ELEC-C9430 Electromagnetism (week 1)

Electromagnetic quantities

Vector algebra

Coordinate systems

Vector calculus

Electromagnetic quantities

Electric field $\bar{E}$ $[\bar{E}] = \frac{V}{m}$

$[f]$ = the unit of the quantity f

Magnetic field $\bar{H}$ $\frac{A}{m}$

Electric displacement (flux density) $\bar{D}$ $\frac{As}{m^2}$

Magnetic displacement (flux density) $\bar{B}$ $\frac{Vs}{m^2} = T$

Current density $\bar{J}$ $\frac{A}{m^2}$

Charge density ρ_v $\frac{As}{m^3}$

del ∇ (nabla)

differentiation with respect to time $\frac{\partial}{\partial t}$

MAXWELL EQUATIONS

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$\nabla \cdot \bar{D} = \rho_v$$

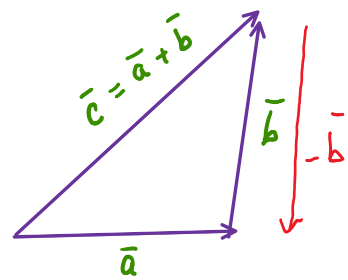
$$\nabla \cdot \bar{B} = 0$$

Vector algebra

scalar c

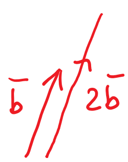
vector $\bar{b}$

$$\bar{c} = \bar{a} + \bar{b}$$
$$\bar{a} = \bar{c} - \bar{b}$$
$$= \bar{c} + (-\bar{b})$$



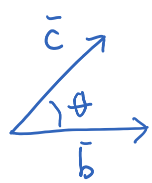
Multiplication by scalar

$$2\bar{b} = \bar{c}$$
$$(-1)\bar{b} = -\bar{b}$$



Dot product (scalar product)

$$\bar{b} \cdot \bar{c} = |\bar{b}||\bar{c}| \cos\theta$$



$$\bar{b} \cdot \bar{b} = |\bar{b}||\bar{b}| \overbrace{\cos 0}^{1} = |\bar{b}|^2 = b^2 \quad \Rightarrow \quad |\bar{b}| = \sqrt{\bar{b} \cdot \bar{b}}$$

Unit vector:

$$\frac{\bar{b}}{|\bar{b}|} = \bar{a}_b \qquad \bar{b} = |\bar{b}|\, \bar{a}_b = b\, \bar{a}_b$$

UNIT VECTOR $\bar{a}$:
- magnitude 1
- dimensionless, no unit

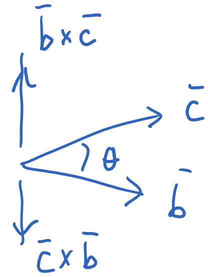
dot product commutative:

$$\bar{b} \cdot \bar{c} = \bar{c} \cdot \bar{b}$$

Cross product (vector product)

$$\bar b \times \bar c = ?$$

$$|\bar b \times \bar c| = |\bar b||\bar c|\sin\theta$$



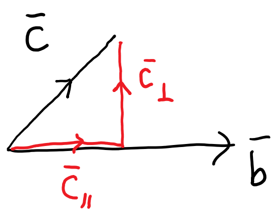
cross product anticommutative:

$$\bar b \times \bar c = -\bar c \times \bar b$$

"bac-cab" rule

$$\bar a \times (\bar b \times \bar c) = \bar b(\bar a \cdot \bar c) - \bar c(\bar a \cdot \bar b)$$

