Lecture Notes on Continuum Thermodynamics

Collection of the main results of the appendices

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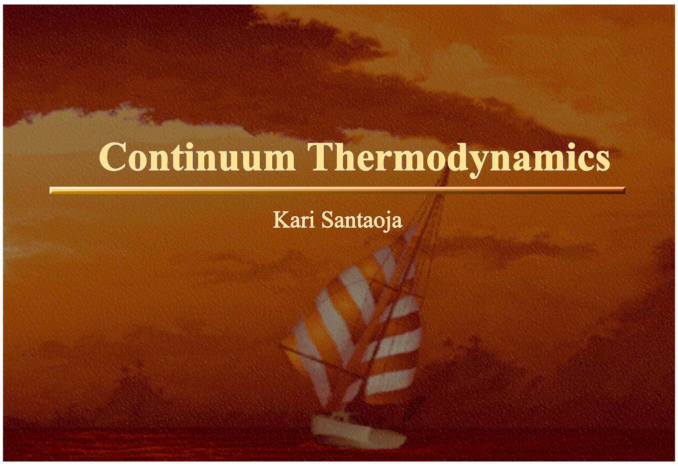
Kari Santaoja



For the reader

The present publication is a collection of the main results of the appendices of the Lecture Notes on Continuum Thermodynamics; January 2023. The idea of the collection is that a student can have it with her/him in the examination.

APPENDIX A

Double-dot product of a skew-symmetric third-order tensor and of a symmetric second-order tensor

Theorem 1: The following expressions holds:

$$\mathbf{c}:\mathbf{h} \equiv \vec{0}, \tag{1}$$

where $\mathbf{c}$ is a third-order tensor which is skew-symmetric in the last two indices and $\mathbf{h}$ is a symmetric second-order tensor. The notation $\vec{0}$ refers to a zero vector, which means that the values of all the components of the vector $\vec{0}$ vanish.

APPENDIX B

Legendre transformation

The investigation is started with a given scalar-valued function F of m second-order tensorial variables $\mathbf{u}^1$,...,$\mathbf{u}^2$, viz.

$$F = F(\mathbf{u}^1, \ldots, \mathbf{u}^m). \tag{1}$$

A new set of second-order tensors $\boldsymbol{\gamma}^1$,...,$\boldsymbol{\gamma}^m$ is introduced by means of the following transformation:

$$\boldsymbol{\gamma}^i := a \frac{\partial F(\boldsymbol{u}^1, \ldots, \mathbf{u}^m)}{\partial \mathbf{u}^i} \qquad i = 1, \ldots, m, \tag{2}$$

where the coefficient a is independent of the variables $\mathbf{u}^i$ and $\boldsymbol{\gamma}^i$ (i = 1,...,m). In the present derivation variables $\mathbf{u}^i$ and $\boldsymbol{\gamma}^i$ are assumed to be second-order tensors, but certainly can be tensors of any order.

The so-called "Hessian" - i.e. the determinant formed by the second partial derivatives of F - is assumed to be different from zero, guaranteeing the independence of the m variables $\boldsymbol{\gamma}^i$. In that case, Equations (2) are solvable for $\mathbf{u}^i$ as a function of $\boldsymbol{\gamma}^i$.

The Legendre transformation Ω of the function F is defined as

$$b\,\Omega := \sum_{i=1}^{m} \mathbf{u}^i : \boldsymbol{\gamma}^i - a F, \tag{3}$$

where b is a coefficient independent of the variables $\mathbf{u}^i$ and $\boldsymbol{\gamma}^i$ (i = 1,...,m). The variables $\mathbf{u}^i$ as expressed in terms of tensors $\boldsymbol{\gamma}^i$ [Equation (2)] are substituted into Equation (3). The function Ω can then be expressed in terms of the new variables $\boldsymbol{\gamma}^i$ alone as follows:

$$\Omega = \Omega(\boldsymbol{\gamma}^i, \ldots, \boldsymbol{\gamma}^m). \tag{4}$$

