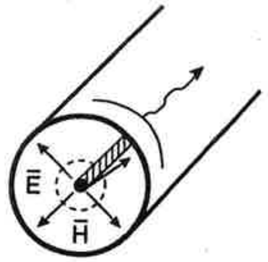
# Transmission lines

- Electromagnetic waves can propagate in free space ← review this! based on Maxwell's equations!
  * but also they can be guided by conducting or dielectric boundaries
- Transmission-line behaviour: occurs when $\lambda <<$ length of transmission line

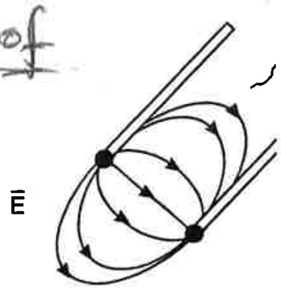
Transmission lines = guiding devices for the electromagnetic field

- The electromagnetic fields are TEM (transverse electromagnetic mode) if the conductors are ideal (zero-resistance); otherwise there will be a small axial component of the electromagnetic field.

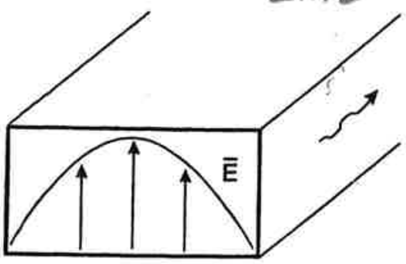
## Types of transmission lines



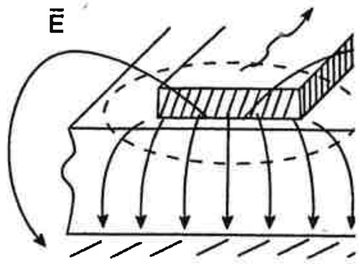

(a) = COAXIAL LINE



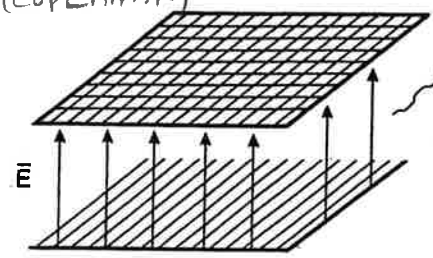

(b) = TWO-WIRE TRANSMISSION LINE



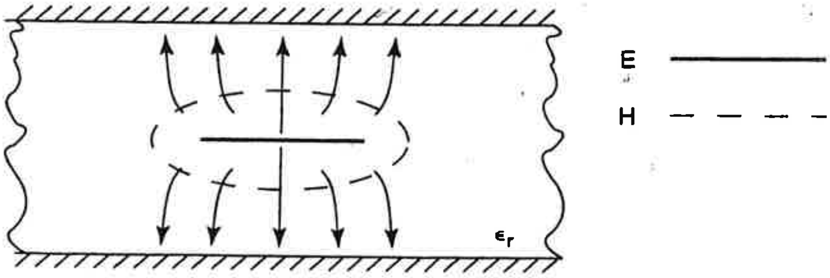

(c) = WAVEGUIDE (COPLANAR)




