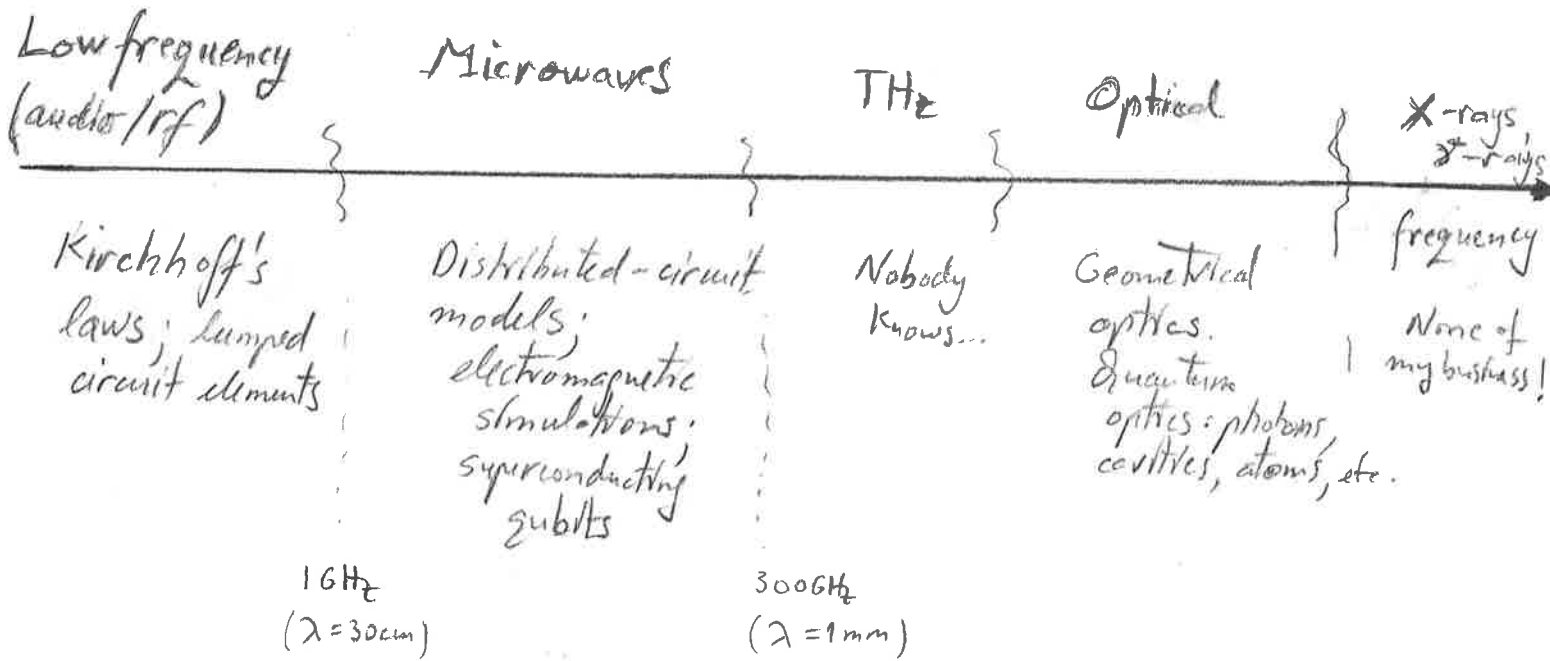
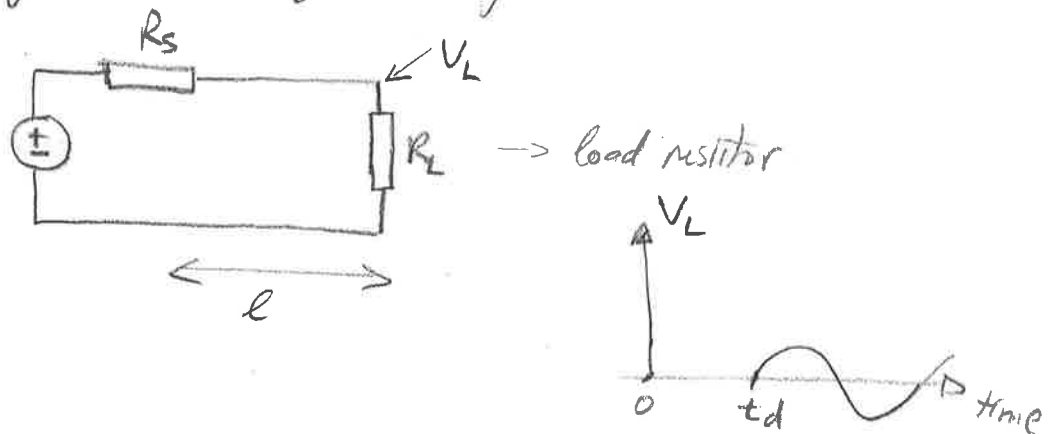