Decomposition of a vector:



$$\bar c = \bar c_\parallel + \bar c_\perp$$

$$\bar c = \alpha \bar b + \bar b \times \bar d$$

$$\bar b \cdot \bar c = \alpha\, \bar b \cdot \bar b + \underbrace{\bar b \cdot (\bar b \times \bar d)}_{=0} \quad \Rightarrow \quad \alpha = \frac{\bar b \cdot \bar c}{\bar b \cdot \bar b}$$

$$\bar b \times \bar c = \alpha \underbrace{\bar b \times \bar b}_{=0} + \bar b \times (\bar b \times \bar d) = \bar b \underbrace{(\bar b \cdot \bar d)}_{=0} - \bar d(\bar b \cdot \bar b)$$

$$\bar d = \frac{-\bar b \times \bar c}{\bar b \cdot \bar b} = \frac{\bar c \times \bar b}{\bar b \cdot \bar b}$$

$$\bar c = \frac{(\bar b \cdot \bar c)\bar b + \bar b \times (\bar c \times \bar b)}{\bar b \cdot \bar b}$$

Vector: bar above the symbol: $\bar b$, $\bar E$

In printed text usually: boldface vectors:

We know that a vector ha
written as

$$\mathbf{A} = \mathbf{a}_A A,$$

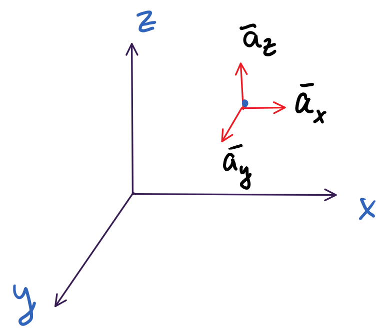
where A is the magnitude

$$A = |\mathbf{A}|,$$

Cheng, p. 14

Coordinate systems

Cartesian coordinate system



$$dV = dx\,dy\,dz$$

$$dS = dx\,dy$$
$$dy\,dz$$
$$dz\,dx$$

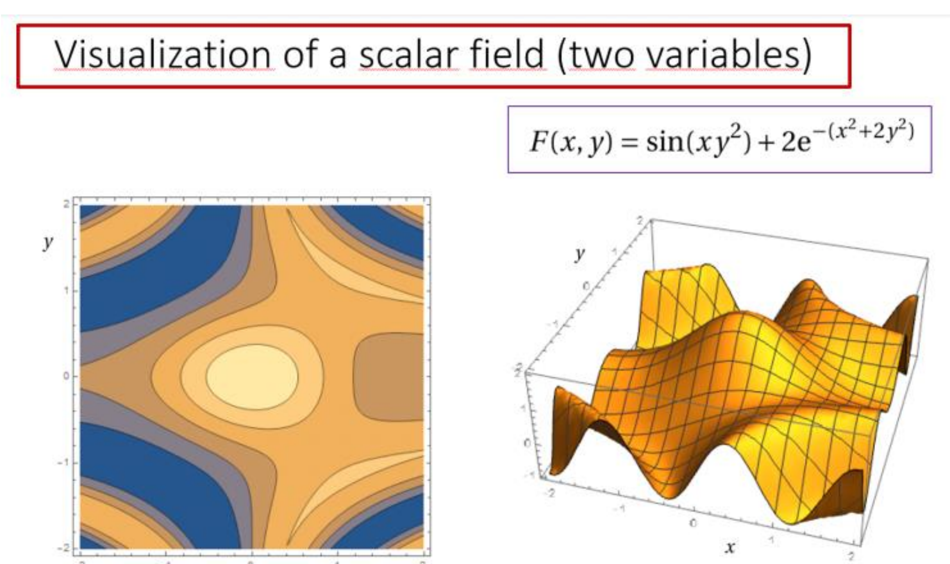
$$d\bar{l} = \bar{a}_x\,dx + \bar{a}_y\,dy + \bar{a}_z\,dz$$

$$\bar{f}\cdot\bar{g} = (\bar{a}_x f_x + \bar{a}_y f_y + \bar{a}_z f_z)\cdot(\bar{a}_x g_x + \bar{a}_y g_y + \bar{a}_z g_z) = f_x g_x + f_y g_y + f_z g_z$$

$$\bar{f}\times\bar{g} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \bar{a}_x(f_y g_z - f_z g_y) + \bar{a}_y(f_z g_x - f_x g_z) + \bar{a}_z(f_x g_y - f_y g_x)$$

