ELEC-C9430 Electromagnetism (week 2)

Electrostatics, Coulomb's law

Scalar potential

Laplace' and Poisson's equations

Static electric dipole

Image principle

Materials: conductors, insulators, permittivity

Electrostatics: $\frac{\partial}{\partial t} = 0$

$$\nabla \times \bar{E} = \cancel{-\frac{\partial \bar{B}}{\partial t}}$$
$$\nabla \times \bar{H} = \bar{J} + \cancel{\frac{\partial \bar{D}}{\partial t}}$$
$$\nabla \cdot \bar{D} = \rho_v$$
$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = 0$$
$$\nabla \cdot \bar{D} = \rho_v$$

$$\bar{D}(\bar{R}) = \varepsilon_0 \bar{E}(\bar{R})$$

Free-space permittivity $\varepsilon_0 \simeq 8.854 \cdot 10^{-12} \frac{As}{Vm}$

Coulomb's law

q_1 q_2 d

$$F_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 d^2}$$

$$[F] = \frac{As \cdot \cancel{As}}{\frac{\cancel{As}}{V\cancel{m}} m^{\cancel{2}}} = \frac{\overbrace{VAs}^{W}}{m} = \frac{J}{m} = \frac{Nm}{m} = N$$

$$\frac{F_{12}}{q_2} = E_1$$

q R $\bar{a}_R$

$$\bar{E}(\bar{R}) = \frac{q}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$\nabla \times \bar{E} = 0$?

$$\nabla \times \bar{E} = \nabla \times \left(E_R \bar{a}_R + 0 \cdot E_\theta + 0 \cdot E_\phi \right)$$

$E_R(R)$ E_θ E_ϕ

$= 0$

$$\frac{\partial}{\partial\theta} E_R(R) = 0$$

$$\frac{\partial}{\partial\phi} E_R(\phi) = 0$$

Spherical coordinates

$$\nabla f = \bar{a}_R \frac{\partial}{\partial R} f + \bar{a}_\theta \frac{1}{R}\frac{\partial}{\partial\theta} f + \bar{a}_\phi \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} f$$

$$\nabla \times \bar{f} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \bar{a}_R & R\bar{a}_\theta & R\sin\theta\bar{a}_\phi \\ \frac{\partial}{\partial R} & \cancel{\frac{\partial}{\partial\theta}} & \cancel{\frac{\partial}{\partial\phi}} \\ f_R & \cancel{Rf_\theta} & \cancel{R\sin\theta f_\phi} \end{vmatrix} \rightarrow 0$$

$$\nabla \cdot \bar{f} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 f_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} f_\phi$$

$$\nabla^2 f = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial f}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$\nabla \cdot \bar{D} = \rho_v$?

q $\vec{E}$

$\bar{D} = \varepsilon_0 \bar{E}$

$\bar{a}_n = \bar{a}_R$

$d\bar{S} = \bar{a}_R dS$

R q V

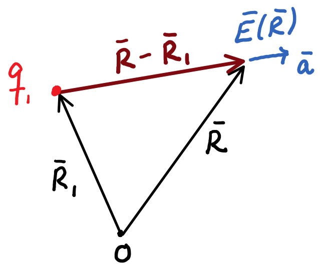
$\bar{a}_R dS$

$$\int \underbrace{\nabla \cdot \bar{D}}_{\rho_v} \, dV = \oint \bar{D} \cdot \underbrace{d\bar{S}}_{a_R \, dS} = \oint D(R) \, dS = D(R) \oint dS = 4\pi R^2 D(R)$$

$$= q \qquad\qquad \bar{D}(\bar{R}) = \hat{a}_R D(R) \qquad\qquad 4\pi R^2 D(R) = q$$

$$\bar{E}(\bar{R}) = \bar{a}_R \frac{D(\bar{R})}{\varepsilon_0} = \bar{a}_R \frac{q}{4\pi\varepsilon_0 R^2}$$