# Circuit elements

- The electromagnetic spectrum as seen by a quantum engineer:



- Why lumped-circuit models don't work at high frequencies?  
The speed of light  $c$  is large but finite.



$$t_d = \frac{l}{c} = \text{delay time}$$

becomes non-negligible if  
 $\lambda \sim l \sim \text{cm}$   
 frequency  $\sim 6\text{GHz}$

# Some basic concepts - electrical circuits

• electric current

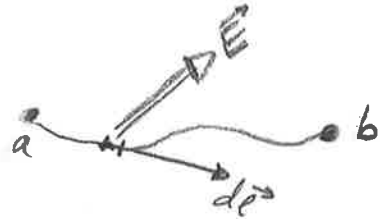
$$I = \frac{dQ}{dt} \quad Q = \int_0^t I dt$$



by convention, the direction of the current is the direction of motion of positive charges.

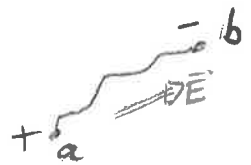
• work done by an electric field

$$W_{ba} = q \int_a^b \vec{E} \cdot d\vec{l}$$



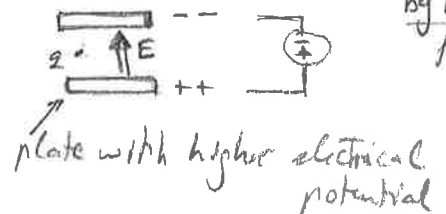
• voltage

$$V = V_{ba} = V_b - V_a = - \frac{dW_{ba}}{dq} = - \int_a^b \vec{E} \cdot d\vec{l}$$



$W_{ba}$  = work done to transport the charge  $q$  against the field,  $W = -W_{ba}$  = work done by the field

E.g. capacitor:



• magnetic flux

$$\text{Node flux: } \phi(t) = \int_{-\infty}^t d\sigma V(\sigma)$$

$$\text{so } V(t) = \frac{d\phi(t)}{dt}$$

• power

$$W(t) = \int_0^t P(\sigma) d\sigma$$

$$P(t) = \frac{dW(t)}{dt} = \frac{dW(t)}{dq(t)} \cdot \frac{dq(t)}{dt} = V(t) \cdot I(t)$$

$$\text{so } W(t) = \int_0^t V(\sigma) \cdot I(\sigma) d\sigma$$

• phasors

Useful concept if: - the circuit is linear  
- all independent sources are sinusoidal  
- only steady-state response is desired

$$x(t) = A \cos(\omega t + \phi) = \text{Re}(A e^{i\phi} e^{i\omega t})$$

$X \equiv A e^{i\phi} \equiv$  phasor = transformation of a sine waveform from time-domain into frequency domain.