The following result is obtained:

$$\mathbf{u}^i \; = \; b \, \frac{\partial \Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m)}{\partial \boldsymbol{\gamma}^i} \qquad i \; = \; 1,\ldots,m \, . \tag{8}$$

APPENDIX C

Legendre partial transformation

In this case, the scalar-valued function F is assumed to be a function of two independent sets of tensorial variables, which are $\mathbf{u}^1,...,\mathbf{u}^m$ and $\mathbf{w}^1,...,\mathbf{w}^n$, i.e.

$$F \; = \; F(\mathbf{u}^1,\ldots,\mathbf{u}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n) \, . \tag{1}$$

The new independent set of second-order tensorial variables $\boldsymbol{\gamma}^1,...,\boldsymbol{\gamma}^m$ is assumed to be defined by

$$\boldsymbol{\gamma}^i \; := \; a \, \frac{\partial F(\mathbf{u}^1,\ldots,\mathbf{u}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n)}{\partial \mathbf{u}^i} \qquad i \; = \; 1,\ldots,m \, , \tag{2}$$

where a is a coefficient independent of $\mathbf{u}^i$, $\mathbf{w}^j$ and $\boldsymbol{\gamma}^i$ ($i = 1,...,m$ and $j = 1,...,n$). The variables $\mathbf{u}^i$ are called the active variables and the variables $\mathbf{w}^j$ are called the passive variables of the transformation. A new function Ω, called the Legendre partial transformation, is introduced. It is defined by

$$b \, \Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n) \; := \; \sum_{i=1}^{m} \boldsymbol{\gamma}^i : \mathbf{u}^i \; - \; a \, F(\mathbf{u}^1,\ldots,\mathbf{u}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n) \, . \tag{3}$$

The following results are obtained:

$$\mathbf{u}^i \; = \; b \, \frac{\partial \Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n)}{\partial \boldsymbol{\gamma}^i} \qquad i \; = \; 1,\ldots,m \tag{6}$$

and

$$a \, \frac{\partial F(\mathbf{u}^1,\ldots,\mathbf{u}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n)}{\partial \mathbf{w}^i} \; = \; -b \, \frac{\partial \Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n)}{\partial \mathbf{w}^j} \qquad j \; = \; 1,\ldots,n \, . \tag{7}$$

APPENDIX D

Divergence of the dot product of a second-order tensor and a vector

Theorem 1: The following expression holds:

$$\vec{\nabla}\cdot(\mathbf{h}\cdot\vec{e}) \;=\; (\vec{\nabla}\cdot\mathbf{h})\cdot\vec{e} \;+\; \mathbf{h}:\vec{\nabla}\vec{e}\,, \tag{1}$$

where $\vec{\nabla}$ is the vector operator del, $\mathbf{h}$ is a second-order tensor, and $\vec{e}$ is a vector.

APPENDIX E

Stress power per unit volume

Theorem 1: The following expression holds:

$$\boldsymbol{\sigma}^{\mathrm{c}}:\vec{\nabla}\vec{v} \;=\; \boldsymbol{\sigma}^{\mathrm{c}}:\mathbf{d}\,. \tag{1}$$

Theorem 2: In the case of small displacements and displacement gradients the following holds:

$$\boldsymbol{\sigma}:\vec{\nabla}\vec{v} \;=\; \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\,. \tag{6}$$

APPENDIX F

Legendre transformation of a homogeneous function

A scalar-valued function F of m different tensorial variables $\mathbf{u}^1,...,\mathbf{u}^m$ is studied. Function is F expressed as follows:

$$F \;=\; F(\mathbf{u}^1,\ldots,\mathbf{u}^m) \tag{1}$$

is assumed to be a homogeneous function of degree ω and therefore it satisfies the following definition and equation:

$$F(k\mathbf{u}^1,\ldots,k\mathbf{u}^m) \;:=\; k^{\omega}\,F(\mathbf{u}^1,\ldots,\mathbf{u}^m) \tag{2}$$

and

$$\omega\,F(\mathbf{u}^1,\ldots,\mathbf{u}^m) \;=\; \left(\frac{\partial F}{\partial \mathbf{u}^1}:\mathbf{u}^1 \;+\;\ldots\;+\; \frac{\partial F}{\partial \mathbf{u}^m}:\mathbf{u}^m\right), \tag{3}$$

where k is an arbitrary positive real number. [see e.g. Widder (1989, p. 19 and 20)].

