$$

$$ad = bc$$

1) dos $c, d \neq 0$.

$$\frac{a}{c} = \frac{b}{d} = \lambda$$

$$\begin{pmatrix} a \\ c \end{pmatrix} = c \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} b \\ d \end{pmatrix} = d \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & 1 & \ddots & & 0 \\ 0 & 0 & & & \\ 0 & & 1 & & \\ 0 & 0 & 0 & & 1 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix} = 1$$

$$\det\begin{pmatrix} a & & & \\ & b & 0 & \\ & 0 & c & \\ & & & d \end{pmatrix} = abcd.$$

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & A & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix} = \det A$$

$$|A| = \left|\begin{pmatrix} B \\ C \end{pmatrix}\right| = \det\begin{pmatrix} 0 & & B & & & \\ 0 & & & & & \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & \\ 0 & & C & & & \\ 0 & & & & & \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} k-1 \\ \\ \\ \end{matrix} \cdot (-1)^{k-1}$$

$$\left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right| = \left|\begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{pmatrix}\right| + \left|\begin{pmatrix} 0 & b & c \\ d & 0 & 0 \\ 0 & h & i \end{pmatrix}\right|$$

$$+ \left|\begin{pmatrix} 0 & b & c \\ 0 & e & f \\ g & 0 & 0 \end{pmatrix}\right|$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}.$$

$$\mathbb{u} \times \mathbb{v} = \begin{vmatrix} e_1 & u_1 & v_1 \\ e_2 & u_2 & v_2 \\ e_3 & u_3 & v_3 \end{vmatrix} = (u_2 v_3 - v_2 u_3) e_1$$

$$+ (u_3 v_1 - u_1 v_3) e_2$$

$$+ (u_1 v_2 - u_2 v_1) e_3$$

TENTTI 24.2.

16:30-19:30

SALIT A-E

TUO: HENKILÖ-KORTTI

KYNÄT, PYYHEKUMIT

1 SIVU KÄSIN KIRJOITETTUJA MUISTINPANOJA (TOISELLE PUOLELLE)

ÄLÄ TUO: KIRJOJA

LASKIMIA

JOS TARVITSETTE LISÄAIKAA / ERI HUONE:

TÄYTTÄKÄÄ LOMAKE

16:30-20:30

(21:30)

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot5\cdot9 + 2\cdot6\cdot7 + 3\cdot4\cdot8 - 7\cdot5\cdot3 - 4\cdot2\cdot9 - 8\cdot6\cdot1$$