Why it is useful? Simple rules:

variable	phasor
$x(t)$	$Ae^{i\phi}$
$dx(t)/dt$	$i\omega \cdot Ae^{i\phi}$
$\int dt x(t)$	$\frac{Ae^{i\phi}}{i\omega}$

check them based on the definition

### • impedance and admittance

$$Z = \frac{V}{I} \quad \text{with } V \text{ and } I \text{ phasors}$$

$$Z = R + iX$$

$Z$  = impedance  
 $R$  = resistance  
 $X$  = reactance

units:  $\Omega$   
 (Ohm)

$$Y = \frac{1}{Z} = G + iB$$

$Y$  = admittance  
 $G$  = conductance  
 $B$  = susceptance

units: S  
 (Siemens)

### • ac power and decibels

Suppose  $V(t) = V_0 \cos(\omega t + \phi_V) = \text{Re} [V_0 e^{i\phi_V} e^{i\omega t}]$

• phasor  $V_0 e^{i\phi_V} = V$

$I(t) = I_0 \cos(\omega t + \phi_I) = \text{Re} [I_0 e^{i\phi_I} e^{i\omega t}]$

• phasor  $I_0 e^{i\phi_I} = I$

$$\Rightarrow \overset{\substack{\uparrow \\ \text{instantaneous power}}}{P(t)} = V(t) \cdot I(t) = \frac{1}{2} I_0 V_0 \cos(\phi_V - \phi_I) + \frac{1}{2} I_0 V_0 \cos(2\omega t + \phi_V + \phi_I)$$

Average power:

$$\bar{P} = \frac{1}{T} \int_0^T P(t) dt = \frac{1}{2} I_0 V_0 \cos(\phi_V - \phi_I)$$

This term oscillates and cancels out when averaging

$$\bar{P} = \frac{1}{2} \text{Re}[V \cdot I^*]$$

average power delivered to a load can be changed by changing the phases!

# Root-mean square of a periodic signal

$$I(t) = I \cos \omega t \rightarrow I_{\text{rms}}^2 \equiv \frac{1}{T} \int_0^T I^2(t) dt = \frac{I_0^2}{2}$$

use  $\cos^2 \omega t = \frac{1 + \cos 2\omega t}{2}$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}}$$

decibel:

$$N(\text{dB}) = 10 \log_{10} \frac{P}{P_{\text{reference}}}$$

↓  
power in  
decibels

$P$  = power  
 $P_{\text{reference}}$  = a reference power,  
usually 1mW

If  $P_{\text{reference}} = 1\text{mW}$ , then  $N(\text{dBm}) = 10 \log_{10} \frac{P}{1\text{mW}}$

↓  
units are "dBm".

Since  $P \sim V^2$ , we

have  $20 \log_{10} \frac{V}{V_{\text{reference}}}$  (in dBV) as another way to express this.

## Examples:

30dB is an increase in power by 1000

20dB ——— " ——— 100

10dB ——— " ——— 10

3dB ——— " ——— 2

0dB ——— " ——— 1

-3dB is a decrease in power by 2

-10dB ——— " ——— by 10

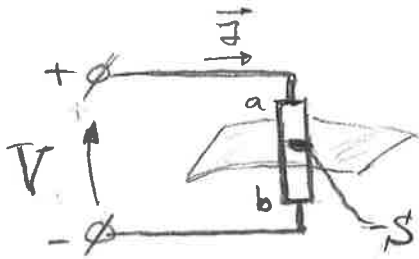
... etc.

