Lecture 2

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I. INTRODUCTION

The electromagnetic spectrum as seen by a quantum engineer:

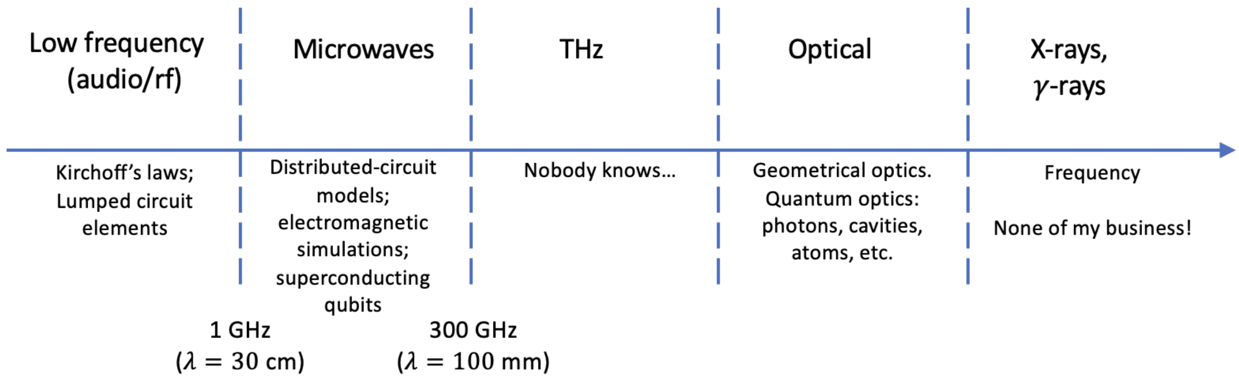


FIG. 1.

- Why don't lumped circuit models work at high frequencies?

The speed of light c is large but finite.

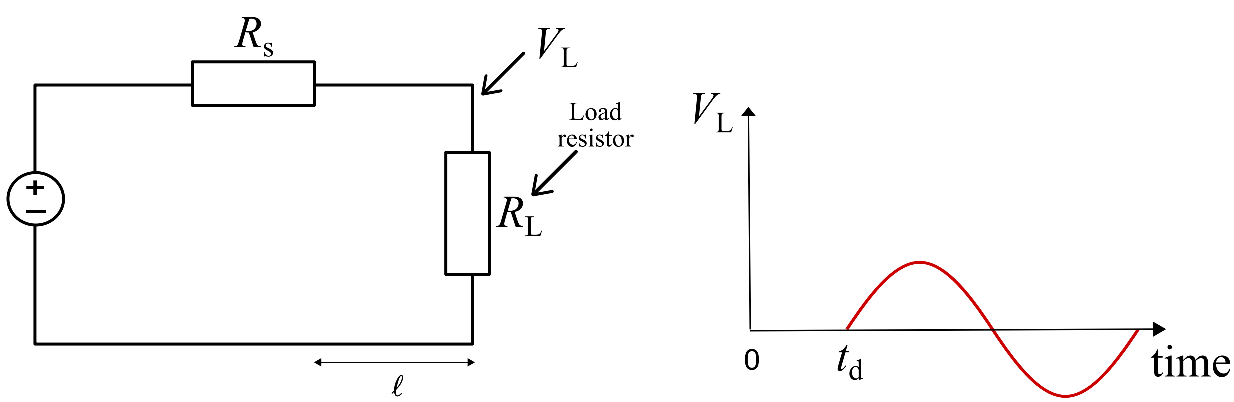


FIG. 2.

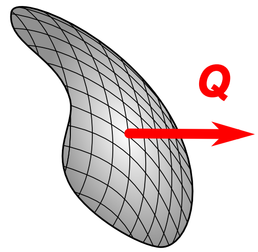
The delay time $t_d = \frac{\ell}{c}$ becomes non-negligible if $\lambda \sim \ell \sim$ cm, as frequency $\sim$ GHz. In other words, in order to apply the lumped circuit models we need to ensure that $t_d \ll T$ (T is the period of the electromagnetic field), or $l \ll \lambda$ (λ is the wavelength).

II. SOME BASIC CONCEPTS – ELECTRICAL CIRCUITS

- Electric current

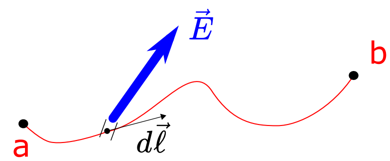
  $I = \frac{dQ}{dt}, \quad Q = \int_0^t I dt$

  By convention, the direction of current is the direction of the motion of positive charges.



