Solutions to Exploratory Exercises

Problem 1. Many composed statements (in mathematics and elsewhere) can be written using the logical symbols $\wedge$ ("and"), $\vee$ ("or"), $\to$ ("if...then..."), $\neg$ ("not"), $\leftrightarrow$ ("if and only if", "is equivalent to"). Whether these composed statements are true or false depends on whether or not the *elementary* statements that they are composed of are true. For example, the composed statement "it rains and it is cold" is true precisely if the elementary statements "it rains" and "it is cold" are both true. Thus, we can *define* the truth value of the composed statement $A \wedge B$ by the following *truth table*, where T and F denote the truth values "true" and "false" respectively:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

(a) Make sure you understand the meaning of this truth table.
(b) Write similar truth tables that define the symbols $\wedge$ ("and"), $\vee$ ("or"), $\neg$ ("not"), $\leftrightarrow$ ("if and only if", "is equivalent to").
(c) Please agree with me that the composed statement $x > 3 \to x^2 > 9$ should be true for all values of x. What is the truth value of the elementary statements $x > 3$ and $x^2 > 9$, when $x = 0$? When $x = 4$? When $x = -4$?
(d) Using your answer in (c), write down a truth table for the implication symbol $\to$.

Solution:
(a) The statement $A \wedge B$ is True if A and B are both True and False otherwise.

(b)

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

A	B	$\neg A$	$\neg B$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

(c) If the statement $x > 3$ is true then the statement $x^2 > 9$ is also true. Hence the implication is true.
i) When $x = 0$, the truth value is *False* for both statements.
ii) When $x = 4$, the truth value is *True* for both statements.
ii) When $x = -4$, the truth value is *False* for the first statement and *True* for the second.
(d) Let A and B be the statements $A = \{x > 3\}$ and $B = \{x^2 > 9\}$

A	B	$A \to B$
T	T	T
F	T	T
F	F	T

Problem 2. Many mathematical statements also contain quantifiers such as $\forall$ ("for all") and $\exists$ ("there exists").
(a) Let $F(x, y)$ be the predicate "x and y are friends". Interpret the following two statements in natural language. Are they different?
- $\forall x \exists y : F(x, y)$
- $\exists y \forall x : F(x, y)$

(b) What are the negations (opposites) of the statements in the previous question?
(c) What is the negation of an "all-quantified" statement $\forall x P(x)$ (where P is an arbitrary predicate)?
(d) What is the negation of an "exists-quantified" statement $\exists x P(x)$?

Solution:

(a)
- Everyone has at least one friend.
- There exist someone who is friends with everyone.

No, they are not same.

(b)
- There is someone who doesn't have a friend
- There is no one who is friends with everyone.

(c) $\exists x \neg P(x)$
(d) $\forall x \neg P(x)$

Problem 3. A set is nothing more than a collection of things (*elements*). Examples of sets include the set $\mathbb{R}$ of all real numbers, the set of all blue cars, the set of subsets of $\mathbb{R}$, etcetera. The language of sets contains the symbols $\in$ ("is a member of") and $\subseteq$ ("is contained in").

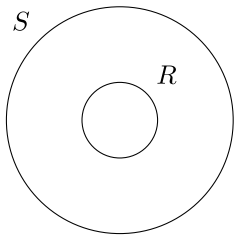
(a) What does it mean that two sets are equal? Formulate this as a logical statement using the logical symbols $\forall$, $\leftrightarrow$, and the set theory symbol $\in$.
(b) Draw a Venn diagram that describes the statement

$$\forall x : x \in R \rightarrow x \in S,$$

where R and S are sets. Can you formulate this statement purely in the language of set theory?

Solution:

(a) Let A and B be two sets, $A = B \leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$. For extra detail the universal set could be taken to be $A \cup B$ and we could write $\forall x \in A \cup B(...)$.
(b) $R \subseteq S$



Problem 4. The *power set* $P(S)$ consists of all subsets of the set S. The *cartesian product* $S \times T$ of two sets is the set $\{(s,t) : s \in S, t \in T\}$.

(a) Let $S = \{a, b, c\}$, $T = \{1, 2\}$. Write down $S \times T$, $P(S)$ and $P(T)$.
(b) Let S be a finite set with $|S| = n$ elements and let T be a finite set with $|T| = m$ elements. How many elements does $S \times T$ have?
(c) How many elements does $P(S)$ have, if S has n elements?

Solution:

(a) $S \times T = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$,
$P(T) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$
(b) $|S \times T| = m \times n$
(c) $|P(T)| = 2^n$. See, for example, the lectures notes for a proof.

<u>Solutions to Additional Exercises</u>

Exercise 1

(i) Come up with three sentences (that can be true or false), and denote them by p, q and r. Formulate the following eight composed sentences in natural language, and convince yourself that they "should be" pairwise equivalent, according to your intuition.

p: It is sunny
q: The places are warm
r: John is happy

(a) $p \rightarrow q$: If it is sunny then the place is warm. $\neg q \rightarrow \neg p$: If the place is not warm, then it is not sunny. The others follows in similar manner.(If not clear,I could detail all the other phrases).