# Circuit elements

Note: an rf-circuit can be constructed from discrete (lumped) elements if the size of each component  $\ll$  wavelength of the rf field

## RESISTOR

Typically a film of conductive material evaporated on a chip.



remember that  $\vec{E} = -\nabla V$

$$V \equiv V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{l} = \int_a^b \frac{\vec{J}}{\sigma} \cdot d\vec{l}$$

$$\vec{J} = \sigma \cdot \vec{E} \quad \text{Ohm's law}$$

$\sigma = \text{conductivity}$

If the frequency is not too high,

then  $\vec{J}$  = uniform over the cross-section  $S$  of the resistor

$$\vec{J} = I/S$$

$$\text{So } V = I \int_a^b \frac{1}{\sigma S} dl = IR$$

$$R = \frac{1}{\sigma S} \int_a^b dl$$

$$\Rightarrow R = \rho \frac{l}{S}$$

and  $\boxed{V = IR}$

$$\rho = \frac{1}{\sigma} = \text{resistivity}$$

$$\int_a^b dl = l = \text{length of the device}$$

$$G = \frac{1}{R} = \text{conductance}$$

$$R = \text{resistance}$$

and  $\boxed{Z(\omega) = R}$  - real and frequency independent

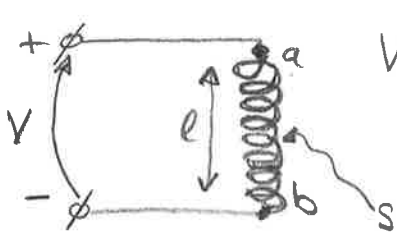
• Instantaneous dissipated power

$$P(t) = V(t) \cdot I(t) = R \cdot I^2(t) = G \cdot V^2(t)$$

• Average dissipated power (for harmonic excitations)  $V(t) = V_0 \cos \omega t$

$$\bar{P} = \frac{1}{2} R |I_0|^2 = \frac{1}{2} G |V_0|^2$$

# INDUCTOR



$$V = V_{ab} = V_a - V_b = - \int_a^b \vec{E} \cdot d\vec{l} = \iint \frac{d\vec{B}(t)}{dt} \cdot d\vec{S} = L \frac{dI}{dt}$$

Maxwell-Faraday equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

+ Stokes Theorem

Here we assumed  $\vec{B}$  uniform over the surface of area  $S$ .

from Ampere's circuital law applied to a solenoid.  $\leftarrow B = \frac{\mu_0 \mu_r N I}{l} \rightarrow$  and there are  $N$  surfaces of area  $S$

$N$  = no. of turns of the solenoid  
 $\mu_0$  = free-space magnetic permeability  
 $\mu_r$  = relative permeability

$$L = \frac{\mu_0 \mu_r N^2 S}{l}$$

• Instantaneous energy stored

$$W_L(t) = \frac{1}{2} L I^2(t)$$

• Average energy stored in ac-harmonic fields

$$\overline{W}_L = \frac{1}{4} L I_0^2$$

So 
$$V = L \frac{dI}{dt}$$

therefore 
$$Z_L(\omega) = iL\omega$$

because 
$$Z_L(\omega) = \frac{V(\omega)}{I(\omega)}$$

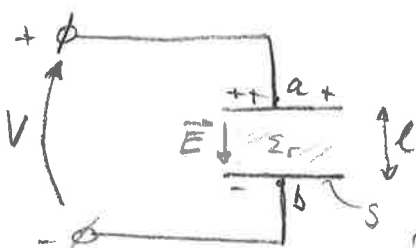
Note: The flux variable  $\phi \equiv \iint \vec{B} \cdot d\vec{S}$  can be used to define a flux at a node with potential  $V$ ,

$$\int_{-\infty}^t V(\tau) d\tau = \phi(t) \quad \text{since} \quad V(t) = \frac{d\phi(t)}{dt}$$

We also have

$$I(t) = \frac{\phi(t)}{L}$$

# CAPACITOR



$$V = - \int_b^a \vec{E} \cdot d\vec{l} = \frac{Q}{C}$$

$C$  = capacitance

$$C = \frac{\epsilon_0 \epsilon_r S}{l}$$

Proof: 
$$E = \frac{Q}{S \epsilon_0 \epsilon_r} = \frac{V}{l} \Rightarrow V = \frac{Q}{\frac{\epsilon_0 \epsilon_r S}{l}}$$