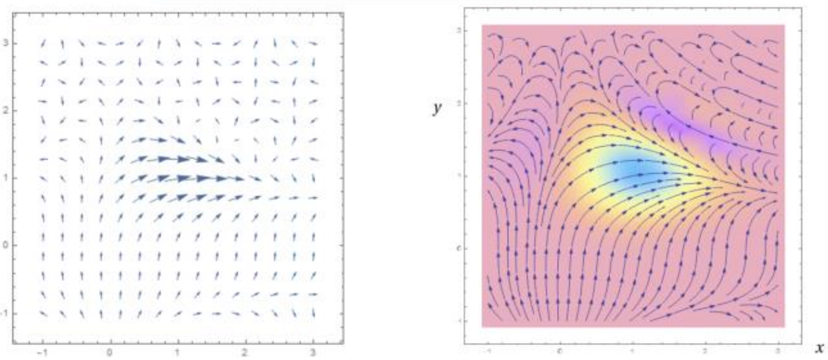
$$\bar{h} = \bar{a}_x\cdot\bar{h} = h_x \quad \left(= \bar{a}_x\cdot(\bar{a}_x h_x + \bar{a}_y h_y + a_z h_z)\right)$$

visualization in two dimensions?

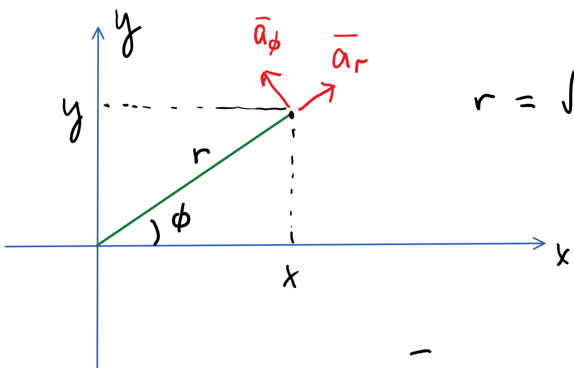


Visualization of a vector field (two variables)

$$\mathbf{G}(x,y) = \mathbf{a}_x\left[\sin(xy^2) + 2e^{-(x-1)^2-2(y-1)^2}\right] + \mathbf{a}_y\cos(xy^2)$$



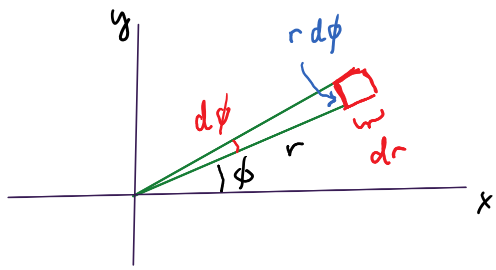
Cylindrical coordinate system (r, ϕ, z) $\bar{a}_r,\ \bar{a}_\phi,\ \bar{a}_z$