(d) = MICROSTRIP LINE

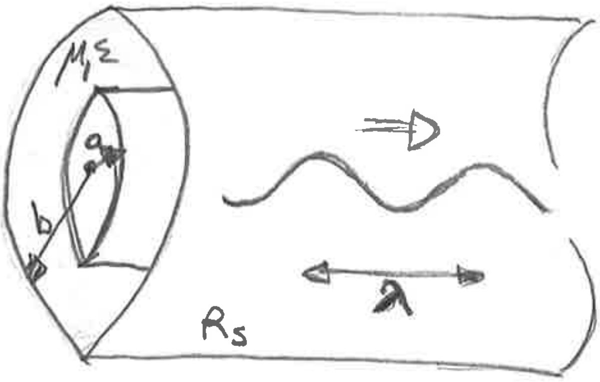



(e) = PARALLEL-PLATE WAVEGUIDE

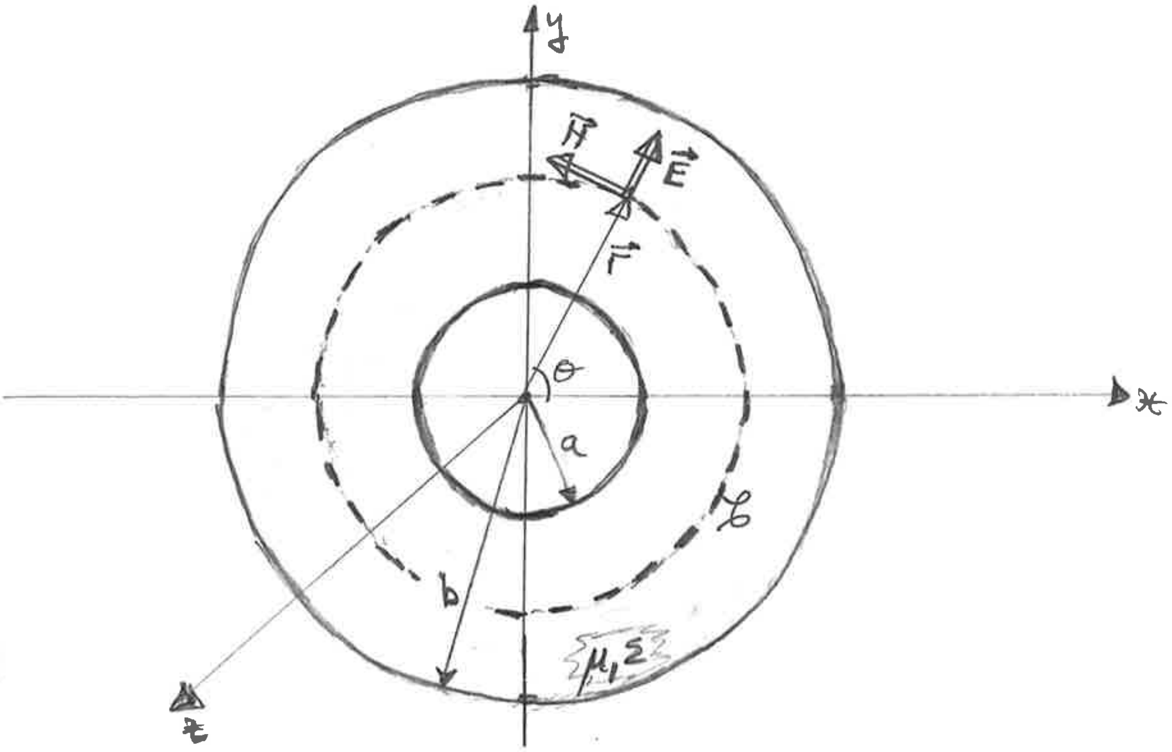



(f) = STRIPLINE

EXAMPLE: the coaxial line

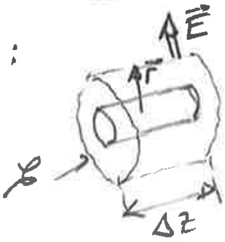



- How to calculate the $\vec{E}$, $\vec{H}$ fields inside?




Electric field $\boxed{\vec{E} = \frac{V_0}{\ln\frac{b}{a}} \frac{\hat{r}}{r}}$

Proof:




$\underline{\nabla \cdot \vec{D} = \rho} \Rightarrow \oint d\vec{s} \cdot \vec{E} = \int \frac{\rho}{\varepsilon} dV$ (Gauss' law)

$2\pi r (\Delta z) \cdot E \qquad \frac{1}{\varepsilon}(\Delta z) \cdot \rho \cdot \pi a^2$

Therefore

$$E = \frac{1}{r} \cdot \frac{\rho a^2}{2\varepsilon}$$

Also $V_0 = \int_a^b dr \cdot E$

$$= \int_a^b \frac{dr}{r} \cdot \frac{\rho a^2}{2\varepsilon} = \frac{\rho a^2}{2\varepsilon} \ln\frac{b}{a} \rightarrow \frac{\rho a^2}{2\varepsilon} = \frac{V_0}{\ln\frac{b}{a}}$$