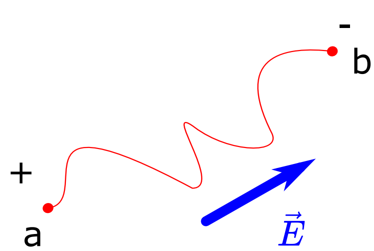
- Work done by an electric field

  $W_{ba} = q \int_a^b \vec{E} \cdot d\vec{\ell}$



- Voltage

  $V = V_{ba} = V_b - V_a = -\frac{dW_{ba}}{dq} = -\int_a^b \vec{E} \cdot d\vec{\ell}$



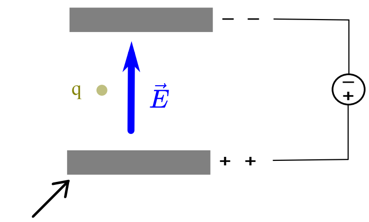
  W_{ba} = work done to transport the charge q against the field.

  $W = -W_{ba}$ = work done by the field.

  e.g. Capacitor

- Magnetic flux

  Node flux: $\phi(t) = \int_{-\infty}^{t} d\tau V(\tau), \quad \therefore V(t) = \frac{d}{dt}\phi(t).$



- Power

  $W(t) = \int_0^t P(\tau) d\tau$

  $P(t) = \frac{dW(t)}{dt} = \frac{dW(t)}{dq(t)} \cdot \frac{dq(t)}{dt} = V(t) \cdot I(t)$

  Therefore, $W(t) = \int_0^t V(\tau) \cdot I(\tau) d\tau$.

- Phasors

  Useful concept if:

– The circuit is linear

– all independent sources are sinusoidal

– only steady-state response is desired.

  $X(t) = A\cos(\omega t + \phi) = \text{Re}(Ae^{i\phi}e^{i\omega t})$

  $X = Ae^{i\phi} \equiv$ phasor = transformation of a sine waveform from time-domain to frequency domain.

  Why is it useful? Simple rules:

variable	**phasor**
$X(t)$	$Ae^{i\phi}$
$\frac{dX(t)}{dt}$	$i\omega \cdot Ae^{i\phi}$
$\int dtX(t)$	$\frac{Ae^{i\phi}}{i\omega}$

- Impedance and admittance

  $Z = V/I$ with V and I phasors.

$$Z = R + iX \begin{cases} Z = \text{impedance} \\ R = \text{resistance} \qquad \text{units: } \Omega \text{ (Ohm)} \\ X = \text{reactance} \end{cases} \tag{1}$$

$$Y = \frac{1}{Z} = G + iB \begin{cases} Y = \text{admittance} \\ G = \text{conductance} \qquad \text{units: } S \text{ (Siemens)} \\ B = \text{susceptance} \end{cases} \tag{2}$$

- AC power and decibels

  Suppose $V(t) = V_0 \cos(\omega t + \phi_V) = \text{Re}[V_0 e^{i\phi_V} e^{i\omega t}]$ phasor: $V_0 e^i \phi_V \equiv V$

  $I(t) = I_0 \cos(\omega t + \phi_I) = \text{Re}[I_0 e^{i\phi I} e^{i\omega t}]$ phasor: $I_0 e^{i\phi_I} \equiv I$.

  Instantaneous power: $P(t) = V(t) \cdot I(t) = \frac{1}{2} I_0 V_0 \cos(\phi_V - \phi I) + \frac{1}{2} I_0 V_0 \cos(2\omega t + \phi_V + \phi_I)$

  Average power: $\overline{P} = \frac{1}{T} \int_0^T P(t) dt = \frac{1}{2} I_0 V_0 \cos(\phi_V - \phi_I)$

  $\overline{P} = \frac{1}{2} \text{Re}[V \cdot I^*]$

  Root mean square of a periodic signal:

  $I(t) = I_0 \cos \omega t \longrightarrow I^2_{rms} \equiv \frac{1}{T} \int_0^T I^2(t) dt = \frac{I_0^2}{2}$, where we have used $\cos^2 \omega t = \frac{1+\cos 2\omega t}{2}$.

  Therefore,

  $$I_{rms} = \frac{I_0}{\sqrt{2}} \, . \tag{3}$$

  The decibel:

  Power in decibels: $N(\text{dB}) = 10 \log_{10} \frac{P}{P_{ref}}$, where P = power and P_{ref} = a reference power, usually 1 mW.

