Some More Math for Economists: The Value Function

Spring 2023

The Value Function

The problems we solve in economics often have two types of variables

- Endogenous variables - The variables economic agents determine (e.g. the controls of the maximization problem)
- Exogenous variables - Parameters that are not determined by decision makers

Understanding how the endogenous variables change when exogenous variables change is a crucial part of economic analysis

Consumer Theory

Recall the standard consumer problem

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ s.t. } p \cdot x \leq m$$

- The x's are endogenous.
- The p's and m are exogenous.

We'd like to understand the properties of

- The value function $V(p, m)$, which gives the value of the objective at the maximum.
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- The value function $V(p, m)$, which gives the value of the objective at the maximum.
  - In the consumer problem, this is called indirect utility.
- The policy function $x(p, m)$, which gives the values of the endogenous variables that solve the maximization problem.
  - In the consumer problem, this is called demand.
  - This is not necessarily a function. We'll talk about that more in math camp.

Continuity

In general, the value and policy functions are well-behaved. Recall:

Definition

A set $X \subseteq \mathbb{R}^n$ is (sequentially) compact if if every sequence in X has a convergent subsequence.

Definition

A function is continuous at x at x if for every sequence $(x_n) \to x$, $\lim_{n\to\infty} f(x_n) = f(x)$.

Theorem of the Maximum

Theorem (A baby theorem of the maximum)

Suppose $f : X \times \Theta \to \mathbb{R}$ is strictly concave in X and continuous, $X \subseteq \mathbb{R}^n$ is non-empty, compact and convex and $g : X \times \Theta \to \mathbb{R}^m$ is quasiconvex in each component, continuous and $\{x : g(x; \theta) \leq 0\}$ is compact for all θ. Then the value and policy functions for the maximization problem

$$\max_{x \in X} f(x; \theta) \text{ s.t. } g(x; \theta) \leq 0$$

are continuous.

Concavity - Maximum theorem proof

Since f is continuous and the constraint is compact we know a max exists for each θ.

Concavity - Maximum theorem proof

Since f is continuous and the constraint is compact we know a max exists for each θ.

Since $f(x;\theta)$ is strictly concave, we know there is a unique maximum at each θ. A natural conjecture would be that the limit of any sequence of maximizers is a maximum, which would give us continuity.

Proof

Fix a sequence $\theta_n \to \theta$. We want to show that $V(\theta_n) \to V(\theta)$ and $\phi(\theta_n) \to \phi(\theta)$ where V, ϕ are the value and policy functions.

- Let x_n be the $\arg\max$ at θ_n.
- Take any convergent subsequence x_{n_k}, let $x^* = \lim x_{n_k}$

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- Let x_n be the $\arg\max$ at θ_n.
- Take any convergent subsequence x_{n_k}, let $x^* = \lim x_{n_k}$
- Note that x^* must be feasible at θ. So $V(\theta) \geq f(x^*; \theta)$.

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We have that $V(\theta) \geq f(x^*; \theta)$. We want to show $V(\theta) = f(x^*; \theta)$.

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Suppose not

- If $f(x^*) < V(\theta)$ then there exists x s.t. $f(x; \theta) > f(x^*, \theta)$.

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Suppose not

- If $f(x^*) < V(\theta)$ then there exists x s.t. $f(x; \theta) > f(x^*, \theta)$.
- But then there exists an $\epsilon, \delta > 0$ s.t. if $||(x', \theta') - (x, \theta)|| < \epsilon$ then $f(x', \theta') - f(x^*, \theta) > \delta$.
- But, then $\lim f(x_{n_k}, \theta_{n_k}) \geq f(x^*, \theta) + \delta$

Proof

We have that $V(\theta) \geq f(x^*;\theta)$. We want to show $V(\theta) = f(x^*;\theta)$.