$\epsilon_0$  = free-space electric permittivity

$\epsilon_r$  = relative permittivity

from Gauss's law

Also  $I(t) = \frac{dq(t)}{dt} \Rightarrow I(t) = C \cdot \frac{dV(t)}{dt}$

• Instantaneous energy stored

$$W_c(t) = \frac{1}{2} CV^2(t)$$

• Average energy stored for ac-harmonic fields

$$\overline{W_c} = \frac{1}{4} CV_0^2$$

So  $I = C \frac{dV}{dt}$

therefore

$$Z_c(\omega) = \frac{1}{i\omega C}$$

again from

$Z_c(\omega) = \frac{V(\omega)}{I(\omega)}$  and using the properties of phasors.

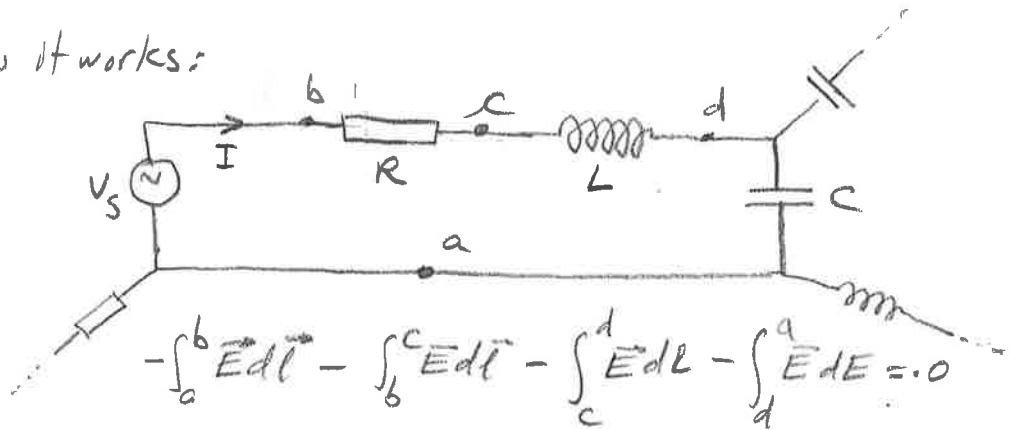
More complex networks of inductors, capacitors, resistors

= Kirchhoff's voltage and current laws

• Kirchhoff's voltage law: FOR ANY CLOSED LOOP OF A CIRCUIT, THE ALGEBRAIC SUM OF VOLTAGES OF THE INDIVIDUAL BRANCHES IS ZERO

$$\sum V_k = 0$$

How it works:

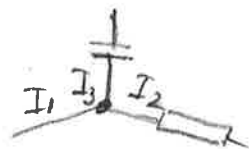


or  $V_s(t) - RI(t) - L \frac{dI(t)}{dt} - \frac{1}{C} \int_{-\infty}^t d\sigma I(\sigma) = 0$

• Kirchhoff's current law: THE ALGEBRAIC SUM OF ALL BRANCH CURRENTS CONFLUENT IN THE SAME NODE IS ZERO

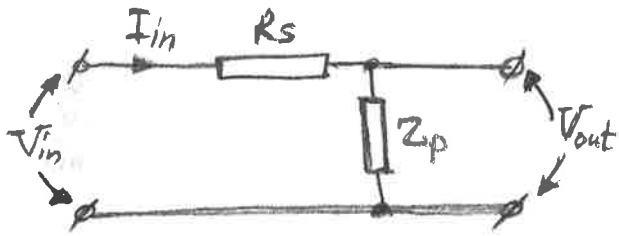
$$\sum I_k = 0$$

How it works:



$I_1 + I_2 + I_3 = 0$  ← no charge accumulation in the node!