So $\vec{E} = \frac{V_0}{\ln\frac{b}{a}} \cdot \frac{\hat{r}}{r}$

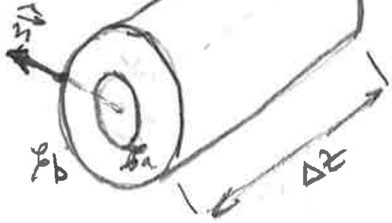
Magnetic field $\boxed{\vec{H} = \frac{I_0}{2\pi r} \cdot \hat{e}_\theta}$

$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \longrightarrow \oint_{\mathcal{C}} \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{s} = I_0$ or $2\pi r \cdot H = I_0 \Rightarrow \vec{H} = \frac{I_0}{2\pi r} \cdot \hat{e}_\theta$

# Towards a distributed model of inductances, capacitances, resistances, conductances

Problem: how to connect the electric and magnetic fields to circuit-elements.

Answer: via stored or dissipated energy




## ① Inductance per unit length

magnetic energy $= \frac{\mu}{4}\int ds\cdot(\Delta z)\, H^2 = \frac{(L'\Delta z)\, I_0^2}{4} \Rightarrow L' = \frac{\mu}{I_0^2}\int ds\, H^2$

$$L' = \frac{\mu}{I_0^2}\int ds\, H^2 = \frac{\mu}{I_0^2}\cdot I_0^2 \int_0^{2\pi} d\theta \int_a^b dr\cdot r\cdot \frac{1}{(2\pi r)^2} = \frac{\mu}{2\pi}\ln\frac{b}{a}$$

$$\boxed{L' = \frac{\mu}{2\pi}\ln\frac{b}{a}}$$

(measured in H/m)

## ② Capacitance per unit length

electrostatic energy $= \frac{\varepsilon}{4}\int ds\cdot(\Delta z)\cdot E^2 = \frac{(C'\Delta z)\, V_0^2}{4} \Rightarrow C' = \frac{\varepsilon}{V_0^2}\cdot\int ds\cdot E^2$

$$C' = \frac{\varepsilon}{V_0^2}\int ds\, \vec{E}^2 = \frac{\varepsilon}{V_0^2}\cdot V_0^2\cdot\frac{1}{\ln^2\frac{b}{a}}\int_0^{2\pi} d\theta\int_0^b dr\cdot r\cdot\frac{1}{r^2} \Rightarrow$$

$$\boxed{C' = \frac{2\pi\varepsilon}{\ln\frac{b}{a}}}$$

(measured in F/m)

## ③ Resistance per unit length

power dissipated in the lossy conductors $= \frac{R_s}{2}\int_{C_a+C_b} dl\cdot\Delta z\cdot J_s^2 = \frac{R_s}{2}\Delta z\cdot\int_{C_a+C_b} dl\cdot H^2 = \frac{R'\Delta z}{2} I_0^2$

$$\Downarrow$$

$$R' = \frac{R_s}{I_0^2}\int_{C_a+C_b} dl\cdot \vec{H}^2$$

Here $R_s$ = surface resistance (measured in Ω)

$\vec{J}_s = \hat{n}\times\vec{H}$ = surface current density (measured in A/m)

$\hat{n}$ = vector unit pointing outwards (normal to the conducting surface)

$$R' = \frac{R_s}{I_0^2}\int_{C_a+C_b} dl\cdot H^2 = \frac{R_s}{(2\pi)^2}\left[\int_0^{2\pi} d\theta\cdot a\cdot\frac{1}{a^2} + \int_0^{2\pi} d\theta\cdot b\cdot\frac{1}{b^2}\right] = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

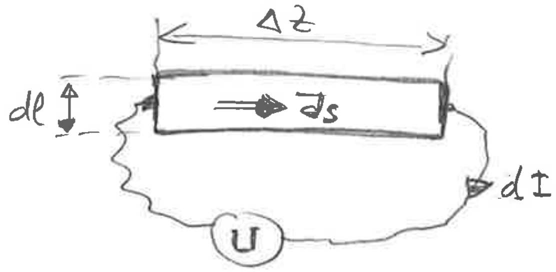
we integrate over the perimeters of circles of radius a or b

$$\Rightarrow \boxed{R' = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

(measured in Ω/m)