  If $P_{ref} = 1$ mW, then $N(\text{dBm}) = 10 \log_{10} \frac{P}{1 \text{ mW}}$. Note that the units are "dBm".

  Since $P \propto V^2$, we have $20 \log_{10} \frac{V}{V_{ref}}$ (in dBV), as another way to express this.

  Examples:

  30 dB is an increase in power by 1000

  20 dB is an increase in power by 100

10 dB is an increase in power by 10

3 dB is an increase in power by 2

0 dB is an increase in power by 1

-3 dB is a decrease in power by 2

-10 dB is a decrease in power by 10

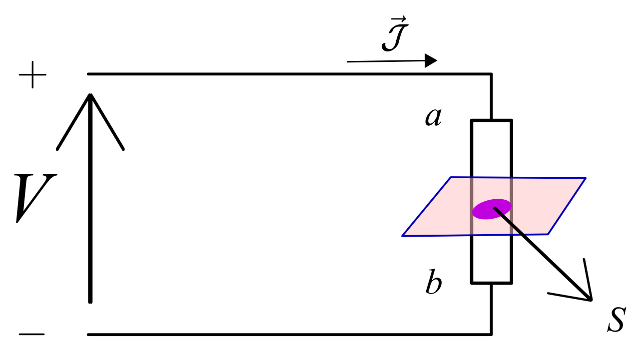
$\cdots$ etc.

III. CIRCUIT ELEMENTS

Note: An rf-circuit can be constructed from discrete (lumped) elements if the size of each component $\ll$ wavelength of the rf field.

- Resistor

  Typically a film of conductive material evaporated on a chip.
  $V \equiv V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{\ell} = \int_a^b \frac{\vec{\mathcal{J}}}{\sigma} d\vec{\ell}$, where we have used $\vec{E} = -\vec{\nabla} V$ and $\vec{\mathcal{J}} = \vec{\sigma} \cdot \vec{E}$ (Ohm's law), where $\vec{\sigma}$ = conductivity.
  If the frequency is not too high, then $\vec{\mathcal{J}}$ is uniform over the cross-section S of the resistor, i.e. $\vec{\mathcal{J}} = \vec{I}/S$.



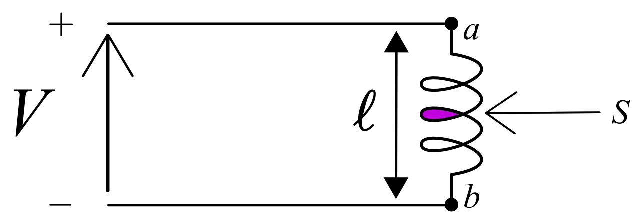
  Therefore, $V = I \int_a^b \frac{1}{\sigma S} d\ell = IR$, where $R = \frac{1}{\sigma S} \int_a^b d\ell$, $\quad \int_a^b d\ell = \ell$ = length of device.
  $R = \frac{1}{\sigma S} \int_a^b d\ell \implies R = \rho \frac{\ell}{S}$, where $\rho = \frac{1}{\sigma}$ = resistivity,
  & $\underline{V = IR}$.
  $G = \frac{1}{R}$ = conductance, where R = resistance, & $Z(\omega) = R$ — real and frequency-independent.

- Instantaneous dissipated power:
  $P(t) = V(t) \cdot I(t) = R \cdot I^2(t) = G \cdot V^2(t)$.

- Average dissipated power (for harmonic excitations: $V(t) = V_0 \cos \omega t$):
  $\overline{P} = \frac{1}{2} R |I_0|^2 = \frac{1}{2} G |V_0|^2$.

- Inductor

  $V = V_{ab} = V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{\ell} = \iint \frac{d\vec{B}(t)}{dt} d\vec{S} = L \frac{dI}{dt}$,
  where we have used the Maxwell-Faraday equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ + Stoke's theorem.



Here we assumed $\vec{B}$ is uniform over the surface area S.
From Ampère's law for a solenoid we have $B = \frac{\mu_0 \mu_r N I}{\ell} \rightarrow$ and there are N surfaces of area S, where $N =$ no. of turns of the solenoid, $\mu_0 =$ free-space magnetic permeability, $\mu_r =$ relative permeability.

$$L = \frac{\mu_0 \mu_r N^2 S}{\ell} \ , \tag{4}$$

so

$$V = L \frac{dI}{dt} \ , \tag{5}$$

therefore

$$Z_L(\omega) = i L \omega; \ , \tag{6}$$

because $Z_L(\omega) = \frac{V(\omega)}{I(\omega)}$.