# Examples: a series-shunt circuit



Simple equations for phasors:

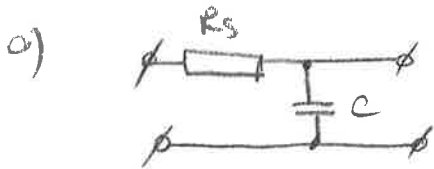
$$V_{out} = Z_p I_{in}$$

$$V_{in} = (R_s + Z_p) I_{in}$$

$$\Rightarrow \frac{V_{out}}{V_{in}} = \frac{Z_p}{R_s + Z_p} = \text{gain or attenuation}$$

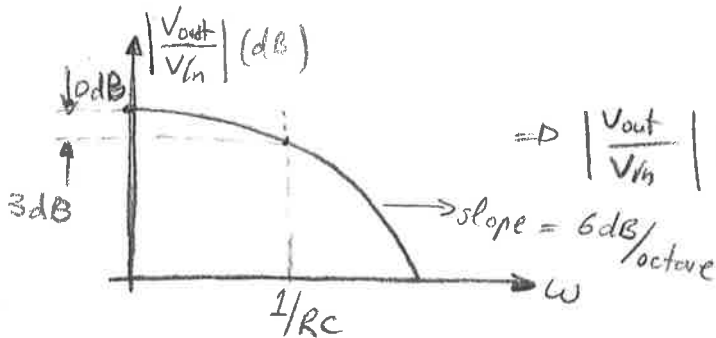
It is convenient to express this in dB:  $\left| \frac{V_{out}}{V_{in}} \right| (dB) = 20 \log_{10} \left| \frac{V_{out}}{V_{in}} \right|$

A few interesting cases:



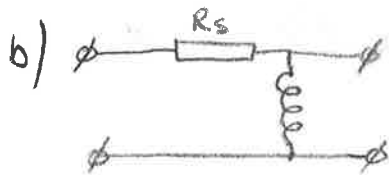
$$Z_p = \frac{1}{i\omega C} \Rightarrow \frac{V_{out}}{V_{in}} = \frac{1}{1 + i\omega C R_s}$$

$$\left| \frac{V_{out}}{V_{in}} \right| = \frac{1}{\sqrt{1 + \omega^2 C^2 R_s^2}}$$



$$\Rightarrow \left| \frac{V_{out}}{V_{in}} \right| (dB) = -10 \log_{10} [1 + \omega^2 C^2 R_s^2]$$

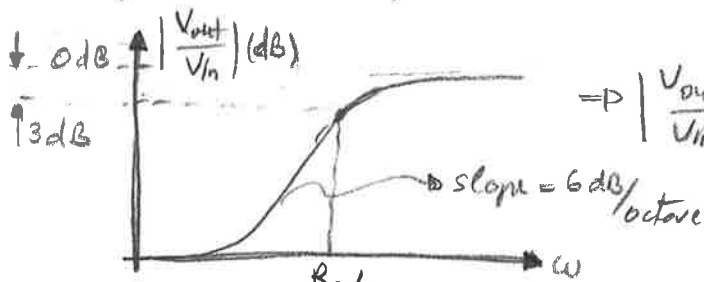
- works like a low-pass filter with cutoff  $\sim 1/RC$



$$Z_p = i\omega L$$

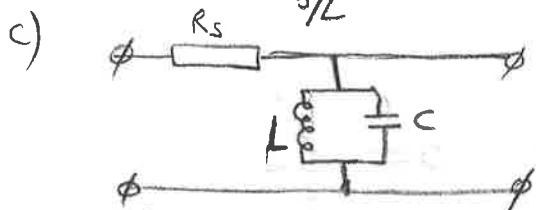
$$\frac{V_{out}}{V_{in}} = \frac{1}{1 - \frac{iR_s}{\omega L}}$$

$$\left| \frac{V_{out}}{V_{in}} \right| = \frac{1}{\sqrt{1 + \left(\frac{R_s}{\omega L}\right)^2}}$$