• Resistance per unit length → a bit more in-depth if you are interested

# Take a slab of metal :




We have defined
the current density $J = \frac{dI}{dS} = \frac{\text{charge/time}}{\text{area}}$

but for sheets of metals we can define

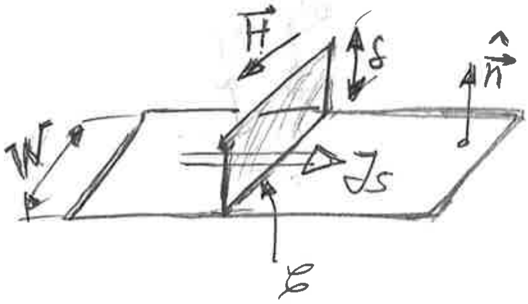
$$J_s = \frac{dI}{dl} = \frac{\text{charge/time}}{\text{unit length across the sheet}}$$

Then $R_s \equiv \frac{U/\Delta z}{J_s}$ = surface resistance ⇐ units of Ohm

So power dissipated $= \frac{1}{2}\int dI \cdot U = \frac{1}{2}\Delta z \cdot \int dl \cdot (R_s J_s) \cdot J_s$

$$= \frac{1}{2}\Delta z \int dl \cdot R_s J_s^2$$

# We also have




Let us take $\vec{H}$ like in this figure, which is the case for the cooxial transmission line

Then $\oint_{\mathcal{C}} \vec{H} \cdot d\vec{l} = H \cdot W$

$$= \int_S \Big(\vec{J} + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{=0}\Big) d\vec{s} = \underbrace{J \cdot \delta}_{=J_s} \cdot W$$

⇒

$J_s = H$,

More generally, for an arbitrary vector $\vec{H}$,

$$\vec{J_s} = \hat{\vec{n}} \times \vec{H}$$

4 Conductance (radial) per unit length

$$\varepsilon = \varepsilon' - i\varepsilon'' = \varepsilon_0 \varepsilon_r (1 - i \tan\delta)$$

$\varepsilon' = \varepsilon_0 \varepsilon_r$

$\varepsilon'' = \varepsilon \tan\delta$ → dissipation in the dielectric between the core metal and the outside shield.

power dissipated $= \dfrac{\omega \varepsilon''}{2} \int ds \cdot \Delta z \cdot E^2 = \dfrac{G' V_0^2}{2} \longrightarrow G' = \dfrac{\omega \varepsilon''}{V_0^2} \cdot \int ds \cdot E^2$

$$\Rightarrow G' = \frac{\omega \varepsilon''}{V_0^2} \int ds \cdot E^2 = \frac{\omega \varepsilon''}{V_0^2} \cdot \int_0^{2\pi} d\theta \int_a^b dr \cdot r \cdot \frac{V_0^2}{r^2 \ln\frac{b}{a}} \Rightarrow \boxed{G' = \frac{2\pi\omega\varepsilon''}{\ln\frac{b}{a}}}$$

(measured in S/m)

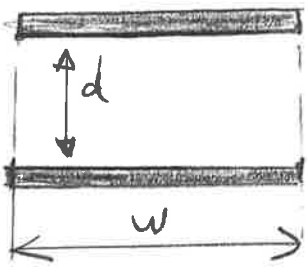
- Examples of materials used in coaxes

| Conductor | Copper Cu | Aluminum Al | Silver Ag | Gold Au |
|---|---|---|---|---|
| Resistivity $\rho$ [nΩ·m] | 16.9 | 26.7 | 16.3 | 22.0 |

n → $10^{-9}$ "nano"

| Dielectric | Dry air | Polyethylene | PTFE | PVC |
|---|---|---|---|---|
| $\varepsilon_r$ | 1,0006 | 2,2 | 2,1 | 3,2 |
| $\tan\delta$ | low | 0,0002 | 0,0002 | 0,001 |
| Resistivity (Ω·m) | high | $10^{15}$ | $10^{15}$ | $10^{15}$ |
| Breakdown voltage (MV/m) | 3 | 47 | 59 | 34 |

