MEC-E8003 Beam, plate and shell models, examples 3

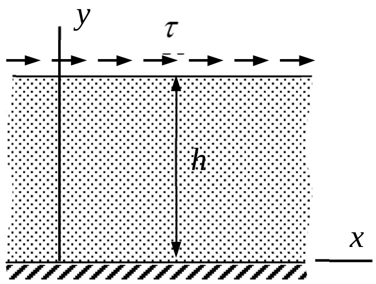
1. The elasticity matrices for *an isotropic material* are the same no matter the *orthonormal* coordinate systems. Consider the elasticity tensor of plane stress in Cartesian $(x.y)$ – and polar (r,ϕ) – coordinate systems and show that

$$\vec{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix} \text{ where}$$

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Solution Discussed during the calculation examples session

2. External shear stress τ is acting on a layer of elastic isotropic material. Young's modulus E and Poisson's ratio ν of the material are constants. Determine stress and displacement in the layer. Assume that stress and displacement components depend on y only and that the external volume force is negligible. Use the component forms of plane stress



$$\begin{Bmatrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + f_x \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + f_y \end{Bmatrix} = 0 , \quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}.$$

Answer $\sigma_{xx} = 0$, $\sigma_{yx} = \tau$, $\sigma_{yy} = 0$, $u = \dfrac{\tau}{G} y$, $v = 0$,

3. Let us consider the principle of virtual work *without* 'a priori' symmetry assumption $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$, when the displacement gradient is expressed as the sum of its symmetric and antisymmetric parts so that $\delta\vec{\vec{\varepsilon}} = \nabla\delta\vec{u} - \delta\vec{\vec{\phi}}$ in which $\delta\vec{\vec{\phi}} = -\delta\vec{\vec{\phi}}_{\mathrm{c}}$. Show that

$$\delta W = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\mathrm{c}}) dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} ,$$

implies e.g. the balance laws of continuum mechanics $\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}} = 0$ and $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$ in V.

Solution Discussed during the calculation examples session

4. Derive the component forms of the thin slab equilibrium equation $\nabla\cdot\vec{N}+\vec{b}=0$ in the polar coordinate system.

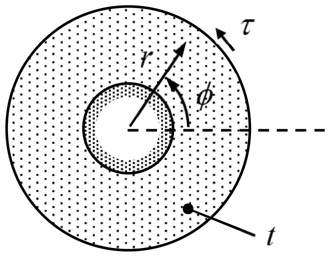
Answer $\begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r}+\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial\phi})+b_r \\ \dfrac{\partial N_{r\phi}}{\partial r}+\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial\phi})+b_\phi \end{Bmatrix}=0$

5. Derive the component form of the thin slab model constitutive equation $f(\vec{N},\vec{u})=0$ in the polar coordinate system starting from the stress resultant definition, stress-strain relationship, and elasticity tensor of the plane stress

$$\vec{N}=\int\vec{\sigma}dn\,,\quad \vec{\sigma}=\vec{\vec{E}}:\nabla\vec{u}\,,\quad \vec{\vec{E}}=\begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}}[E]_\sigma\begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix},$$

Answer $\begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t[E]_\sigma\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial v}{\partial\phi}+u) \\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u}{\partial\phi}-v) \end{Bmatrix},\ N_{\phi r}=N_{r\phi}\,.$

6. A thin slab of inner radius $r=\varepsilon R$ and outer radius $r=R$ is loaded by tangential traction $\vec{t}=\tau\vec{e}_\phi$ on the outer edge $r=R$ (shear stress τ is constant). Assuming rotation symmetry i.e. that stress and displacement components depend only on the distance r from the center point and $u_r\equiv u=0$, solve for the stress and displacement. Material is linearly elastic and isotropic with material parameters E and ν. External distributed forces vanish. Use the component forms



$$\begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r}+\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial\phi})+b_r \\ \dfrac{\partial N_{r\phi}}{\partial r}+\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial\phi})+b_\phi \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t[E]_\sigma\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial v}{\partial\phi}+u) \\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u}{\partial\phi}-v) \end{Bmatrix}.$$

Answer $N_{rr}=N_{\phi\phi}=0\,,\ N_{r\phi}=\tau t\dfrac{R^2}{r^2}\,,\ \ u=0\,,\ v=-\dfrac{\tau}{2G}(\dfrac{R^2}{r}-\dfrac{r}{\varepsilon^2})$

7. Virtual work expression of a linearly elastic bar supported by a spring at the right end $x = L$ ($n = 1$) is given by

$$\delta W = \int_0^L -(EA\frac{du}{dx}\frac{d\delta u}{dx})dx + \int_0^L (b\delta u)dx - (ku\delta u)_{x=L},$$

in which $EA = EA(x)$ and k, b are constants. Displacement vanishes at the left end $x = 0$ ($n = -1$) of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W = 0 \; \forall \delta u \in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x = 0$.