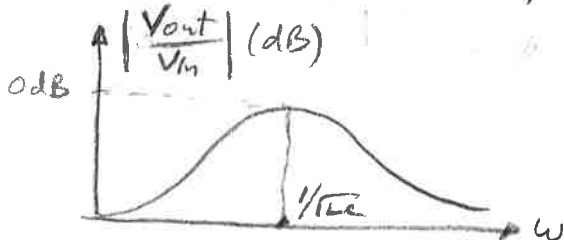
$$\Rightarrow \left| \frac{V_{out}}{V_{in}} \right| (dB) = -10 \log_{10} [1 + (R_s/\omega L)^2]$$

- works as a high-pass filter  $\omega \geq R_s/L$



$$Z_p = \frac{i\omega L \cdot \frac{1}{i\omega C}}{i\omega L + \frac{1}{i\omega C}} = \frac{i\omega L}{1 - \omega^2 LC}$$

$$\frac{V_{out}}{V_{in}} = \frac{i\omega L}{R_s(1 - \omega^2 LC) + i\omega L}$$

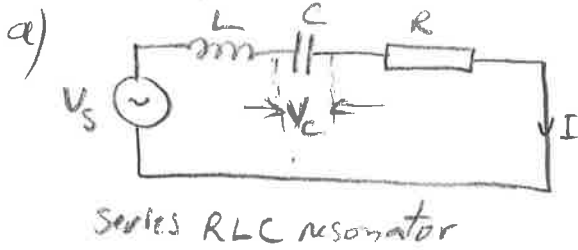


$$\left| \frac{V_{out}}{V_{in}} \right| (dB) = 20 \log_{10} \frac{\omega L}{\sqrt{R_s^2(1 - \omega^2 LC)^2 + (\omega L)^2}}$$



# Resonators based on lumped circuit elements

R L C components can be used to realize resonators.



$$V_s(\omega) = Z(\omega) \cdot I(\omega)$$

$$Z(\omega) = R + iL\omega + \frac{1}{iC\omega}$$

$$= R + i \frac{L}{\omega} (\omega^2 - \omega_0^2) \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Resonance:  $\omega = \omega_0$

$Z(\omega_0) = R$   $\rightarrow$  The impedance is real (resistive)

the reactive part is zero, meaning that the inductor and capacitor reactances cancel each other.

Indeed

$$\begin{cases} \overline{W}_L = \frac{1}{2} L |I|^2 \\ \overline{W}_C = \frac{1}{2} C |V_c|^2 = \frac{1}{2} \frac{|I|^2}{C\omega^2} \end{cases}$$

$$\text{but } |V_c| = \frac{|I|}{C\omega}$$

so at resonance

$$\omega = \omega_0 \Rightarrow \underline{\overline{W}_L = \overline{W}_C}$$

due to this, the energy oscillates between the capacitor and inductor and the source has to provide only what is lost through R.

## • Quality factor

- suppose we put some energy  $W(0)$  in the resonator. Due to the resistance R, this will be dissipated.

$$W(t) = W(0) e^{-\omega_0 t / Q}$$

$Q = \text{quality factor}$  - it measures how well the resonator stores energy.

Now

$$-\frac{dW}{dt} = \frac{\omega_0 W}{Q}$$

$\overline{P}$  = average loss in a period

$$\frac{2\pi}{\omega_0} = T = \text{period}$$

$$\overline{P} = \frac{1}{T} \int_0^T -\frac{dW}{dt} dt$$

$$\text{or } \overline{P} = \frac{\omega_0}{Q} \left( \frac{1}{T} \int_0^T W(t) dt \right) = \overline{W}$$