$r = \sqrt{x^2 + y^2}$ $\tan\phi = \dfrac{y}{x}$

$$\bar{a}_r = \cos\phi\ \bar{a}_x + \sin\phi\ \bar{a}_y$$

$$\bar{a}_\phi = -\sin\phi\ \bar{a}_x + \cos\phi\ \bar{a}_y$$

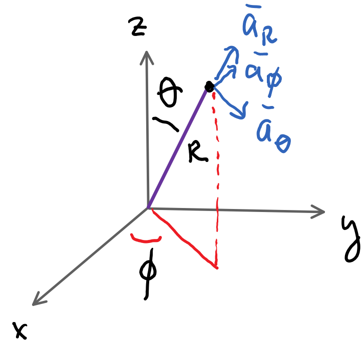


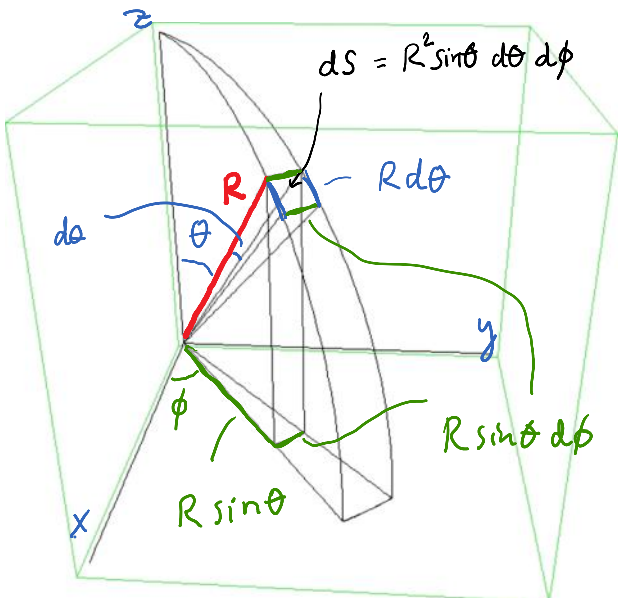
$$dS = r\,d\phi\,dr$$

$$dV = dz\,dS = r\,dr\,d\phi\,dz$$

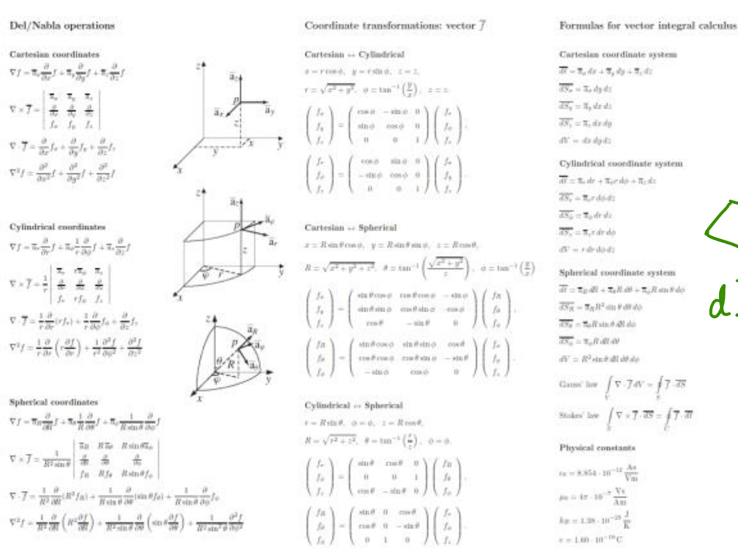
Spherical coordinate system (R, θ, ϕ)

(R, θ, ϕ)





$$dV = R^2 \sin\theta \, dR \, d\theta \, d\phi$$



Del/Nabla operations

Cartesian coordinates

$\nabla f = \bar{a}_x \frac{\partial}{\partial x} f + \bar{a}_y \frac{\partial}{\partial y} f + \bar{a}_z \frac{\partial}{\partial z} f$

$\nabla \times \bar{f} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$

$\nabla \cdot \bar{f} = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$

$\nabla^2 f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$

Cylindrical coordinates

$\nabla f = \bar{a}_r \frac{\partial}{\partial r} f + \bar{a}_\phi \frac{1}{r} \frac{\partial}{\partial \phi} f + \bar{a}_z \frac{\partial}{\partial z} f$

$\nabla \times \bar{f} = \frac{1}{r} \begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ f_r & r f_\phi & f_z \end{vmatrix}$

$\nabla \cdot \bar{f} = \frac{1}{r} \frac{\partial}{\partial r} (r f_r) + \frac{1}{r} \frac{\partial}{\partial \phi} f_\phi + \frac{\partial}{\partial z} f_z$