Answer $\frac{d}{dx}(EA\frac{du}{dx}) + b = 0$ in $(0, L)$, $EA\frac{du}{dx} + ku = 0$ at $x = L$, and $u = 0$ at $x = 0$

8. Virtual work expression of a torsion bar is given by

$$\delta W = \int_0^L (-\frac{d\delta\phi}{dx}GJ\frac{d\phi}{dx} - \delta\phi k\phi + \delta\phi c)dx + (T\delta\phi)_{x=L}$$

in which $c(x)$ and T represent the given loading. Deduce *in detail* the differential equation for the rotation $\phi(x)$ and the boundary conditions implied by principle of virtual work and the fundamental lemma of variation calculus. The unknown $\phi(x)$ and the given $GJ(x)$, $k(x)$ and $c(x)$ are assumed to have continuous derivatives of all orders. In addition, $\delta\phi$ and ϕ are assumed to vanish at $x = 0$.

Answer $\frac{d}{dx}(GJ\frac{d\phi}{dx}) - k\phi + c = 0$ in $(0, L)$, $GJ\frac{d\phi}{dx} - T = 0$ at $x = L$, $\phi = 0$ at $x = 0$.

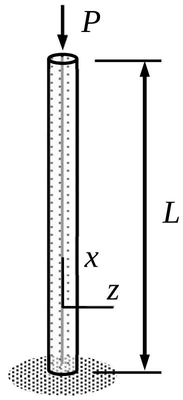
9. Virtual work expression of a Bernoulli beam, clamped at the left end $x = 0$ and loaded by force F and moment R at the right end $x = L$ of solution domain $\Omega = (0, L)$, is given by

$$\delta W = \int_0^L (M\frac{d^2\delta w}{dx^2} + b\delta w)dx + (F\delta w - R\frac{d\delta w}{dx})_{x=L}.$$

Use the principle of virtual work $\delta W = 0 \; \forall \delta w \in U$ to derive the beam equilibrium equation in Ω, natural boundary conditions on $x = L$, and essential boundary conditions on $x = 0$. Functions of set U have continuous derivatives up to the fourth order in Ω. In addition, a function of U vanishes at $x = 0$ as does also its first derivative.

Answer $\frac{d^2M}{dx^2} + b = 0$ in $(0, L)$, $-\frac{dM}{dx} + F = 0$ and $M - R = 0$ at $x = L$, $w = \frac{dw}{dx} = 0$ at $x = 0$

10. When displacement is confined to the $xz-$plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx \,.$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus.

Answer $EI\dfrac{d^4w}{dx^4} + P\dfrac{d^2w}{dx^2} = 0$ in $(0,L)$, $EI\dfrac{d^3w}{dx^3} + P\dfrac{dw}{dx} = EI\dfrac{d^2w}{dx^2} = 0$ at $x = L$,

$w = \dfrac{dw}{dx} = 0$ at $x = 0$.

The elasticity matrices for *an isotropic material* are the same no matter the *orthonormal* coordinate systems. Consider the elasticity tensor of plane stress in Cartesian $(x.y)-$ and polar $(r,\phi)-$ coordinate systems and show that

$$\vec{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix} \text{ where}$$

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Solution

Let us start with the elasticity tensor in the polar coordinate system and the relationship between the basis vectors of the Cartesian and polar coordinate systems

$$\vec{\vec{E}} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix} \text{ and } \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi \\ -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}.$$

Using

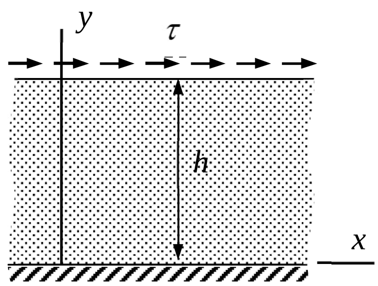
$$\begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix} = \begin{bmatrix} \mathrm{c}^2\phi & \mathrm{s}^2\phi & \mathrm{c}\phi\mathrm{s}\phi \\ \mathrm{s}^2\phi & \mathrm{c}^2\phi & -\mathrm{c}\phi\mathrm{s}\phi \\ -2\mathrm{c}\phi\mathrm{s}\phi & 2\mathrm{c}\phi\mathrm{s}\phi & \mathrm{c}^2\phi-\mathrm{s}^2\phi \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix} = [T]\begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix}$$

gives the expression of the elasticity matrix in the Cartesian $(x.y)-$ coordinate system:

$$\vec{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} [T]^{\mathrm{T}}[E]_\sigma[T] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix}.$$

What remains is showing that $[T]^{\mathrm{T}}[E]_\sigma[T]=[E]_\sigma$. See Mathematica notebook for the simplification of the left-hand side.

External shear stress τ is acting on a layer of elastic isotropic material. Young's modulus E and Poisson's ratio ν of the material are constants. Determine stress and displacement in the layer. Assume that stress and displacement components depend on y only and that the external volume force is negligible. Use the component forms of plane stress



$$\begin{Bmatrix} \partial\sigma_{xx}/\partial x+\partial\sigma_{yx}/\partial y+f_x \\ \partial\sigma_{xy}/\partial x+\partial\sigma_{yy}/\partial y+f_y \end{Bmatrix}=0,\quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y+\partial v/\partial x \end{Bmatrix}.$$

Solution

Solution to stress and displacement follows from the equilibrium equations, constitutive equations, and boundary conditions. In the layer problem, $u_x=u(y)$, $u_y=v(y)$, $f_x=f_y=0$. At the lower edge $u(0)=0$ and at the upper edge $\sigma_{yx}(h)=\sigma_{xy}(h)=\tau$ (stress is symmetric).