- Instantaneous energy stored:

$$W_L(t) = \frac{1}{2} L I^2(t) \ . \tag{7}$$

- Average energy stored in AC-harmonic fields:

$$\overline{W_L} = \frac{1}{4} L I_0^2 \ . \tag{8}$$

Note: The flux variable $\phi = \iint \vec{B} \cdot d\vec{S}$ can be used to define a flux at a mode with potential V,

$$\int_{-\infty}^{t} V(\tau)d\tau = \phi(t) \ , \tag{9}$$

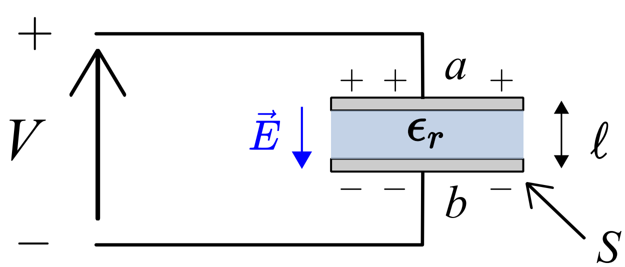
since $V(t) = \frac{d\phi(t)}{dt}$.

- Capacitor

$$V = -\int_{b}^{a} \vec{E} \cdot d\vec{\ell} = \frac{Q}{C} \ , \tag{10}$$

where C = capacitance and $C = \frac{\epsilon_0 \epsilon_r S}{\ell}$.

ϵ_0 = free-space electric permittivity and ϵ_r = relative permittivity.



Proof: From Gauss's law applied to the electric field between infinite plates

$E = \frac{Q}{S\epsilon_0\epsilon_r} = \frac{V}{\ell} \implies V = \frac{Q}{\frac{\epsilon_0\epsilon_r S}{\ell}}$.

Also, $I(t) = \frac{dQ(t)}{dt}$ implies

$$I(t) = C \cdot \frac{dV(t)}{dt} \ . \tag{11}$$

so

$$I = C\frac{dV}{dt} \ , \tag{12}$$

therefore

$$Z_C(\omega) = \frac{1}{i\omega C}, \tag{13}$$

Again from $Z_C(\omega) = \frac{V(\omega)}{I(\omega)}$ and using the properties of phasors.

- Instantaneous energy stored:

$$W_C(t) = \frac{1}{2}CV^2(t) \ . \tag{14}$$

- Average energy stored in AC-harmonic fields:

$$\overline{W_C} = \frac{1}{4} C V_0^2 \ . \tag{15}$$

IV. MORE COMPLEX NETWORKS OF INDUCTORS, CAPACITORS, RESISTORS ···

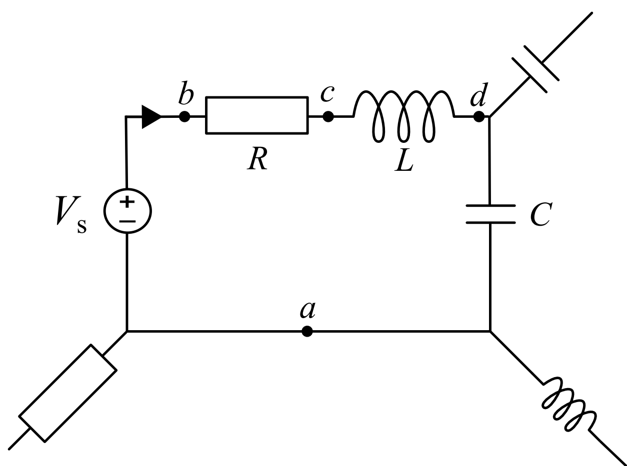
Kirchhoff's voltage and current laws:

- Kirchhoff's voltage law

  For any closed loop of a circuit, the algebraic sum of voltages of the individual branches is zero, i.e.,

$$\sum V_k = 0 \ . \tag{16}$$

  How it works:



$$-\int_a^b \vec{E} \cdot d\vec{\ell} - \int_b^c \vec{E} \cdot d\vec{\ell} - \int_c^d \vec{E} \cdot d\vec{\ell} - \int_d^a \vec{E} \cdot d\vec{\ell} = 0 \ , \tag{17}$$

