Q:- Transform the vector $\bar{B} = y\,\bar{a}_x - x\,\bar{a}_y + z\,\bar{a}_z$ into cylindrical coordinates

Sol:-

From the formula sheet:-

$$\begin{pmatrix} f_r \\ f_\phi \\ f_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$B_r = \cos\phi\, B_x + \sin\phi\, B_y$$

$$B_\phi = -\sin\phi\, B_x + \cos\phi\, B_y$$

$$\bar{B} = \underbrace{y}_{B_x}\,\bar{a}_x - \underbrace{x}_{B_y}\,\bar{a}_y + \underbrace{z}_{B_z}\,\bar{a}_z$$

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$B_r = y\cos\phi - x\sin\phi$$

$$B_r = r\,\cancel{\sin\phi\cos\phi} - r\,\cancel{\cos\phi\sin\phi}$$

$$\boxed{B_r = 0}$$

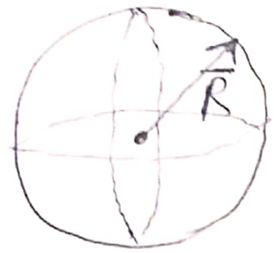
$$B_\phi = -y\sin\phi - x\cos\phi = -r\sin^2\phi - r\cos^2\phi = -r$$

Thus:-

$$\boxed{\bar{B} = -r\,\bar{a}_\phi + z\,\bar{a}_z}$$

Q: Prove divergence theorem for

$\bar{D} = R\bar{a}_R$



$$\int \bar{\nabla}\cdot\bar{D}\, dV = \oint \bar{D}\cdot d\bar{S}$$

from formula sheet:-

$$\bar{\nabla}\cdot\bar{F} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 f_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta)$$

$$+ \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} f_\phi$$

$$f_R = R, \quad f_\theta = 0, \quad f_\phi = 0$$

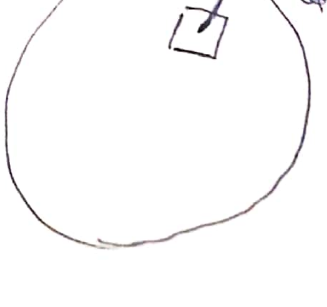
$$\bar{\nabla}\cdot\bar{D} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 \cdot R) = \frac{1}{R^2}\frac{\partial}{\partial R}R^3 = \frac{3R^2}{R^2}$$

$$\boxed{\bar{\nabla}\cdot\bar{D} = 3}$$

$$\int \bar{\nabla}\cdot\bar{D}\, dV = \int 3\, dV = 3\int dV = 3\left(\frac{4\pi R^3}{3}\right)$$

$$= \boxed{4\pi R^3}$$

$\oint \bar{D}\cdot d\bar{S} = ?$



$d\bar{S} = \bar{a}_R R^2 \sin\theta\, d\theta\, d\phi$

$$\oint \bar{D}\cdot d\bar{S} = \oint (R\bar{a}_R)\cdot(R^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_R)$$

$$= \oint R^3 \sin\theta\, d\theta\, d\phi$$

$$= R^3 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi$$

$$= R^3 \int_0^{2\pi} -\cos\theta \Big|_0^{\pi}\, d\phi$$

$$= R^3 \int_0^{2\pi} 2\, d\phi$$

$$= \boxed{4\pi R^3}$$

Hence:-

$$\int \bar{\nabla}\cdot\bar{D}\, dV = \oint \bar{D}\cdot d\bar{S} = 4\pi R^3$$