$$= 45 + 84 + 96 - 105 - 72 - 48$$

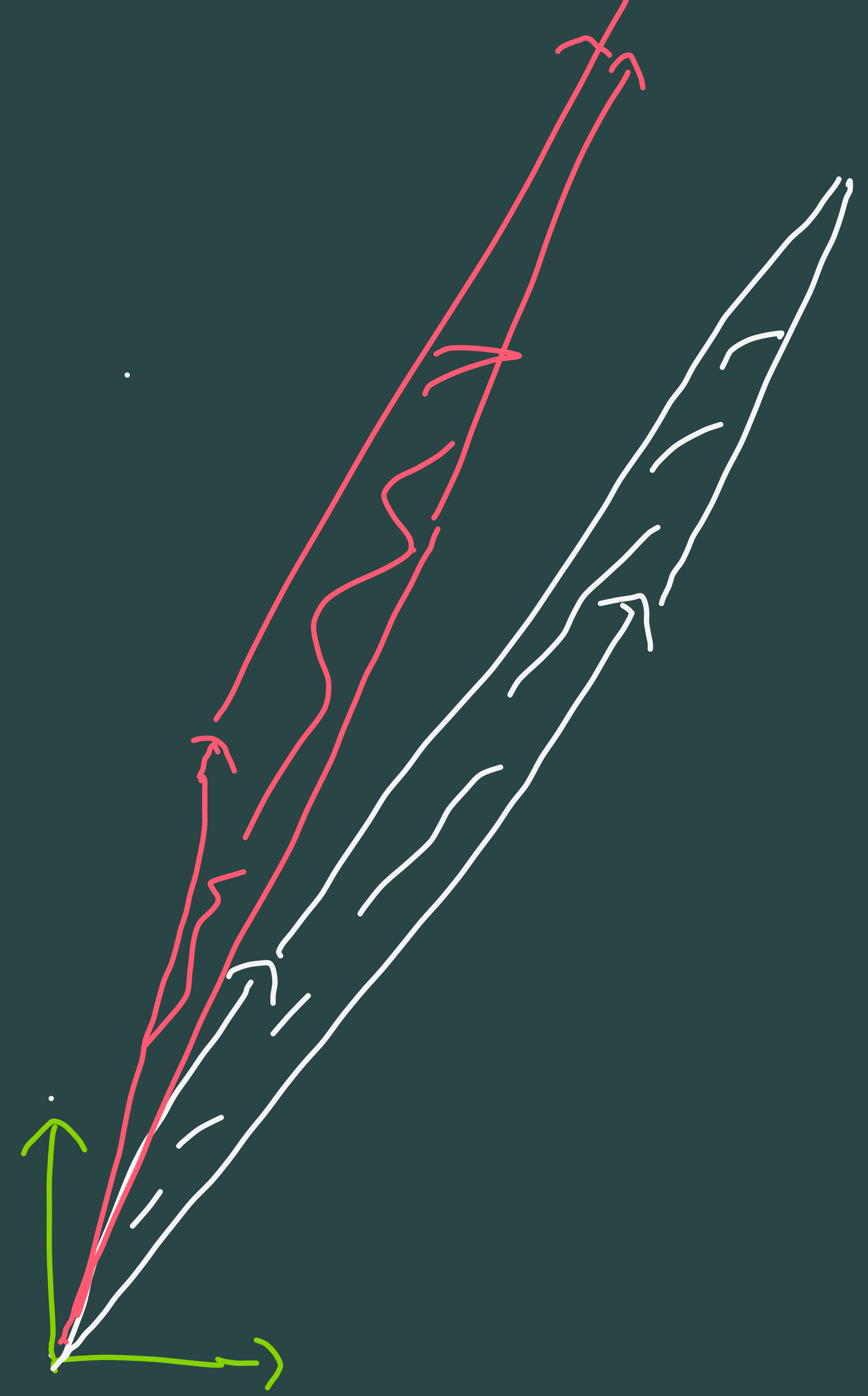
$$= 225 - 225 = 0$$

$$\begin{array}{ccc} \multicolumn{2}{c}{\overbrace{\hspace{4em}}^{-2}} & \\ \left|\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}\right| \\ \multicolumn{3}{c}{\underbrace{\hspace{6em}}_{-3}} \end{array} = \begin{array}{c} \overbrace{\hspace{4em}}^{-2} \\ \left|\begin{matrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -12 \end{matrix}\right| \end{array}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 4 & -3 & 0 \\ 7 & -6 & 0 \end{vmatrix} = 1 \cdot (-3) \cdot 0 = 0$$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$



$$\begin{matrix} \\ -4 \\ -7 \end{matrix}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{matrix} \\ \\ -2 \end{matrix}\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot (-3) \cdot 0 = 0$$

$$\begin{vmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & 0 & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{vmatrix} = 1$$

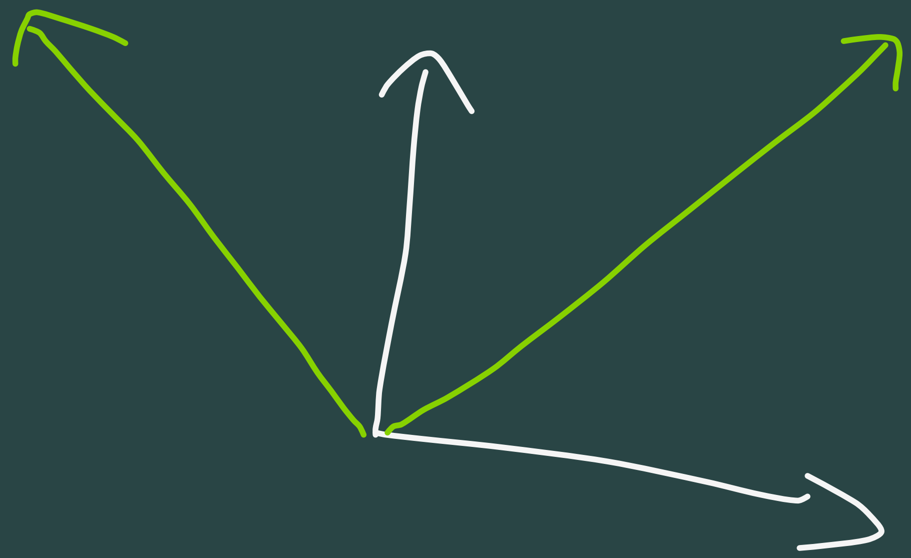
$$\begin{vmatrix} a & & & * \\ & b & & \\ 0 & & c & \\ & & & d \end{vmatrix} = \;= abcd \;= \begin{vmatrix} a & & & 0 \\ & b & & \\ * & & c & \\ & & & d \end{vmatrix}$$

$$\begin{vmatrix} 1 & * & * & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\begin{pmatrix} l_{i+1} \\ j_{i+1} \end{pmatrix} = \begin{pmatrix} 0{,}9 & 0{,}1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} l_i \\ j_i \end{pmatrix}$$