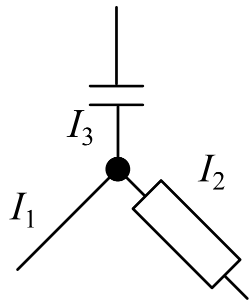
  or

$$V_s - RI(t) - L\frac{dI(t)}{dt} - \frac{1}{C}\int_{-\infty}^{t} d\tau I(\tau) = 0 \ . \tag{18}$$

- Kirchhoff's current law

  The algebraic sum of all branch currents confluent in the same node is zero.

$$\sum I_k = 0 \ . \tag{19}$$



How it works:

$$I_1 + I_2 + I_3 = 0 \ , \tag{20}$$

i.e., no charge accumulates in the node!

A. Application: impedances in series and in parallel

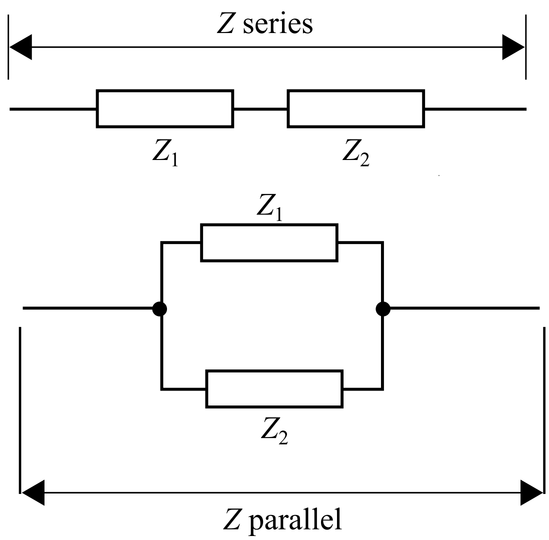


FIG. 3. Impedances in series and parallel.

The equivalent impedance of two impedances Z_1 and Z_2 placed in series is

$Z_{\text{series}} = Z_1 + Z_2$.

The equivalent impedance of two impedances Z_1 and Z_2 placed in parallel is

$Z_{\text{parallel}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$.

Convince yourself that this is the case by using Kirchhoff's laws.

B. Example: series-shunt circuits

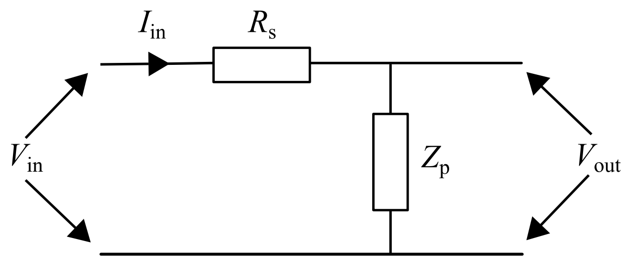


FIG. 4. A generic series-shunt circuit

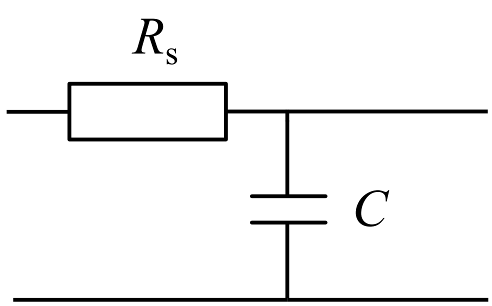
Simple equations for phasors: $V_{\text{out}} = Z_p I_{\text{in}} \quad V_{\text{in}} = (R_s + Z_p) I_{\text{in}}$

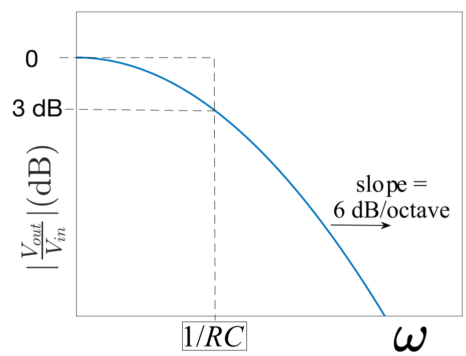
$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_p}{R_s + Z_p} =$ gain or attenuation

It is convenient to express this in dB: $\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| (dB) = 20 \log_{10} \left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|$.