# Other transmission lines

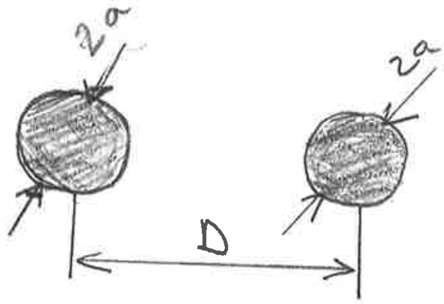



$$L' = \frac{\mu d}{w}$$

$$C' = \frac{\varepsilon' w}{d}$$

$$R' = \frac{2R_s}{w}$$

$$G' = \frac{\omega \varepsilon'' w}{d}$$




$$L' = \frac{\mu}{\pi} \cosh^{-1} \frac{D}{2a}$$

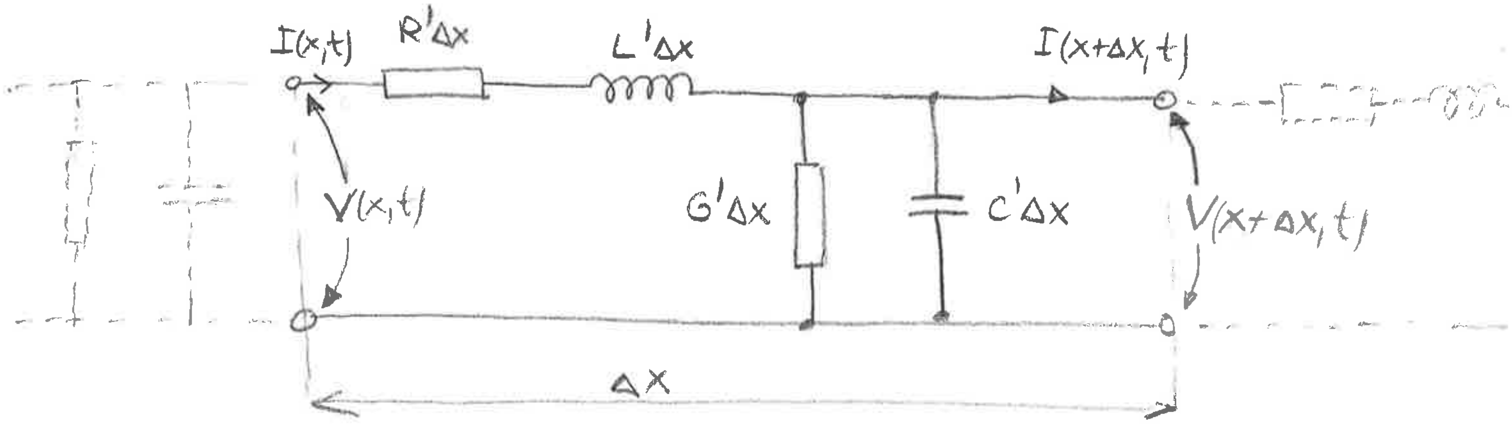
$$C' = \frac{\pi \varepsilon'}{\cosh^{-1} \frac{D}{2a}}$$

$$R' = \frac{R_s}{\pi a}$$

$$G' = \frac{\pi \omega \varepsilon''}{\cosh^{-1} \frac{D}{2a}}$$

# TRANSMISSION LINES – general models

- If the length of a circuit is $\gtrsim \lambda$ we have to use either a simulator of Maxwell's equations or a distributed model of lumped elements.
- TLs = two parallel conductors that guide the electromagnetic field
  examples: two-wire lines, striplines, microstrip lines




$R', L', G', C'$ = resistance, inductance, conductance, capacitance per unit length.

KIRCHHOFF says:

$$\begin{cases} V(x,t) = I(x,t) R'\Delta x + L'\Delta x \cdot \dfrac{\partial I(x,t)}{\partial t} + V(x+\Delta x,t) \\ I(x,t) = V(x+\Delta x,t) G'\Delta x + C'\Delta x \dfrac{\partial V(x+\Delta x,t)}{\partial t} + I(x+\Delta x,t) \end{cases}$$

$\Delta x \to 0$

$$\begin{cases} -\dfrac{\partial V(x,t)}{\partial x} = R' I(x,t) + L' \dfrac{\partial I(x,t)}{\partial t} \\ -\dfrac{\partial I(x,t)}{\partial x} = G' V(x,t) + C' \dfrac{\partial V(x,t)}{\partial t} \end{cases}$$