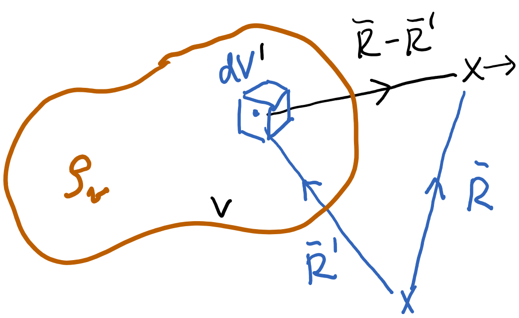
Charge distribution



$$\bar{E}(\bar{R}) = \frac{q_1}{4\pi\varepsilon_0 |\bar{R} - \bar{R}_1|^2} \, \bar{a}$$

$$\bar{a} = \frac{\bar{R} - \bar{R}_1}{|\bar{R} - \bar{R}_1|}$$

$$\bar{E}(\bar{R}) = \frac{q_1 (\bar{R} - \bar{R}_1)}{4\pi\varepsilon_0 |\bar{R} - \bar{R}_1|^3}$$



$$\bar{E}(\bar{R}) = \int_V \frac{\rho_v(\bar{R}') \, (\bar{R} - \bar{R}')}{4\pi\varepsilon_0 \, |\bar{R} - \bar{R}'|^3} \, dV'$$

$$\nabla \cdot \bar{E}(\bar{R}) = \frac{\rho_v(\bar{R})}{\varepsilon_0}$$

(Scalar) potential $V(\bar{R})$

$\nabla \times \bar{E} = 0$

$$\bar{E}(\bar{R}) = -\nabla V(\bar{R}) \Rightarrow \nabla \times \bar{E} = 0$$

(NULL IDENTITY)

$-\nabla \times (\nabla V) = 0$

$[\bar{E}] = \frac{V}{m}$

$[\nabla] = \frac{1}{m}$

Laplace's and Poisson's equations

$$\nabla \cdot \bar{D} = \rho_v \Rightarrow -\nabla \cdot \varepsilon_0 \nabla V = \rho_v \Rightarrow \nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon_0}$$

$$\nabla = \bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z} \qquad \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$$

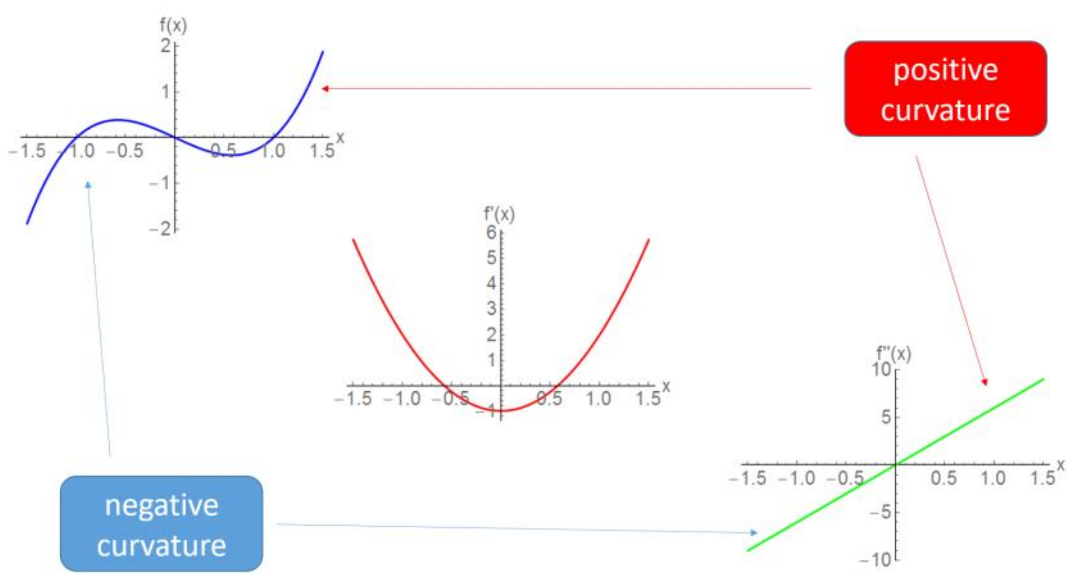
LAPLACE'S OPERATOR

$(= \Delta)$

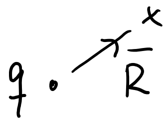
$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_0} \qquad \text{(POISSON)}$$

$$\nabla^2 V = 0 \qquad \text{(LAPLACE)}$$

Second derivative = measure of curvature:

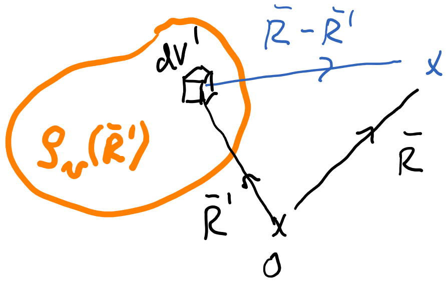


Potential of a point charge (monopole)



$$\bar{E}(\bar{R}) = \frac{q}{4\pi\varepsilon_0 R^2}\bar{a}_R = -\nabla V = -\bar{a}_R \frac{\partial V}{\partial R}$$