$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical coordinates

$\nabla f = \bar{a}_R \frac{\partial}{\partial R} f + \bar{a}_\theta \frac{1}{R} \frac{\partial}{\partial \theta} f + \bar{a}_\phi \frac{1}{R \sin\theta} \frac{\partial}{\partial \phi} f$

$\nabla \times \bar{f} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \bar{a}_R & R\bar{a}_\theta & R\sin\theta \bar{a}_\phi \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_R & R f_\theta & R \sin\theta f_\phi \end{vmatrix}$

$\nabla \cdot \bar{f} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 f_R) + \frac{1}{R \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta f_\theta) + \frac{1}{R \sin\theta} \frac{\partial}{\partial \phi} f_\phi$

$\nabla^2 f = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial f}{\partial R} \right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}$

Coordinate transformations: vector $\bar{f}$

Cartesian ↔ Cylindrical

$x = r\cos\phi, \; y = r\sin\phi, \; z = z,$

$r = \sqrt{x^2 + y^2}, \; \phi = \tan^{-1}\left(\frac{y}{x}\right), \; z = z$

$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_r \\ f_\phi \\ f_z \end{pmatrix},$

$\begin{pmatrix} f_r \\ f_\phi \\ f_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}.$

Cartesian ↔ Spherical

$x = R\sin\theta\cos\phi, \; y = R\sin\theta\sin\phi, \; z = R\cos\theta,$

$R = \sqrt{x^2 + y^2 + z^2}, \; \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \; \phi = \tan^{-1}\left(\frac{y}{x}\right)$

$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} f_R \\ f_\theta \\ f_\phi \end{pmatrix},$

$\begin{pmatrix} f_R \\ f_\theta \\ f_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}.$

Cylindrical ↔ Spherical

$r = R\sin\theta, \; \phi = \phi, \; z = R\cos\theta,$

$R = \sqrt{r^2 + z^2}, \; \theta = \tan^{-1}\left(\frac{r}{z}\right), \; \phi = \phi.$

$\begin{pmatrix} f_r \\ f_\phi \\ f_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} f_R \\ f_\theta \\ f_\phi \end{pmatrix},$

$\begin{pmatrix} f_R \\ f_\theta \\ f_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_r \\ f_\phi \\ f_z \end{pmatrix}.$

Formulas for vector integral calculus

Cartesian coordinate system

$\overline{dl} = \bar{a}_x dx + \bar{a}_y dy + \bar{a}_z dz$

$\overline{dS_x} = \bar{a}_x dy\, dz$

$\overline{dS_y} = \bar{a}_y dx\, dz$

$\overline{dS_z} = \bar{a}_z dx\, dy$

$dV = dx\, dy\, dz$

Cylindrical coordinate system

$\overline{dl} = \bar{a}_r dr + \bar{a}_\phi r\, d\phi + \bar{a}_z dz$

$\overline{dS_r} = \bar{a}_r r\, d\phi\, dz$

$\overline{dS_\phi} = \bar{a}_\phi dr\, dz$

$\overline{dS_z} = \bar{a}_z r\, dr\, d\phi$

$dV = r\, dr\, d\phi\, dz$

Spherical coordinate system

$\overline{dl} = \bar{a}_R dR + \bar{a}_\theta R\, d\theta + \bar{a}_\phi R\sin\theta\, d\phi$

$\overline{dS_R} = \bar{a}_R R^2 \sin\theta\, d\theta\, d\phi$

$\overline{dS_\theta} = \bar{a}_\theta R\sin\theta\, dR\, d\phi$

$\overline{dS_\phi} = \bar{a}_\phi R\, dR\, d\theta$

$dV = R^2 \sin\theta\, dR\, d\theta\, d\phi$

Gauss' law $\int_V \nabla \cdot \bar{f}\, dV = \oint_S \bar{f} \cdot \overline{dS}$

Stokes' law $\int_S \nabla \times \bar{f} \cdot \overline{dS} = \oint_C \bar{f} \cdot \overline{dl}$