Therefore,

$$\begin{cases} -\dfrac{\partial^2 V(x,t)}{\partial x^2} = -R'\left(G'V(x,t) + C'\dfrac{\partial V(x,t)}{\partial t}\right) - L'\left(G'\dfrac{\partial V(x,t)}{\partial t} + C'\dfrac{\partial^2 V(x,t)}{\partial t^2}\right) \\ -\dfrac{\partial^2 I(x,t)}{\partial x^2} = -G'\left(R'I(x,t) + L'\dfrac{\partial I(x,t)}{\partial t}\right) - C'\left(R'\dfrac{\partial I(x,t)}{\partial t} + L'\dfrac{\partial^2 I(x,t)}{\partial t^2}\right) \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 V(x,t)}{\partial x^2} = L'C'\dfrac{\partial^2 V(x,t)}{\partial t^2} + (R'C' + L'G')\dfrac{\partial V(x,t)}{\partial t} + R'G'V(x,t) \\ \dfrac{\partial^2 I(x,t)}{\partial x^2} = L'C'\dfrac{\partial^2 I(x,t)}{\partial t^2} + (R'C' + L'G')\dfrac{\partial I(x,t)}{\partial t} + R'G'I(x,t) \end{cases}$$

Harmonic signals:

$$V(x,t) = V(x)\,e^{i\omega t}$$
$$I(x,t) = I(x)\,e^{i\omega t}$$

$V(x), I(x)$ = phasors

$$\Rightarrow \begin{cases} \dfrac{d^2V(x)}{dx^2} - \gamma^2 V(x) = 0 \\ \dfrac{d^2I(x)}{dx^2} - \gamma^2 I(x) = 0 \end{cases}$$

$$\gamma = \alpha + i\beta = \sqrt{(R' + i\omega L')(G' + i\omega C')}$$

$\gamma$ = propagation constant

$\alpha$ = attenuation constant

$\beta$ = phase constant

$$\Rightarrow \boxed{V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x}}$$ general solution

From $-\dfrac{\partial V(x,t)}{\partial x} = R' I(x,t) + L' \dfrac{\partial I(x,t)}{\partial t}$ we get $I(x) = -\dfrac{1}{R' + i\omega L'}\dfrac{dV(x)}{dx}$

or

$$I(x) = \frac{1}{Z_0} V^+ e^{-\gamma x} - \frac{1}{Z_0} V^- e^{\gamma x} = I^+ e^{-\gamma x} + I^- e^{\gamma x}$$

where $I^\pm = \pm \dfrac{V^\pm}{Z_0}$

where $Z_0 = \sqrt{\dfrac{R' + i\omega L'}{G' + i\omega C'}}$ = characteristic impedance of the transmission line

Lossless transmission case: $R' = G' = 0$

$$\gamma = i\beta = i\omega\sqrt{L'C'}$$

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{L'}{C'}}$$ → now independent of frequency!

Note: free-space impedance $= 377\,\Omega$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$ = phase velocity

Exercise: Show that for the low-loss case $R' \ll \omega L'$, $G' \ll \omega C'$ we have

$$\beta \simeq \omega\sqrt{L'C'}$$
$$\alpha \simeq \frac{1}{2}\sqrt{L'C'}\left(\frac{R'}{L'} + \frac{G'}{C'}\right)$$

- Standardized values:

| $Z_0$ | Application |
|---|---|
| $50\,\Omega$ | Instrumentation, communication |
| $75\,\Omega$ | TV, VHF radio |
| $300\,\Omega$ | RF |
| $600\,\Omega$ | audio |

- A note on the meaning of $V^+e^{-\gamma x}$ and $V^-e^{+\gamma x}$.

These are phasors of waves propagating in the positive direction of the x-axis, respectively negative direction.