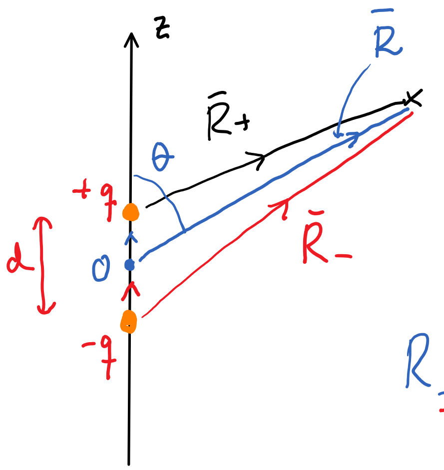
$$V = \frac{q}{4\pi\varepsilon_0 R}$$



$$V(\bar{R}) = \int_V \frac{\rho_v(\bar{R}')\,dV'}{4\pi\varepsilon_0|\bar{R}-\bar{R}'|}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_0}$$

Static electric dipole



$$V(\bar{R}) = ?$$

$$V(\bar{R}) = V_+(\bar{R}) + V_-(\bar{R}) = \frac{+q}{4\pi\varepsilon_0 R_+} + \frac{-q}{4\pi\varepsilon_0 R_-}$$

$$R_{\pm} = \sqrt{\left(\bar{R} \mp \bar{a}_z\frac{d}{2}\right)\cdot\left(\bar{R} \mp \bar{a}_z\frac{d}{2}\right)} = \sqrt{R^2 \mp R\,d\cos\theta + \left(d/2\right)^2}$$

$$= R\sqrt{1 \mp \frac{d\cos\theta}{R} + \cancel{\left(\frac{d}{2R}\right)^2}} \qquad d/R \ll 1$$

$$\simeq R\left(1 \mp \frac{d\cos\theta}{2R}\right)$$

$\bar{a}_R\cdot\bar{a}_z = \cos\theta$

$$\sqrt{1-x} \simeq 1 - \frac{x}{2}$$

$$\frac{1}{1-x} \simeq 1 + x$$

$$V_d(\bar{R}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R\left(1-\frac{d\cos\theta}{2R}\right)} - \frac{1}{R\left(1+\frac{d\cos\theta}{2R}\right)}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 R}\left(1 + \frac{d\cos\theta}{2R} - \left(1 - \frac{d\cos\theta}{2R}\right)\right) = \frac{\overbrace{q\,d\cos\theta}^{p\cos\theta = \bar{p}\cdot\bar{a}_R}}{4\pi\varepsilon_0 R^2}$$

$$= \frac{q}{4\pi\varepsilon_0 R}\left(\left(1 + \frac{d\cos\theta}{2R}\right) - \left(1 - \frac{d\cos\theta}{2R}\right)\right) = \frac{1}{4\pi\varepsilon_0 R^2}$$

Spherical coordinates

$$\nabla f = \bar{a}_R \frac{\partial}{\partial R} f + \bar{a}_\theta \frac{1}{R}\frac{\partial}{\partial \theta} f + \cancel{\bar{a}_\phi \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} f}$$

$$\nabla \times \bar{f} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \bar{a}_R & R\bar{a}_\theta & R\sin\theta\bar{a}_\phi \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f_R & Rf_\theta & R\sin\theta f_\phi \end{vmatrix}$$

$$\nabla\cdot\bar{f} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 f_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}f_\phi$$

$$\nabla^2 f = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

DIPOLE MOMENT

$$\bar{p} = qd\,\bar{a}_z$$

$$\bar{E}_d(\bar{R}) = -\nabla V_d = -\nabla\frac{p\cos\theta}{4\pi\varepsilon_0 R^2}$$

$$= -\bar{a}_R\frac{\partial V_d}{\partial R} - \frac{1}{R}\bar{a}_\theta\frac{\partial V_d}{\partial\theta}$$