Mikä on $\begin{pmatrix} l_{100} \\ j_{100} \end{pmatrix} = \begin{pmatrix} 0.9 & 0{,}1 \\ -3 & 4 \end{pmatrix}^{100} \begin{pmatrix} l_0 \\ j_0 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$



$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1$$

$$= 1 + \lambda^2 - 2\lambda + 1 = \lambda^2 - 2\lambda + 2$$

$$= 0$$

$$\Leftrightarrow$$

$$\lambda = 1 \pm \sqrt{1-2} = 1 \pm i$$

Ominaisvektorit

$$A - iI = 0$$

$$\frac{-1}{1-i} \curvearrowright \left(\begin{array}{cc|c} 1-i & -1 & 0 \\ 1 & 1-i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1-i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_2 = t, \quad x_1 = \frac{t}{1-i} = \frac{t(1+i)}{2}$$

$\lambda = 1+i$

O. vek.

$$t\begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix}$$

Om. vek. $\lambda = 1 - i$

$$: \quad t\begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}^{1000} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^{1000} & 0 \\ 0 & 2^{1000} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} L_{100} \\ J_{100} \end{pmatrix} = \underbrace{\begin{pmatrix} 0{,}9 & 0{,}1 \\ -3 & 4 \end{pmatrix}^{100}}_{} \begin{pmatrix} L_0 \\ J_0 \end{pmatrix}$$

$$\underbrace{V}_{\text{Ominais-vektorit}} \; \underbrace{D}_{\text{om.arvot}} \; V^{-1}$$

$$A = \begin{pmatrix} 0.9 & 0.1 \\ -3 & 4 \end{pmatrix}$$

$$\chi_A(\lambda) = \begin{vmatrix} 0.9-\lambda & 0.1 \\ -3 & 4-\lambda \end{vmatrix} =$$

$$= (4-\lambda)(0.9-\lambda) + 0.3$$

$$= 3.6 + \lambda^2 - 4.9\lambda + 0.3$$

$$= \lambda^2 - 4.9\lambda + 3.9$$

λ ominaisarvo

$$\Longleftrightarrow$$

$$\chi_A(\lambda) = 0$$

$$\Longleftrightarrow$$

$$\lambda = 2.45 \pm \overbrace{\sqrt{2.45^2 - 3{,}9}}^{1.45} = \begin{cases} 1 \\ 3{,}9 \end{cases}$$

Ominaisvektorit:

$$\lambda = 1 \quad : \quad \left(\begin{array}{cc|c} 0.9-1 & 0.1 & 0 \\ -3 & 4-1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -0.1 & 0.1 & 0 \\ -3 & 3 & 0 \end{array}\right)$$

$$v = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 3.9$$

$$\left(\begin{array}{cc|c} 0.9-3.9 & 0.1 & 0 \\ -3 & 4-3.9 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -3 & 0.1 & 0 \\ -3 & 0.1 & 0 \end{array}\right)$$

$$V = t\begin{pmatrix} 1/30 \\ 1 \end{pmatrix} = s\begin{pmatrix} 1 \\ 30 \end{pmatrix}$$

$$\begin{pmatrix} 0.9 & 0.1 \\ -3 & 4 \end{pmatrix} = \frac{1}{1-30} \underbrace{\begin{pmatrix} 1 & 1 \\ 30 & 1 \end{pmatrix}}_{V} \begin{pmatrix} 3.9 & 0 \\ 0 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} 1 & -1 \\ -30 & 1 \end{pmatrix}}^{V^{-1}}$$

$$\begin{pmatrix} 0.9 & 0.1 \\ -3 & 4 \end{pmatrix}^{100} = \frac{-1}{29} \begin{pmatrix} 1 & 1 \\ 30 & 1 \end{pmatrix} \begin{pmatrix} 3.9^{100} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -30 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