Next, m second-order tensors $\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m$ are introduced by defining

$$\boldsymbol{\gamma}^i \ := \ a \, \frac{\partial F}{\partial \mathbf{u}^i} \qquad\qquad i = 1,\ldots,m \,, \tag{4}$$

where a is an arbitrary coefficient independent of both $\mathbf{u}^i$ and $\boldsymbol{\gamma}^i$ $(i = 1,\ldots,m)$.

The Legendre transformation Ω of the function F is defined as in Appendix B, i.e.

$$b\,\Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m) \ := \ \sum_{i=1}^{m} \mathbf{u}^i{:}\boldsymbol{\gamma}^i \ - \ a\,F(\mathbf{u}^1,\ldots,\mathbf{u}^m)\,, \tag{5}$$

where the coefficient b does not depend on the tensorial variables $\mathbf{u}^i$ and $\boldsymbol{\gamma}^i$ $(i = 1,\ldots,m)$.

The result can be written in the form

$$t^{\frac{\omega}{\omega-1}}\,\Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m) \ = \ \Omega(t\,\boldsymbol{\gamma}^1,\ldots,t\,\boldsymbol{\gamma}^m)\,. \tag{16}$$

Equation (16) therefore shows the Legendre transformation $\Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m)$ to be a homogeneous function of degree $\omega/(\omega - 1)$, where ω is the degree of the original function F. This does not hold for the case $\omega = 1$, as can be seen in Equation (16).

If the original function F were a homogeneous function of degree $\mu = 1/\kappa$, the function Ω would be a homogeneous function of degree $1/(1 - \mu)$. As above, this does not hold for the case $\mu = 1$.

APPENDIX G

Normality rule for the non-separated dissipation $\boldsymbol{\Phi}$

The non-separated form of the Clausius-Duhem Inequality is given by Equation (13.1), viz.

$$\Phi \ = \ \boldsymbol{\sigma}{:}\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}} \ + \ \boldsymbol{\beta}{:}\dot{\boldsymbol{\alpha}} \ - \ \frac{\vec{\nabla}T}{T}\cdot\vec{q} \qquad (\ \geq\ 0\ )\,. \tag{1}$$

By following the concept given by Section 13.1 the principle of maximum dissipation for non-separated dissipation Φ is written in the following mathematical form:

maximise with respect to the fluxes $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}},\dot{\boldsymbol{\alpha}},\vec{q})$

$$\Phi \ = \ \boldsymbol{\sigma}{:}\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}} \ + \ \boldsymbol{\beta}{:}\dot{\boldsymbol{\alpha}} \ - \ \frac{\vec{\nabla}T}{T}\cdot\vec{q} \tag{2}$$

subject to:

$$\tau \ = \ \rho\,\varphi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}},\dot{\boldsymbol{\alpha}},\vec{q};\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathrm{i}},\boldsymbol{\alpha},T,h) \ - \ \Phi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}},\dot{\boldsymbol{\alpha}},\vec{q},\boldsymbol{\sigma},\boldsymbol{\beta},\vec{\nabla}T/T) \ = \ 0\,,, \tag{3}$$

where $\tau = 0$ is a constraint and φ is the specific dissipation function.

As a result the following normality rule was achieved:

$$\boldsymbol{\sigma} = \mu \rho_0 \frac{\partial \varphi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}, \vec{q}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h)}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}} \tag{7}$$

and

$$\boldsymbol{\beta} = \mu \rho_0 \frac{\partial \varphi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}, \vec{q}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h)}{\partial \dot{\boldsymbol{\alpha}}} . \tag{8}$$

and finally

$$-\frac{\vec{\nabla} T}{T} = \mu \rho_0 \frac{\partial \varphi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}, \vec{q}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h)}{\partial \vec{q}} . \tag{9}$$