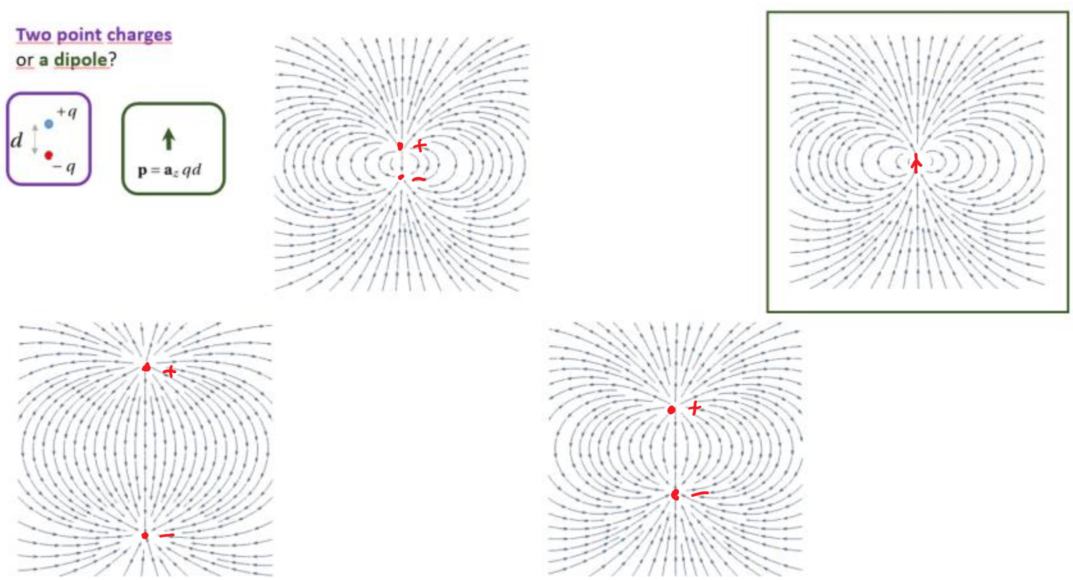
$$= \frac{p}{4\pi\varepsilon_0}\left(\bar{a}_R\, 2R^{-3}\cos\theta + \frac{1}{R}\bar{a}_\theta R^{-2}\sin\theta\right)$$

$$= \frac{p}{4\pi\varepsilon_0 R^3}\left(\bar{a}_R\, 2\cos\theta + \bar{a}_\theta\sin\theta\right)$$

$R^{-2}\cos\theta$ $\quad \frac{\partial\cos\theta}{\partial\theta} = -\sin\theta$

f

$\frac{\partial R^{-2}}{\partial R} = -2R^{-3}$

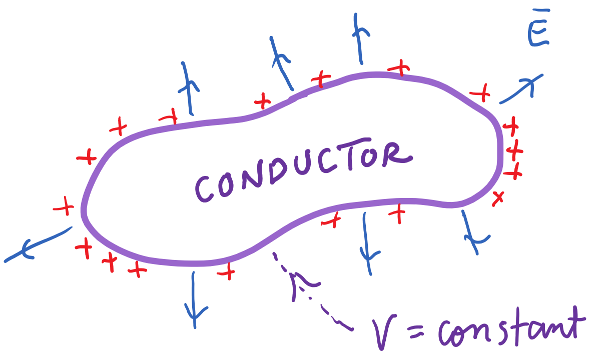


On the accuracy of Taylor series:

!

x	$\frac{1}{1+x}$	$1-x$	error		$\sqrt{1+x}$	$1+\frac{x}{2}$	error
0,1	0,90909090909	0,9	1 %		1,0488088482	1,05	0,114 %
0,01	0,99009900990	0,99	10^{-4}		1,0049875621	1,005	$1,24*10^{-5}$
0,001	0,99900099900	0,999	10^{-6}		1,0004998751	1,0005	$1.25*10^{-7}$
0,0001	0,999900009999	0,9999	10^{-8}		1,0000499988	1,00005	$1.25*10^{-9}$

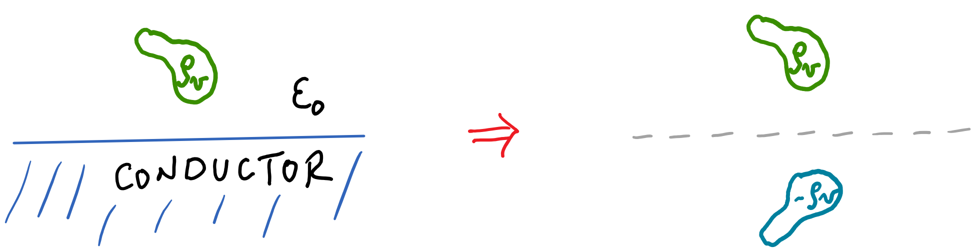
Conductor (perfect electric conductor)