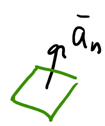
Physical constants

$\epsilon_0 = 8.854 \cdot 10^{-12} \frac{\text{As}}{\text{Vm}}$

$\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{Vs}}{\text{Am}}$

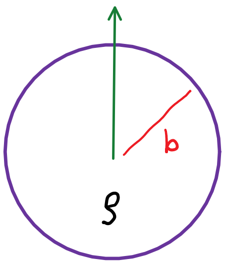
$k_B = 1.38 \cdot 10^{-23} \frac{\text{J}}{\text{K}}$

$e = 1.60 \cdot 10^{-19}\text{C}$



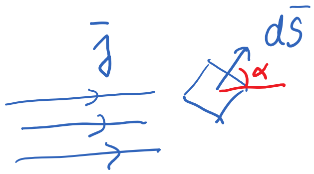
$$d\bar{S} = \bar{a}_n \, dS$$

Example: mass of the Earth

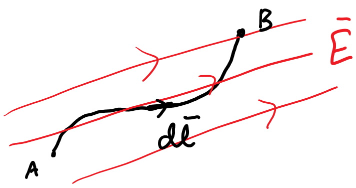


$$M = \underset{\substack{\uparrow \\ \text{CONSTANT}}}{\int \rho \, dV} = \rho \underbrace{\int_0^b R^2 \, dR}_{\left. \frac{R^3}{3} \right|_0^b} \underbrace{\int_0^\pi \sin\theta \, d\theta}_{\left. -\cos\theta \right|_0^\pi = 2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{4\pi}{3} b^3 \cdot \rho$$



$$\int \bar{J} \cdot d\bar{S} = |\bar{J}| \, dS \cos\alpha$$



$$\int_C \bar{E} \cdot d\bar{l}$$

Vector calculus

$$\nabla = \bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}$$

Gradient

$$\nabla f = \bar{a}_x \frac{\partial f}{\partial x} + \bar{a}_y \frac{\partial f}{\partial y} + \bar{a}_z \frac{\partial f}{\partial z}$$

Independence of coordinate system:
Example:

$$f(\bar{R}) = R$$

$$\nabla f = \bar{a}_R \frac{\partial f}{\partial R} = \bar{a}_R \frac{\partial R}{\partial R} = \bar{a}_R$$

$$\nabla f = \bar{a}_x \frac{\partial}{\partial x} \sqrt{x^2+y^2+z^2} + \bar{a}_y \cdots$$

$$= \bar{a}_x \frac{\frac{1}{2} 2x}{\sqrt{}} + \cdots$$

Spherical coordinates

$$\nabla f = \overline{a}_R \frac{\partial}{\partial R} f + \overline{a}_\theta \frac{1}{R} \frac{\partial}{\partial \theta} f + \overline{a}_\phi \frac{1}{R\sin\theta} \frac{\partial}{\partial \phi} f$$

$$\nabla \times \overline{f} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \overline{a}_R & R\,\overline{a}_\theta & R\sin\theta\,\overline{a}_\phi \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_R & R f_\theta & R\sin\theta\, f_\phi \end{vmatrix}$$

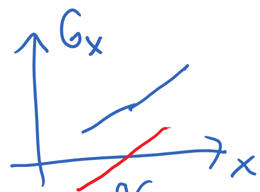
$$\nabla \cdot \overline{f} = \frac{1}{R^2} \frac{\partial}{\partial R}(R^2 f_R) + \frac{1}{R\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, f_\theta) + \frac{1}{R\sin\theta} \frac{\partial}{\partial \phi} f_\phi$$

$$\nabla^2 f = \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial f}{\partial R}\right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}$$

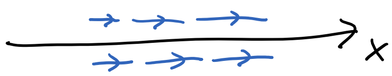
$$= \bar{a}_x \frac{x}{R} + \bar{a}_y \frac{y}{R} + \bar{a}_z \frac{z}{R} = \frac{x\bar{a}_x + y\bar{a}_y + z\bar{a}_z}{R} = \frac{\bar{R}}{R} = \bar{a}_R$$