A few interesting cases:

a) Z_p is a capacitor





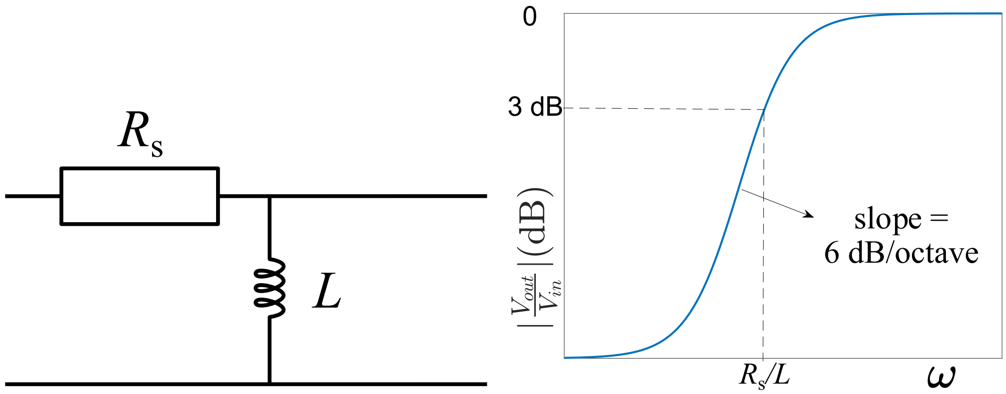
$$Z_p = \frac{1}{i\omega C} \implies \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + i\omega C R_s}.$$

$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = \frac{1}{\sqrt{1 + \omega^2 C^2 R_s^2}} \implies \left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| (\text{dB}) = -10 \log_{10}[1 + \omega^2 C^2 R_s^2].$$

Works like a low-pass filter with cutoff $\sim 1/R_s C$.

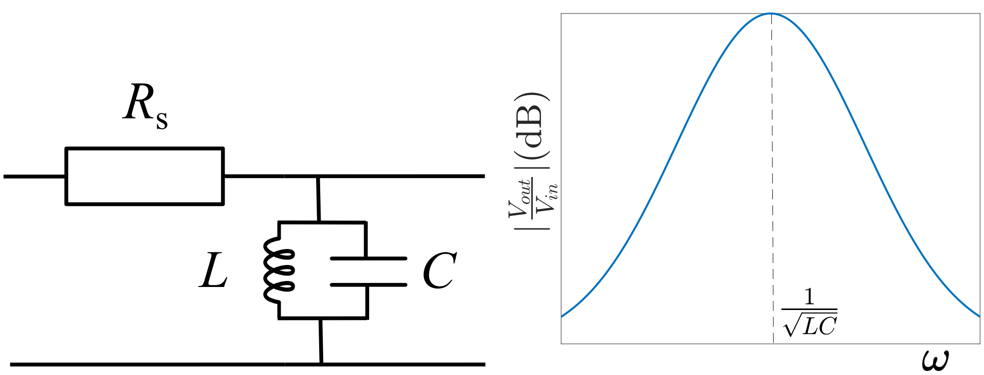
b) Z_p is an inductor

$$Z_p = i\omega L \qquad \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 - \frac{iR_s}{\omega L}}.$$



$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = \frac{1}{\sqrt{1+\left(\frac{R_s}{\omega L}\right)^2}} \implies \left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|(\text{dB}) = -10\log_{10}\left[1+\left(\frac{R_s}{\omega L}\right)^2\right].$$

c) Z_p is an LC-circuit



$$Z_p = \frac{i\omega L\cdot\frac{1}{i\omega C}}{i\omega L+\frac{1}{i\omega C}} = \frac{i\omega L}{1-\omega^2 LC} \qquad \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{i\omega L}{R_s(1-\omega^2 LC+i\omega L)}.$$

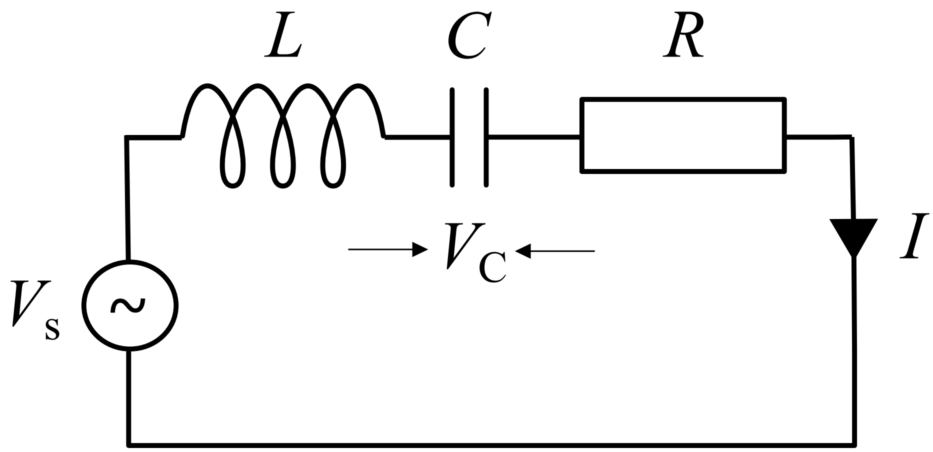
$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right|(\text{dB}) = 20\log_{10}\frac{\omega L}{\sqrt{R_s(1-\omega^2 LC)^2+(\omega L)^2}}.$$