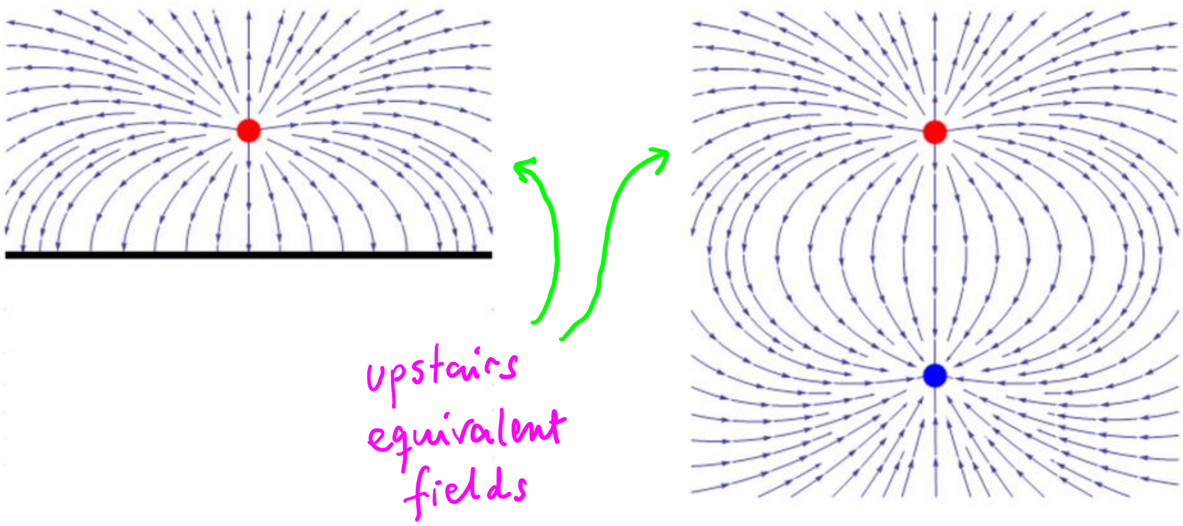


(charges can move freely)

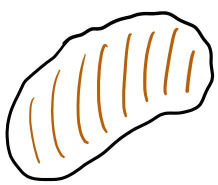
Potential is constant on the conductor surface

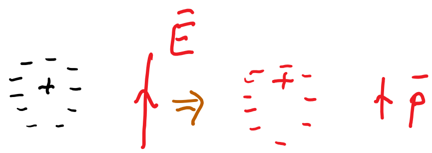
Image principle (mirror image)





Dielectric materials, insulators





$\bar{D} = \varepsilon_0 \bar{E}$

Polarization vector (dipole moment density) $\bar{P} = n\,\bar{p}$ (n: $\frac{1}{m^3}$, $\bar{p}$: Asm)

$[\bar{P}] = \frac{As}{m^2} = [\bar{D}]$

$= [\varepsilon_0][\bar{E}]$

$= \frac{As}{Vm} \cdot \frac{V}{m}$

$$\bar{D} = \varepsilon_0 \bar{E} + \underbrace{\bar{P}}_{\varepsilon_0 \chi_e \bar{E}} = \underbrace{\varepsilon_0 (1 + \chi_e)}_{\varepsilon = \varepsilon_r \varepsilon_0} \bar{E}$$

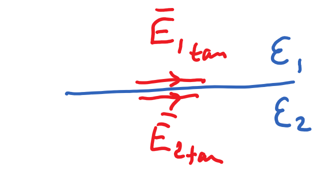
Electric susceptibility χ_e

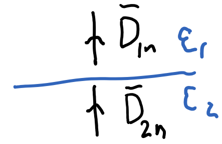
Permittivity ε

Relative permittivity $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$

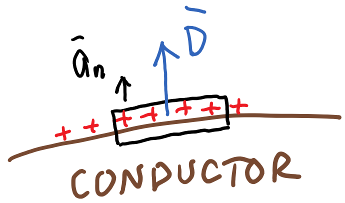
Boundary/interface conditions
(no surface charges)

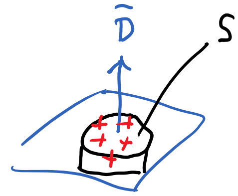
- tangential electric field continuous
- normal flux density continuous





$$\begin{cases} \bar{a}_n \cdot \bar{D}_1 = \bar{a}_n \cdot \bar{D}_2 \\ \bar{a}_n \times \bar{E}_1 = \bar{a}_n \times \bar{E}_2 \end{cases}$$





$\bar{a}_n dS$

$$\int_{\ominus} \underbrace{\nabla \cdot \bar{D}}_{\rho_v} \, dV = \oint \bar{D} \cdot d\bar{S}$$

$$= DS$$

$$= \rho_s S$$

$$\boxed{D = \rho_s}$$

Flux density is the surface charge density on conductor surface