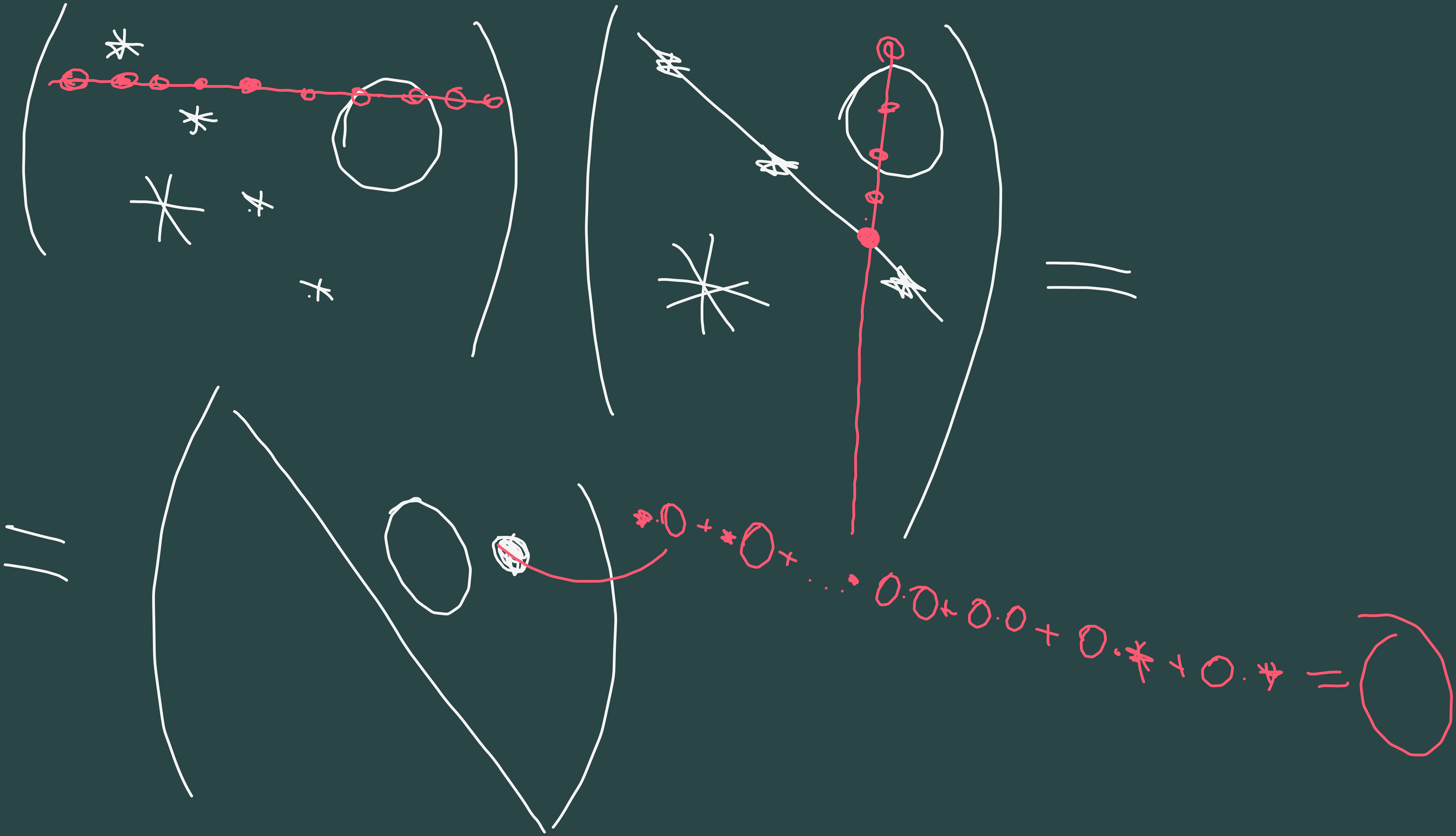
$$\left(\begin{array}{cc|c} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 0 \end{array}\right)$$