To see this, consider the "+" wave

$$V^+e^{-\gamma x}e^{i\omega t} = V^+e^{-\alpha x}e^{i\omega\left(t-\frac{x}{v_p}\right)}$$

where $\gamma = \alpha + i\beta$

$v_p = \frac{\omega}{\beta}$

$t - \frac{x}{v_p} = \varphi$ = constant $\rightarrow$ means

$x = +tv_p - \varphi$

(propagation in the positive direction)

Differently, the "−" wave is

$$V^-e^{+\gamma x}e^{i\omega t} = V^-e^{-\alpha x}e^{i\omega\left(t+\frac{x}{v_p}\right)}$$

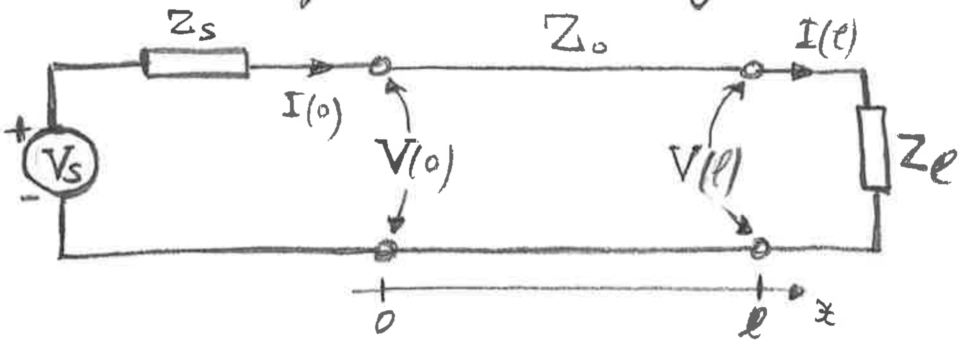
$t + \frac{x}{v_p} = \varphi \Rightarrow x = -tv_p - \varphi$

(propagation in the negative direction)

So $V^+$ = forward wave

$V^-$ = associated with reflection from a load

- Incident and reflected waves along a loaded transmission line




line terminated in a load impedance $Z_\ell$

$$V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x}$$
$$I(x) = I^+ e^{-\gamma x} + I^- e^{\gamma x} \quad , \; I^\pm = \pm \frac{V^\pm}{Z_0}$$

$$\begin{cases} V(0) = V_S - Z_S I(0) & \text{— Kirchhoff's law} \\ V(\ell) = Z_\ell I(\ell) \end{cases}$$

or:

$$\begin{cases} V^+ + V^- = V_S - \dfrac{Z_S}{Z_0}(V^+ + V^-) \\ V^+ e^{-\gamma \ell} + V^- e^{\gamma \ell} = \dfrac{Z_\ell}{Z_0}\left(V^+ e^{-\gamma \ell} - V^- e^{\gamma \ell}\right) \end{cases}$$

- Define a reflection coefficient at the load $x = \ell$ $\quad \Gamma_V^\ell = \dfrac{V^- e^{\gamma \ell}}{V^+ e^{-\gamma \ell}}$

$$\rightarrow \quad 1 + \Gamma_V^\ell = \frac{Z_\ell}{Z_0}\left(1 - \Gamma_V^\ell\right) \quad \Rightarrow \quad \boxed{\Gamma_V^\ell = \frac{Z_\ell - Z_0}{Z_\ell + Z_0}}$$

We can also define a current reflection coefficient at the load

$$\boxed{\Gamma_I^\ell \equiv \frac{I^- e^{\gamma \ell}}{I^+ e^{-\gamma \ell}} = -\Gamma_V^\ell}$$

- Define the transmission coefficient at the load $x = \ell$ $\quad T_V = \dfrac{V^+ e^{-\gamma \ell} + V^- e^{\gamma \ell}}{V^+ e^{-\gamma \ell}}$

so $\boxed{T_V = 1 + \Gamma_V^\ell}$

and for current $\boxed{T_I = \dfrac{I^+ e^{-\gamma \ell} + I^- e^{\gamma \ell}}{I^+ e^{-\gamma \ell}} = 1 + \Gamma_I^\ell}$