The specific dissipation function φ has to satisfy the following condition:

$$\varphi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}, \vec{q}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h) = \mu \left(\frac{\partial \varphi}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{i}} + \frac{\partial \varphi}{\partial \dot{\boldsymbol{\alpha}}} : \dot{\boldsymbol{\alpha}} + \frac{\partial \varphi}{\partial \vec{q}} \cdot \vec{q} \right) . \tag{10}$$

The first-order sufficient condition for the point $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}, \vec{q})$ to be a local maximum is that Equations (7), (8) and (9) hold and that the specific dissipation function φ is a homogeneous function of degree $1/\mu$. The latter property is obtained if the coefficient μ in Expression (10) is a constant. If the multiplier μ is not a constant but $\mu = \mu(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h)$, the specific dissipation function φ is not a homogeneous function and the value for μ is obtained from Equation (10).

APPENDIX H

Normality rule for the separated dissipation $\boldsymbol{\Phi}$

Not included here.

APPENDIX I

Normality rule for the separated specific dissipation function φ having same internal variables in several parts

This appendix evaluates cases when the specific dissipation function φ is separated into several parts but the same internal variable exists in more than one specific dissipation function. In order to make the evaluation easy to follow, the simplest possible case is evaluated here. It is: The specific dissipation function φ_{mech} is assumed to be separated into two parts.

The Clausius-Duhem inequality for the mechanical part was given by Expression $(13.2)_1$. It has the following appearance:

$$\Phi_{\mathrm{mech}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{i}} + \boldsymbol{\beta} : \dot{\boldsymbol{\alpha}} \qquad (\geq 0) . \tag{1}$$

The mechanical part of the specific dissipation function φ_{mech} is assumed to be separated into two parts as follows:

$$\varphi_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h) = \overset{1}{\varphi}_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h) + \overset{2}{\varphi}_{\mathrm{mech}}(\dot{\boldsymbol{\alpha}}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h) . \tag{2}$$

As Separation (1) shows, the internal variable $\dot{\boldsymbol{\alpha}}$ is on both parts of the specific dissipation function φ.

According to Expressions (13.5) and (13.6) of Section 13.1, the principle of maximum dissipation:

maximise with respect to the fluxes $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}})$

$$\Phi_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}, \boldsymbol{\beta}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{i}} + \boldsymbol{\beta} : \dot{\boldsymbol{\alpha}} \tag{3}$$

subject to:

$$\tau_{\mathrm{mech}} = \rho_0 \, \varphi_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T, h) - \Phi_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}, \boldsymbol{\beta}) = 0 , \tag{4}$$

where $\tau_{\mathrm{mech}} = 0$ is a constraint and φ_{mech} is the specific dissipation function for mechanical behaviour.

The investigation can be continued by following the same procedure as in Section 13.1. This means that the normality rule follows Normality Rule (13.20) and (13.21), viz.

$$\boldsymbol{\sigma} = \mu \, \rho_0 \, \frac{\partial \varphi_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}; state)}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}} = \mu \, \rho_0 \, \frac{\partial \overset{1}{\varphi}_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}; state)}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}} . \tag{5}$$

and

$$\boldsymbol{\beta} = \mu \, \rho_0 \, \frac{\partial \varphi_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}; state)}{\partial \dot{\boldsymbol{\alpha}}} = \mu \, \rho_0 \, \frac{\partial \overset{1}{\varphi}_{\mathrm{mech}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{i}}, \dot{\boldsymbol{\alpha}}; state)}{\partial \dot{\boldsymbol{\alpha}}} + \mu \, \rho_0 \, \frac{\partial \overset{2}{\varphi}_{\mathrm{mech}}(\dot{\boldsymbol{\alpha}}; state)}{\partial \dot{\boldsymbol{\alpha}}} . \tag{6}$$

In Normality Rule (5) and (6) the set of state variables $(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{i}}, \boldsymbol{\alpha}, T)$ and function h are expressed by the notation *state*. It is worth noting that the approach given by Expressions (3) and (4) leads to the result that the parameters μ related to the specific dissipation functions $\overset{1}{\varphi}_{\mathrm{mech}}$ and $\overset{2}{\varphi}_{\mathrm{mech}}$ are the same [as shown by Expressions (5) and (6)].