Tiedetään jo, että yhtälöt ovat riippuvat eli riittää ratkaista

$(1-\lambda)x + y = 0$

TAI

$x - \lambda y = 0$



RW

$$LU\mathbf{x} = \mathbf{b}$$

$$\Longleftrightarrow$$

$$L\mathbf{y} = \mathbf{b} \text{ jossa } \mathbf{y} = U\mathbf{x}$$

$$\begin{array}{ccc|c} x & y & z & \\ a & b & c & g \\ & d & e & h \\ \mathrm{O} & & f & i \end{array}$$

$$fx = i$$

$$x = \frac{i}{f} \qquad dy = h - \frac{ei}{f}$$

$$y = \frac{h}{d} - \frac{ei}{fd}$$

$$\begin{pmatrix} 1 & * & * & & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * & * \\ 0 & 0 & 0 & 0 & \textcircled{0} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{pmatrix}\begin{pmatrix} 4 & 5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & 2 & \\ 0 & & 3 \end{pmatrix}\begin{pmatrix} 4 & & 0 \\ & 1 & \\ 0 & & 7 \end{pmatrix}\begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 21 \end{pmatrix} \begin{pmatrix} 1 & 5/4 & 1/4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

LDU-tekijöihinjako.

$$f_B = \begin{pmatrix} f(\mathbf{b}_1)_B, & \cdots, & f(\mathbf{b}_n)_B \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{e}_1)_B & & (\mathbf{e}_n)_B \end{pmatrix} \begin{pmatrix} f(\mathbf{b}_1)_E & \cdots & f(\mathbf{e}_n)_E \end{pmatrix} \begin{pmatrix} (\mathbf{b}_1)_E & \cdots & (\mathbf{b}_n)_E \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= -\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \left(\qquad \qquad \right)$$

$$A^T A = \left(\begin{matrix} \text{On diag} \\ \sigma_1^2 \cdots \sigma_r^2, 0 \ldots 0 \\ \\ \qquad \qquad \diagdown \end{matrix} \right)$$

$\sigma_1^2 \cdots \sigma_r^2, 0 \ldots 0$

$$AA^T = \left(\, \vdots \, \right)$$

$$V^T V = \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_n & — \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$$

$$V^T = V^{-1}$$

$$\Uparrow \qquad \Uparrow$$

$$(V^T)^{-1} = V^{TT}$$

$$\Downarrow$$

V^T:n sarakkeet ortonorm.

sarakkeet $:= \frac{A v_i}{\sigma_i}$

$$A = U \Sigma V^T$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}$$

rivit = $A^T A$:n om.vektorit

(pituus 1)

$$A = \begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \approx \begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 15-\lambda & 13 \\ 13 & 15-\lambda \end{vmatrix}$$

$$A^TA = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 15 & 13 \\ 13 & 15 \end{pmatrix}$$

$$\chi_{A^TA}(\lambda) = (15-\lambda)^2 - 13^2 = 0 \quad \left\{ \begin{array}{l} \lambda = 28 \\ \text{tai} \\ \lambda = 2 \end{array} \right.$$

$$\Longleftrightarrow \quad 15-\lambda = \pm 13 \quad \Longleftrightarrow$$

$$\chi_{A^TA} = \begin{vmatrix} 15-\lambda & 13 \\ 13 & 15-\lambda \end{vmatrix} = (15-\lambda)(15-\lambda) - 13^2$$

$$\sigma_1^2 = 28$$

$$\sigma_2^2 = 2$$

$$A^TAv_1 = 28v_1 \iff$$

$$\left(\begin{array}{cc|c} -13 & 13 & 0 \\ 13 & -13 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 15-28 & 13 & 0 \\ 13 & 15-28 & 0 \end{array}\right)$$

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_1 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A^T A v = \sigma_2^2 v = 2v$$

$$\left(\begin{array}{cc|c} 15-2 & 13 & 0 \\ 13 & 15-2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$v = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} & & \\ & U & \\ & & \end{pmatrix} \begin{pmatrix} \sqrt{28} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{2}} A \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_1 = \frac{\frac{1}{\sqrt{2}} A \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{28}} = \frac{\begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}{\sqrt{28}} = \frac{1}{\sqrt{7}} \begin{pmatrix} 3/\sqrt{2} \\ 2/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 3/\sqrt{14} & 0 \\ 2/\sqrt{14} & 0 \\ 1/\sqrt{14} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{28} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 3/\sqrt{14} & 0 \\ 2/\sqrt{14} & 0 \\ 1/\sqrt{14} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{28} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 3/\sqrt{14} \\ 2/\sqrt{14} \\ 1/\sqrt{14} \\ 0 \end{pmatrix} \sqrt{28} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 512 \times 512 \\ \tilde{A} \end{pmatrix} = U \begin{pmatrix} \sigma_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \sigma_{20} & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 6 & \\ 0 & & & & & & 6 \end{pmatrix} V^T$$

$$u_1 = \frac{A v_1}{\sigma_1}$$

$$u_2 = \frac{A v_2}{\sigma_2}$$