**• Average power delivered to the load**

$$\overline{P}_\ell = \frac{1}{2} R_e \left[ V(\ell) I^*(\ell) \right]$$

$\frac{1}{2}$ — comes from the fact that the field is harmonic

Now
$$1 - \Gamma_V = \frac{I^- e^{\gamma \ell} + I^+ e^{-\gamma \ell}}{I^+ e^{-\gamma \ell}} = \frac{I(\ell)}{I^+ e^{-\gamma \ell}}$$
$$1 + \Gamma_V = \frac{V^+ e^{-\gamma \ell} + V^- e^{\gamma \ell}}{V^+ e^{\gamma \ell}} = \frac{V(\ell)}{V^+ e^{-\gamma \ell}}$$
$\rightarrow$

$$(1 - \Gamma_V^*)(1 + \Gamma_V) = \frac{V(\ell) I^*(\ell)}{I^{+*} V^+ e^{-\gamma \ell} (e^{-\gamma \ell})^*}$$

but $I^+ = \frac{V^+}{Z_0}$

$$V(\ell) I^*(\ell) = \frac{1}{Z_0} \left| V^+ e^{-\gamma \ell} \right|^2 \cdot \underbrace{(1 - \Gamma_V^*)(1 + \Gamma_V)}_{= 1 - \Gamma_V^* + \Gamma_V - |\Gamma_V|^2}$$

$-\Gamma_V^* + \Gamma_V$ → imaginary!

$$\Rightarrow \overline{P}_\ell = \frac{1}{2 Z_0} \cdot \left| V^+ e^{-\gamma \ell} \right|^2 \left( 1 - |\Gamma_V|^2 \right)$$

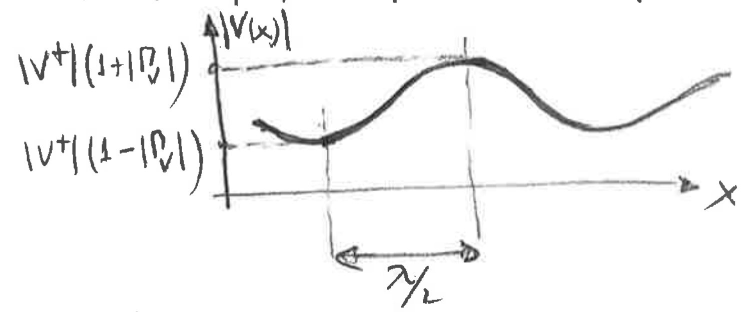
**• VSWR** (voltage standing-wave ratio)

$$V(x) = V^+ e^{-\gamma x} + V^- e^{\gamma x} = V^+ e^{-\gamma x} \left[ 1 + \Gamma_V e^{-2\gamma(\ell - x)} \right]$$

remember that $\Gamma_V \equiv \frac{V^- e^{\gamma \ell}}{V^+ e^{-\gamma \ell}}$

Let's consider a lossless line $\alpha = 0$ $\quad \gamma = i\beta = \frac{2\pi i}{\lambda}$

$$|V(x)| = |V^+| \cdot \left| 1 + \Gamma_V e^{-2i\beta(\ell - x)} \right|$$ — oscillates, min. and max. seperated by $\frac{\pi}{\beta} = \frac{\lambda}{2}$




$$VSWR = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|}$$ = ratio between the max. line voltage and min. line voltage

- Impedance along the line

$$Z(x) = \frac{V(x)}{I(x)} = Z_0 \frac{V^+ e^{-\gamma x} + V^- e^{\gamma x}}{V^+ e^{-\gamma x} - V^- e^{\gamma x}} = Z_0 \frac{1 + \Gamma_V e^{-2\gamma(l-x)}}{1 - \Gamma_V e^{-2\gamma(l-x)}}$$

take $x = 0$ → we get $Z(0) \equiv Z_{in}$ = input impedance of the line, i.e. the impedance seen when looking toward the load.

$$\boxed{Z_{in} = Z_0 \cdot \frac{Z_l + Z_0 \tanh \gamma l}{Z_0 + Z_l \tanh \gamma l}}$$

— can be verified immediately by recalling that $\Gamma_V = \frac{Z_l - Z_0}{Z_l + Z_0}$

Note that in general $Z_{in} \neq Z_0$, so the termination matters!

and also $Z_{in}$ is frequency-dependent

References:


- David M. Pozar – Microwave Engineering
- R.E. Collin – Foundations for Microwave Engineering