APPENDIX J

On partial derivatives of the von Mises operator J_{vM} acting on a tensorial variable

Theorem 1: The following results hold:

$$\frac{\partial J_{vM}(\boldsymbol{\sigma} - \boldsymbol{\beta}^1)}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \frac{\mathbf{s} - \mathbf{b}^1}{J_{vM}(\boldsymbol{\sigma} - \boldsymbol{\beta}^1)} \tag{14}$$

and

$$\frac{\partial J_{vM}(\boldsymbol{\sigma} - \boldsymbol{\beta}^1)}{\partial \boldsymbol{\beta}^1} = -\frac{3}{2} \frac{\mathbf{s} - \mathbf{b}^1}{J_{vM}(\boldsymbol{\sigma} - \boldsymbol{\beta}^1)} \,. \tag{15}$$

In Derivatives (14) and (15) the notations $\mathbf{s}$ and $\mathbf{b}^1$ are deviatoric tensors of $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}^1$.

APPENDIX K

Proof of equality
$(\boldsymbol{\sigma} - \boldsymbol{\beta}^1):(\mathbf{s} - \mathbf{b}^1) = (\mathbf{s} - \mathbf{b}^1):(\mathbf{s} - \mathbf{b}^1)$

Theorem 1: The following expression holds:

$$(\boldsymbol{\sigma} - \boldsymbol{\beta}^1):(\mathbf{s} - \mathbf{b}^1) = (\mathbf{s} - \mathbf{b}^1):(\mathbf{s} - \mathbf{b}^1) \,. \tag{1}$$

APPENDIX L

Miscellaneous expressions

This appendix proves miscellaneous theorems. The proofs are too short for separate appendices.

Theorem 1: The following expression holds:

$$\vec{n} \times \vec{a} = \vec{n} \cdot (\mathbf{1} \times \vec{a}) \,, \tag{1}$$

where $\mathbf{1}$ is second-order identity tensor given by Definition (2.26) and $\vec{n}$ and $\vec{a}$ are arbitrary vectors.

Theorem 2: The following expression holds:

$$\mathbf{1} : \vec{\nabla} \vec{v} = \vec{\nabla} \cdot \vec{v} \,. \tag{6}$$

Theorem 3: The derivative of a second order tensor $\mathbf{a}(\mathbf{e})$ with respect a second-order tensor $\mathbf{c}$, where $\mathbf{e} = f(\mathbf{c})$, is obtained by the following chain rule:

$$\frac{\partial \mathbf{a}(\mathbf{e})}{\partial \mathbf{c}} = \frac{\partial \mathbf{e}}{\partial \mathbf{c}} : \frac{\partial \mathbf{a}(\mathbf{e})}{\partial \mathbf{e}} \,. \tag{12}$$

Theorems 4.1 and 4.2: The following expressions hold:

$$\frac{\partial \vec{a}(\mathbf{e})}{\partial \mathbf{c}} = \frac{\partial \mathbf{e}}{\partial \mathbf{c}} : \frac{\partial \vec{a}(\mathbf{e})}{\partial \mathbf{e}} \quad \text{and} \quad \frac{\partial a(\mathbf{e})}{\partial \mathbf{c}} = \frac{\partial \mathbf{e}}{\partial \mathbf{c}} : \frac{\partial a(\mathbf{e})}{\partial \mathbf{e}} \,. \tag{17}$$

where $\vec{a}$ is a vector, a is a scalar and $\mathbf{e}$ and $\mathbf{c}$ are second-order tensors.

Theorems 5.1 and 5.2: The following expressions hold:

$$\frac{\partial \vec{a}(\vec{e})}{\partial \vec{b}} = \frac{\partial \vec{e}}{\partial \vec{b}} \cdot \frac{\partial \vec{a}(\vec{e})}{\partial \vec{e}} \quad \text{and} \quad \frac{\partial a(\vec{e})}{\partial \vec{b}} = \frac{\partial \vec{e}}{\partial \vec{b}} \cdot \frac{\partial a(\vec{e})}{\partial \vec{e}} \,. \tag{18}$$

where $\vec{a}$ is a vector, a is a scalar and $\vec{e}$ and $\vec{b}$ are vectors.