Divergence

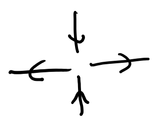
$$\nabla \cdot \bar{G} = \frac{\partial}{\partial x} G_x + \frac{\partial}{\partial y} G_y + \frac{\partial}{\partial z} G_z$$



$$\bar{G} = \bar{a}_x G_x(x)$$

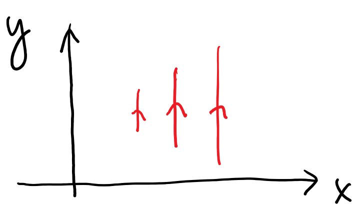


$$\frac{\partial G_x}{\partial x} > 0$$

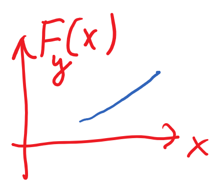


Curl

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \bar{a}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \bar{a}_y \cdots$$



$$\bar{F}(x) = \bar{a}_y F_y(x)$$



$$\frac{\partial F_y}{\partial x} > 0$$

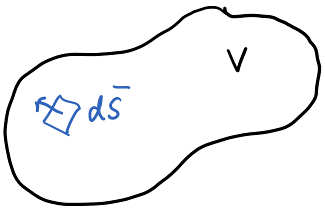
$$\nabla \times \bar{F} = \bar{a}_z \overbrace{\frac{\partial F_y}{\partial x}}^{>0}$$

INTEGRAL THEOREMS

$$\int_a^b h'(x)\, dx = h(b) - h(a)$$

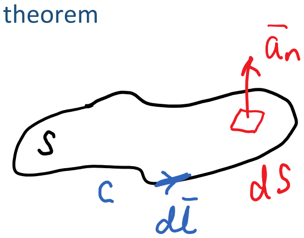
a

Divergence theorem



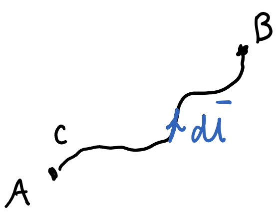
$$\int_V \nabla \cdot \bar{D}\, dV = \oint_S \bar{D} \cdot \underset{\bar{a}_n\, dS}{d\bar{S}}$$

Stokes's theorem



$$\int_S \nabla \times \bar{H} \cdot d\bar{S} = \oint_C \bar{H} \cdot d\bar{l}$$

Gradient theorem



$$\int_C \nabla V \cdot d\bar{l} = V(b) - V(a)$$

Null identities

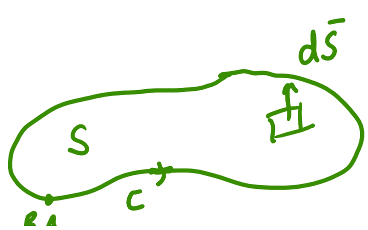
$$\int_V \nabla \cdot (\nabla \times \bar{E})\, dV$$

$$= \oint_S \nabla \times \bar{E} \cdot d\bar{S}$$

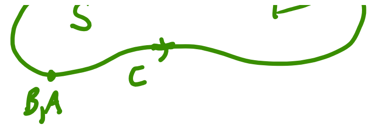
$$= \oint_C \bar{E} \cdot d\bar{l} = 0$$

$$\Rightarrow \nabla \cdot (\nabla \times \bar{E}) = 0$$

$$\int \nabla \times (\nabla g) \cdot d\bar{S}$$



$$\int_S \nabla\times(\nabla g)\cdot d\bar{S}$$



$$= \oint_C \nabla g \cdot d\bar{l}$$

$$= g(B) - g(\overset{\swarrow B}{A}) = 0$$

$$\Rightarrow \nabla\times(\nabla g) = 0$$