(b) $p \leftrightarrow q$: It is Sunny if and only if the place is warm. and
$(p \wedge q) \vee (\neg p \wedge \neg q)$: It is sunny and the places is warm or it is not sunny and the place is not warm. The others follows in similar manner.

(ii) Prove using truth tables that they are indeed equivalent.

(a) $p \rightarrow q$ and $\neg q \rightarrow \neg p$

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$	$p \rightarrow q$
1	1	0	0	1	1
1	0	1	0	0	0
0	1	0	1	1	1
0	0	1	1	1	1

(b) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

p	q	$\neg q$	$\neg p$	$p \wedge q$	$\neg p \wedge \neg q$	$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
1	1	0	0	1	0	1	1
1	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	1	1	1	1

(c) $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge q \to p)$	$p \leftrightarrow q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

(d) $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$

p	q	r	$p \to r$	$q \to r$	$p \wedge q$	$(p \wedge q) \to r$	$(p \to r) \vee (q \to r)$
1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	0
1	0	1	1	1	0	1	1
1	0	0	0	1	0	1	1
0	1	1	1	1	0	1	1
0	1	0	1	0	0	1	1
0	0	1	1	1	0	1	1
0	0	0	1	1	0	1	1

Exercise 2

Are the following sentences true or false?

(a) $0 \in \{0\}$. True
(b) $\{0\} \in 0$. False
(c) $0 \in \mathbb{Z}$. True
(d) $0 \subset \mathbb{Z}$. False
(e) $\{0\} \subset \mathbb{Z}$. True
(f)) $0 \in \{0, \{0\}, \{0, \{0, \{0\}\}\}\}$. True
(g) $0 \in \emptyset$ False
(h) $\emptyset \in \{0\}$ False
(i) $\emptyset \subset \{0\}$ True
(j) $\emptyset = \{0\}$ False

Exercise 3

Show that if A and B are sets such that $A \times B = B \times A$, then either $A = B$ or $A = \emptyset$ or $B = \emptyset$.

If $A = \emptyset$ or $B = \emptyset$ then the statement is immediately true since the conclusion is true. Suppose A and B are both non-empty sets. For all $a \in A$ and for all $b \in B$, by definition of $A \times B$, $(a, b) \in A \times B$. Since $A \times B = B \times A$ (by hypothesis) we also have $(a, b) \in B \times A$ and so $a \in B$ and $b \in A$. Thus $A = B$.

Exercise 4

Show that if $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.

Let $(x, y) \in A \times C$. Then $x \in A$ and $y \in C$ and so $x \in B$ and $y \in D$. Hence $(x, y) \in B \times D$. Therefore, $A \times C \subseteq B \times D$. Or symbolically

$$\forall x \forall y ((x, y) \in A \times C \to x \in A \wedge y \in C \to x \in B \wedge y \in D \to (x, y) \in B \times D)$$

Exercise 5

Write down the truth table of the composed statement $(p \vee q) \to (p \wedge \neg r)$.

p	q	r	$\neg r$	$p \vee q$	$p \wedge \neg r$	$(p \vee q) \to (p \wedge \neg r)$
1	1	1	0	1	1	1
1	1	0	1	0	1	0
1	0	1	0	1	0	1
1	0	0	1	1	0	1
0	1	1	0	1	0	1
0	1	0	1	0	0	1
0	0	1	0	1	0	1
0	0	0	1	1	0	1

Exercise 6

Let $L(x, y)$ be the sentence "x loves y". Write the following sentences using connectives, quantors, equality signs, and the elementary sentence L. Let $r =$Raymond.

a) Everybody loves Raymond. $\forall x L(x, r)$
b) Everybody loves somebody. $\forall x \exists y L(x, y)$
c) There exists somebody who everybody loves. $\exists y \forall x L(x, y)$
d) Nobody loves everybody. $\forall y \exists x \neg L(y, x)$
e) There is some person, whom Raymond does not love. $\exists x \neg L(r, x)$.
f) There is some person, whom nobody loves. $\exists y \forall x \neg L(x, y)$
g) There is exactly one person, whom everybody loves.

$$\exists y(\forall x(L(x, y)) \wedge \forall z(\forall x(L(x, z) \rightarrow z = y)))$$

Here are using the following formulation of the definition of $\exists! =$ "*there exists a unique*".

$$\exists! x P(x) \longleftrightarrow \exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$$

See for example `https://en.wikipedia.org/wiki/Uniqueness_quantification`

h) Raymond loves exactly two persons.

$$\exists x, y(x \neq y \wedge L(r, x) \wedge L(r, y) \wedge \forall z(x \neq z \wedge y \neq z \rightarrow \neg L(r, z)))$$

i) Everybody loves themselves. $\forall x L(x, x)$.
j) There is somebody, who only loves themself. $\exists y(L(y, y) \wedge \forall x(L(y, x) \rightarrow x = y))$.