So

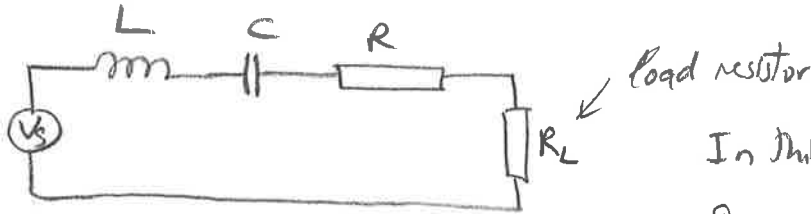
$$Q \equiv \frac{\omega_0 \overline{W}}{\overline{P}}$$

total energy averaged over a period

But 
$$\begin{cases} \overline{W} = \overline{W}_C + \overline{W}_L = 2 \cdot \overline{W}_L = \frac{L |I|^2}{2} \\ \overline{P} = R \cdot \frac{|I|^2}{2} \end{cases}$$

So 
$$Q = \frac{\omega_0 L}{R} = \frac{1}{R} \sqrt{\frac{L}{C}}$$

2 Loading of a resonant circuit =



In this case

$$Q_L = \left( \frac{1}{R} + \frac{1}{R_L} \right) \sqrt{\frac{L}{C}} = \text{loaded } Q$$

or 
$$Q_L^{-1} = Q_{\text{ext}}^{-1} + Q^{-1}$$

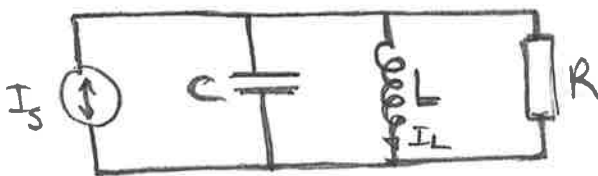
Simply because now

$$\begin{aligned} Q_L^{-1} &= \frac{\overline{P}}{\omega_0 \overline{W}} = \frac{(R + R_L) \frac{|I|^2}{2}}{\omega_0 L \frac{|I|^2}{2}} \\ &= \frac{R}{\omega_0 L} + \frac{R_L}{\omega_0 L} \end{aligned}$$

$$Q_{\text{ext}} = \frac{1}{R_L} \sqrt{\frac{L}{C}} = \text{external } Q$$

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} = \text{internal } Q$$

b)



parallel RLC resonator

$$Z(\omega) = \frac{1}{G + i \frac{C}{\omega} (\omega^2 - \omega_0^2)} = \frac{1}{Y(\omega)}$$

$$G = 1/R$$

A similar idea: 
$$\overline{W} = \overline{W}_C + \overline{W}_L = \frac{1}{4} |V|^2 \cdot C + \frac{1}{4} L |I_L|^2$$

But 
$$I_L = \frac{V}{i\omega L}$$

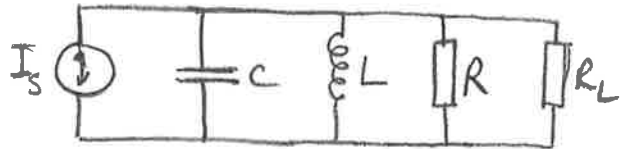
$$= \frac{1}{2} C |V|^2 \text{ at resonance}$$

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\overline{P} = \frac{1}{2} G |V|^2$$

$$\Rightarrow Q = \frac{\omega_0 \overline{W}}{\overline{P}} = \frac{\omega_0 C}{G} = \omega_0 RC = \frac{R}{\omega_0 L} = R \sqrt{\frac{C}{L}}$$

→ Loading of the parallel RLC resonator →



$$Q_L = \left( \frac{1}{R} + \frac{1}{R_L} \right) \sqrt{\frac{C}{L}} = \text{loaded } Q$$

$$Q_{\text{ext}} = \frac{1}{R_L} \sqrt{\frac{C}{L}} = \text{external } Q$$

$$Q = \frac{1}{R} \sqrt{\frac{C}{L}} = \text{internal } Q$$

$$\text{or } Q_L^{-1} = Q_{\text{ext}}^{-1} + Q^{-1}$$

→ This relation is the same as for the series RLC resonator