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- If $f(x^*) < V(\theta)$ then there exists x s.t. $f(x;\theta) > f(x^*,\theta)$.
- But then there exists an $\epsilon, \delta > 0$ s.t. if $||(x',\theta') - (x,\theta)|| < \epsilon$ then $f(x',\theta') - f(x^*,\theta) > \delta$.
- But, then $\lim f(x_{n_k},\theta_{n_k}) \geq f(x^*,\theta) + \delta$
- Thus, x^* must be the maximum. Since the maximum is unique $x_n \to x^*$ and thus V and ϕ are continuous.

Maximum theorem

Note that the convexity assumptions play almost no role. They make the $\arg\max$ unique, and that's it.

One may conjecture this holds much more generally.

- In fact, if the objective is continuous, the constraints are compact and vary "continuously" then the value and policy functions are continuous.
- This is Berge's Theorem of the Maximum.
- We need a theory of multivalued functions to deal with the continuity of the constraints and policy functions that I don't want to spend time on in this course.

Concavity

We can say a number of things about the concavity of the value function

- Suppose $f(x, \theta)$ is (quasi)concave and the other assumptions for the maximum theorem hold. Then $V(\theta)$ is (quasi)concave.
- Suppose we're solving $\max_{x \in X} f(x; \theta)$ for convex Θ and f is linear (or affine) in θ. Then $V(\theta)$ is convex.

The Envelope Theorem

Characterizing the derivative of the value function is surprisingly useful.

Let x_1 solve $\max_{x \in X} f(x, \theta)$ at θ_1 with value function $V(\theta)$. Then θ_1 must solve

$$\min_{\theta \in \Theta} V(\theta) - f(x_1, \theta)$$

so, $V(\theta)$ and $f(x_1, \theta)$ must be tangent

The Envelope Theorem

We can do the same trick for constrained maximization problems,

$$\max f(x;\theta) \text{ s.t. } g(x;\theta) \leq 0$$

then θ_1 solves

$$\min_{\theta \in \Theta} V(\theta) - f(x, \theta) \text{ s.t. } g(x_1;\theta) \leq 0$$

The Envelope Theorem

$$\min_{\theta \in \Theta} V(\theta) - f(x_1, \theta) \text{ s.t. } g(x_1; \theta) \leq 0$$

has KKT conditions

$$V_\theta(\theta_1) = f_\theta(x_1, \theta_1) - \lambda g_\theta(x_1, \theta_1)$$

Envelope Theorem

We can say a little more. Under constraint qualification, we "know" that θ_1 solves

$$\min_{\theta \in \Theta} V(\theta) - f(x_1, \theta) + \lambda(\theta) g(x_1, \theta)$$

where $\lambda(\theta)$ is the λ from the KKT conditions. Applying the envelope theorem and KKT conditions gives

$$V_\theta(\theta_1) = f_\theta(x_1, \theta_1) - \lambda(\theta_1) g_\theta(x_1, \theta_1)$$

This again gives us the intuition that the multiplier is the shadow price of the constraint.

The Envelope Theorem

We're cheating a bit here.

- We need the KKT conditions to hold at a max for this to make any sense.
- It's not obvious that $V(\theta)$ is differentiable. In fact, it's easy to draw a picture where it isn't.

Theorem

Let $V(\theta)$ be the value function and $\phi(\theta)$ be the policy function. If f, g, and ϕ are continuously differentiable and $Dg(\phi(\theta), \theta)$ has full rank then V is differentiable and

$$DV(\theta) = -\lambda' D_\theta g(\phi(\theta), \theta) + D_\theta f(\phi(\theta), \theta)$$

Envelope Theorem

Asking for the policy function to be differentiable sucks.

- If the FOCs are necessary and sufficient, we could apply the implicit function theorem to them to get this.
- Alternatively, properties like monotonicity or concavity almost imply differentiability.
  - If, for instance, we can show the value function is concave, then the envelope theorem basically holds.