V. RESONATORS BASED ON LUMPED CIRCUIT ELEMENTS

RLC components can be used to realize resonators.

A. RLC-series resonators

$$V_s = Z(\omega)\cdot I(\omega)$$
$$Z(\omega) = R + iL\omega + \frac{1}{iC\omega} = R + \frac{iL}{\omega}(\omega^2-\omega_0^2),$$



where $\omega_0 = \frac{1}{\sqrt{LC}}$, and at resonance $\omega = \omega_0$.

$Z(\omega_0) \equiv R \implies$ The impedance is real (resistive). The reactive part is zero, meaning that the inductor and capacitor reactances cancel each other. Due to this, the energy oscillates between the capacitor and the inductor and the source has to provide only what is lost through R.

Indeed

$$\begin{cases} \overline{W_L} = \frac{1}{4}L|I|^2 \\ \overline{W_C} = \frac{1}{4}C|V_C|^2 = \frac{1}{4}\frac{|I|^2}{C\omega^2}, \text{but } |V_C| = \frac{|I|}{C\omega}\ , \end{cases} \tag{21}$$

so at resonance: $\omega = \omega_0 \implies \underline{\overline{W_L} = \overline{W_C}}$.

Quality factor

Suppose we put some energy $W(0)$ in the resonator. Due to the resistance R, this will be dissipated.

$W(t) = W(0)e^{-\omega_0 t/Q}$

Q = quality factor – it measures how well the resonator stores energy.

Now $-\frac{dW}{dt} = \frac{\omega_0 W}{Q}$.

$\overline{P}$ = average loss in a period, $\overline{P} = -\frac{1}{T}\int_0^T \frac{dW}{dt}dt$, where $\frac{2\pi}{\omega_0} = T$ = period, or $\overline{P} = \frac{\omega_0}{Q}\frac{1}{T}\int_0^T W(t)dt$. But $\frac{1}{T}\int_0^T W(t)dt \equiv \overline{W}$ = total energy averaged over a period.

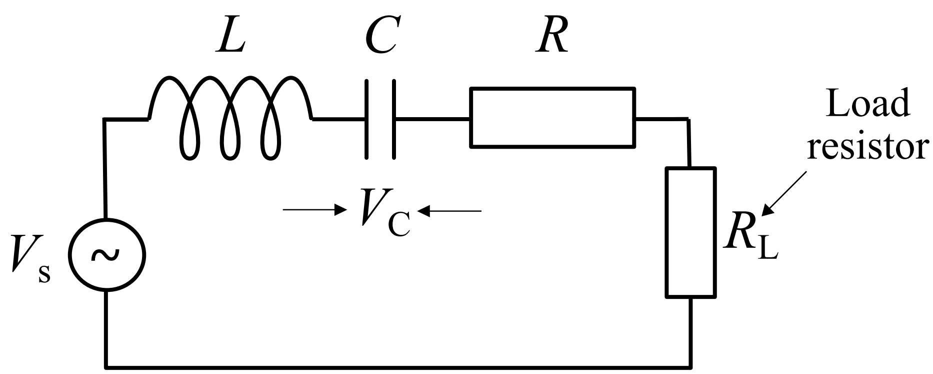
Since $\frac{2\pi}{\omega_0} = T = \text{period}$, $Q \equiv \frac{\omega_0 \overline{W}}{\overline{P}}$.

But

$$\begin{cases} \overline{W} = \overline{W_C} + \overline{W_L} = 2 \cdot \overline{W_L} = \frac{L|I|^2}{2} \\ \overline{P} = R \cdot \frac{|I|^2}{2} \ , \end{cases} \tag{22}$$

$\therefore \underline{Q = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}}$.