Theorem 6: The material derivative of the scalar-valued function $\underline{\theta} = \underline{\theta}(\underline{\vec{x}}(t),t)$ has the form

$$\frac{\mathrm{D}\underline{\theta}(\underline{\vec{x}}(t),t)}{\mathrm{D}t} = \frac{\partial \underline{\theta}(\underline{\vec{x}}(t),t)}{\partial t} + \frac{\mathrm{D}\underline{\vec{x}}(t)}{\mathrm{D}t} \cdot \frac{\partial \underline{\theta}(\underline{\vec{x}}(t),t)}{\partial \underline{\vec{x}}(t)} \,. \tag{19}$$

Theorem 7: The following expression holds:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = \vec{\nabla} \cdot (\vec{v}\overleftarrow{\nabla}) \,. \tag{20}$$

Theorem 8: The following expression holds:

$$\mathbf{1} : \overset{\circ}{\boldsymbol{\varepsilon}}(\vec{x},t) = \vec{\nabla}(\vec{x}) \cdot \vec{v}(\vec{x},t) \quad \Leftrightarrow \quad \mathbf{1} : \overset{\circ}{\boldsymbol{\varepsilon}} = \vec{\nabla} \cdot \vec{v} \,. \tag{28}$$

APPENDIX M

Material derivative of the Jacobian determinant

The following equality holds:

$$\frac{\dot{J}(\vec{X},t)}{J(\vec{X},t)} = \vec{\nabla}(\underline{\vec{x}}) \cdot \underline{\vec{v}}(\underline{\vec{x}}(t)) \,. \tag{12}$$

APPENDIX N

Constitutive tensor C and the compliance tensor S

Theorem 1: The following holds:

$$\mathbf{I}^s : \mathbf{I}^s = \mathbf{I}^s \tag{1}$$

Theorem 2: The following expressions for the compliance tensor $\mathbf{S}$ are coincide:

$$\mathbf{S} := -\frac{\nu}{E}\,\mathbf{1}\,\mathbf{1} + \frac{1+\nu}{E}\,\mathbf{I}^s \qquad \text{and} \qquad \mathbf{C}:\mathbf{S} = \mathbf{S}:\mathbf{C} = \mathbf{I}^s . \tag{4}$$

Theorem 3: Constitutive tensor $\mathbf{C}$ has a major symmetry. This is

$$C_{ijkl} = C_{klij} . \tag{19}$$

Theorem 4: Constitutive tensor $\mathbf{C}$ has a minor symmetry in the first pair of indices and in the second pair of indices. This is

$$C_{ijkl} = C_{jikl} \qquad \text{and} \qquad C_{ijkl} = C_{ijlk} . \tag{29}$$

Theorem 5: The following holds:

$$\mathbf{I}^s:\mathbf{C} = \mathbf{C}:\mathbf{I}^s = \mathbf{C} \qquad \text{and} \qquad \mathbf{I}^s:\mathbf{S} = \mathbf{S}:\mathbf{I}^s = \mathbf{S} . \tag{33}$$

Theorem 6: The following holds:

$$\mathbf{I}^s:\tilde{\mathbf{S}} = \tilde{\mathbf{S}} , \tag{36}$$

where $\tilde{\mathbf{S}}$ is the effective compliance tensor for deformation of a material containing non-interacting spherical microvoids within a matrix material having a linear elastic response.

APPENDIX O

Scalar components of the compliance tensor $\mathbf{S}^{dr}$

Not included here.

APPENDIX P

Change of the coordinate system

Not included here.

APPENDIX Q

Heat equation for solids in terms of the specific Gibbs free energy *g*

Not included here.

APPENDIX R

Clausius-Duhem inequality when the material model is expressed by the specific Helmholtz free energy ψ

Not included here.