Producer Theory

Consider a cost minimization problem:

$$C(r, w, \bar{y}) = \min_{k,l \in \mathbb{R}_+} rk + wl$$
$$\text{s.t. } f(k, l) \geq \bar{y}$$

Intuitively, it seems like observing a firm's costs should give us a lot of information about it's inputs. From the envelope theorem

$$\frac{\partial C}{\partial w} = \frac{\partial [rk + wl]}{\partial w} = l(r, w, \bar{y})$$

So changes in costs identify input demands. We can also see that

$$\frac{\partial C}{\partial \bar{Y}} = \lambda(r, w, \bar{y})$$

the multiplier identifies the marginal cost of increasing output.

Consumer Theory

What about the consumer problem

$$\max_{x \in \mathbb{R}^n_+} u(x)$$
$$\text{s.t. } p \cdot x \leq m$$

Applying the envelope theorem to the indirect utility function

$$\frac{\partial v}{\partial p_i} = \lambda x_i(p, m)$$

and

$$\frac{\partial v}{\partial m} = -\lambda$$

so

$$x(p, m) = \frac{-\nabla_p v(p, m)}{\frac{\partial v}{\partial m}(p, m)}$$

Comparative Statics

Consider the unconstrained maximization problem

$$\max_{x \in X} f(x; \theta).$$

This has first order conditions

$$\nabla_x f(x; \theta) = 0.$$

We can use this to describe $\nabla_\theta x(\theta)$

Comparative Statics

From the implicit function theorem

$$D_\theta x = -(D^2_{x,x} f)^{-1} D^2_{x,\theta} f$$

Moreover, we know that $D^2_{x,x} f$ is negative semi-definite, from the first order conditions. So signing the cross-partial tells us a lot about the policy function.

Profit Maximization

Consider the firm problem

$$\max qP(q;\theta)$$

where $\theta \in \mathbb{R}$ is parameterizes inverse demand so that $dP/d\theta > 0$. From the previous logic,

$$sign\left(\frac{dq}{d\theta}\right) = sign\left(P_\theta + qP_{q\theta}\right)$$

Whether a shift in demand increases or decreases output depends on $P_{q\theta}$.

Risk Comparative Statics

Consider the problem of an individual with in initial wealth W_0 trying to decide how much to invest in a risky asset with return $r \sim f(r)$, $E(r) > 0$, i.e.

$$\max_{x \in [0, W_0]} \int_{-\infty}^{\infty} u(W_0 - x + x(1+r)) f(r)\, dr$$

Assume u is strictly concave, so this has a unique solution.

Risk

This has first order condition

$$g(x, W_0) := \int r u'(W_0 + rx) f(r)\, dr = 0$$

- Clearly $x = 0$ doesn't solve this. A risk averse consumer is willing to take on some risk.
- A natural assumption would be that risk aversion is decreasing in wealth. A natural measure of risk aversion is

  $$A(w) = -\frac{u''(w)}{u'(w)},$$

  assume this is decreasing is w. Would a richer consumer be more willing to take on risk?

Risk

Assuming we're at an interior x, we can use the implicit function theorem to calculate $dx(W_0)/dW_0$.

$$\frac{dx(W_0)}{dW_0} = -g_x^{-1} g_{W_0}$$

where

$$g_x = \int r^2 u''(W_0 + rx) f(r)\, dr$$

and

$$g_{W_0} = \int r u''(W_0 + rx) f(r)\, dr$$

Risk

We have to sign g_{W_0}.

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$$
\begin{aligned}
g_{W_0} &= \int_{-\infty}^{\infty} r u''(W_0 + rx) f(r)\, dr \\
&= \int_{-\infty}^{\infty} -r u'(W_0 + rx) A(W_0 + rx) f(r)\, dr \\
&= \int_{-\infty}^{0} -r u'(W_0 + rx) A(W_0 + rx) f(r)\, dr \\
&+ \int_{0}^{\infty} -r u'(W_0 + rx) A(W_0 + rx) f(r)\, dr \\
&\geq \int_{-\infty}^{0} -r u'(W_0 + rx) A(W_0) f(r)\, dr \\
&+ \int_{0}^{\infty} -r u'(W_0 + rx) A(W_0) f(r)\, dr \\
&= -A(W_0) g(x, W_0) \\
&= 0
\end{aligned}
$$