Loading of the RLC-series resonant circuit:

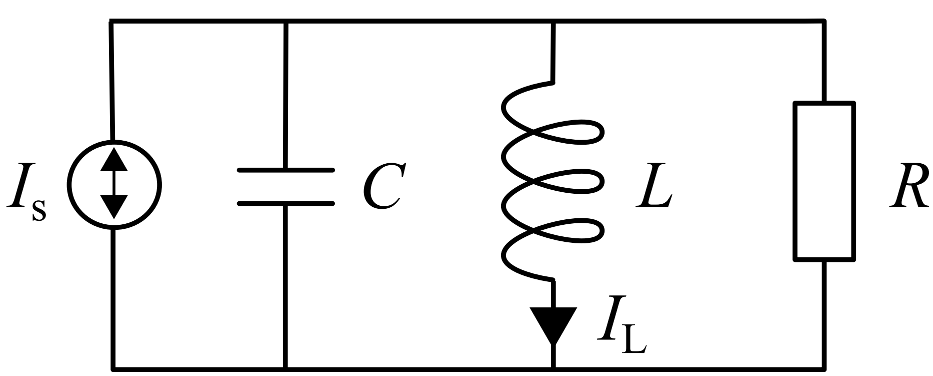


In this case, $Q_L = \left(\frac{1}{R} + \frac{1}{R_L}\sqrt{\frac{L}{C}}\right) = \text{loaded } Q$.

or $Q_L^{-1} = Q_{ext}^{-1} + Q^{-1}$, simply because now $Q_L^{-1} = \frac{\Phi}{\omega_0 \overline{W}} = \frac{(R+R_L)\frac{|I|^2}{2}}{\frac{\omega_0 L|I|^2}{2}} = \frac{R}{\omega_0 L} + \frac{R_L}{\omega_0 L}$.

Also, $Q_{ext} = \frac{1}{R_L}\sqrt{\frac{L}{C}} = \text{external } Q$ & $Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \text{internal } Q$.

B. RLC parallel resonant circuit



$$Z(\omega) = \frac{1}{G + i\frac{c}{\omega}(\omega^2 - \omega_0^2)} = \frac{1}{Y(\omega)} \ , \tag{23}$$

where $G = 1/R$.

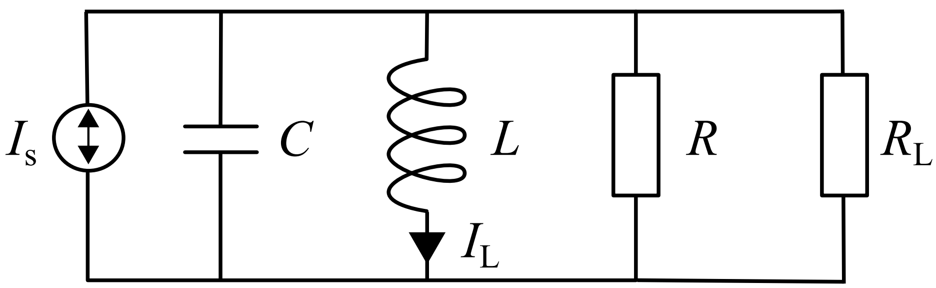
A similar idea: $\overline{W} = \overline{W_C} + \overline{W_L} = \frac{1}{4}|V|^2 \cdot C + \frac{1}{4}L|I_L|^2$, where $I_L = \frac{V}{i\omega L}$.

Furthermore, at resonance ($\omega = \omega_0 = \frac{1}{LC}$), we have $\overline{W} = \frac{1}{2}CV^2$.

Quality factor

$\overline{P} = \frac{1}{2}G|V|^2 \implies Q = \frac{\omega_0 \overline{W}}{\overline{P}} = \frac{\omega_0 C}{G} = \omega_0 RC = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$.

Loading of the Parallel RLC Resonator:



$$Q_L = \left(\frac{1}{R} + \frac{1}{R_L}\right)\sqrt{\frac{C}{L}} = \text{loaded Q} \tag{24}$$

$$Q_{\text{ext}} = \frac{1}{R_L}\sqrt{\frac{C}{L}} = \text{external Q} \tag{25}$$

$$Q = \frac{1}{R}\sqrt{\frac{C}{L}} = \text{internal Q} \tag{26}$$

or $Q_L^{-1} = Q_{\text{ext}}^{-1} + Q^{-1} \rightarrow$ This relation is the same as for the series RLC resonator.