The two equilibrium equations and three constitutive equations simplify to

Equilibrium $\quad \dfrac{d\sigma_{yx}}{dy}=0$ and $\dfrac{d\sigma_{yy}}{dy}=0$

Constitutive $\quad \sigma_{xx}=\dfrac{\nu E}{1-\nu^2}\dfrac{dv}{dy}$, $\sigma_{yy}=\dfrac{E}{1-\nu^2}\dfrac{dv}{dy}$, $\sigma_{xy}=\sigma_{yx}=\dfrac{E}{2(1+\nu)}\dfrac{du}{dy}=G\dfrac{du}{dy}$,

In this case, solution to stress and displacement follows by considering first the equilibrium equations and using, after that, the constitutive equations. Boundary value problem for the stress components are composed of the equilibrium equations and the boundary condition at the upper edge. Boundary value problems and their solutions to the stress components are

$$\frac{d\sigma_{yx}}{dy}=0 \text{ in } (0,h) \text{ and } \sigma_{yx}(h)=\tau \quad\Rightarrow\quad \sigma_{yx}(y)=\tau. \quad \Leftarrow$$

$$\frac{d\sigma_{yy}}{dy}=0 \text{ in } (0,h) \text{ and } \sigma_{yy}(h)=0 \quad\Rightarrow\quad \sigma_{yy}(y)=0. \quad \Leftarrow$$

Knowing the stress, boundary value problems for the displacement components are composed of the constitutive equations and the boundary condition on the lower edge. Boundary value problems and their solutions to the displacement components are

$$G\frac{du}{dy}=\sigma_{yx}=\tau \text{ in } (0,h) \text{ and } u(0)=0 \quad\Rightarrow\quad u(y)=\frac{\tau}{G}y. \quad \Leftarrow$$

$$\frac{E}{1-\nu^2}\frac{dv}{dy}=\sigma_{yy}=0 \text{ in } (0,h) \text{ and } v(0)=0 \quad\Rightarrow\quad v(y)=0. \quad \Leftarrow$$

Finally, the third constitutive equation, not used above, gives

$$\sigma_{xx} = \frac{\nu E}{1-\nu^2}\frac{dv}{dy} = 0\,.$$

Let us consider the principle of virtual work *without* ‘a priori’ symmetry assumption $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$, when the displacement gradient is expressed as the sum of its symmetric and antisymmetric parts so that $\delta\vec{\vec{\varepsilon}} = \nabla\delta\vec{u} - \delta\vec{\vec{\phi}}$ in which $\delta\vec{\vec{\phi}} = -\delta\vec{\vec{\phi}}_{\mathrm{c}}$. Show that

$$\delta W = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\mathrm{c}}) dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ ,$$

Implies, e.g., the balance laws of continuum mechanics $\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}} = 0$ and $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$ in V.

Solution

Symmetry of stress is just the local form of moment of momentum balance law. The condition is often satisfied ‘a priori’ but it can also be embedded in the virtual work expression form. Also now, the fundamental theorem of calculus (integration by parts) and the fundamental lemma of variation calculus are the tools for deriving a boundary value problem starting from a given virtual work expression. In addition, division of displacement gradient into its symmetric and antisymmetric parts

$$\nabla\delta\vec{u} = \frac{1}{2}[\nabla\delta\vec{u} + (\nabla\delta\vec{u})_{\mathrm{c}}] + \frac{1}{2}[\nabla\delta\vec{u} - (\nabla\delta\vec{u})_{\mathrm{c}}] = \delta\vec{\vec{\varepsilon}} + \delta\vec{\vec{\phi}}$$

is needed. According to principle of virtual work without the assumption of stress symmetry (generic form)

$$\delta W = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\mathrm{c}}) dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \qquad \Leftrightarrow$$

$$\delta W = -\int_V \vec{\vec{\sigma}} : (\nabla\delta\vec{u})_{\mathrm{c}} dV + \int_V \vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}_{\mathrm{c}} dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ .$$

Vector identity $\vec{\vec{a}} : (\nabla\vec{b})_{\mathrm{c}} = \nabla \cdot (\vec{\vec{a}} \cdot \vec{b}) - (\nabla \cdot \vec{\vec{a}}) \cdot \vec{b}$ adapted to the present case with $\vec{\vec{a}} = \vec{\vec{\sigma}}$ and $\vec{b} = \vec{u}$ gives the form

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}}) \cdot \delta\vec{u} dV - \int_V \nabla \cdot (\vec{\vec{\sigma}} \cdot \delta\vec{u}) dV + \int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}_{\mathrm{c}}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ .$$

Use of the divergence theorem, also known as Gauss's theorem, in the second term on the right-hand side gives first

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}}) \cdot \delta\vec{u} dV - \int_A (\vec{n} \cdot \vec{\vec{\sigma}}) \cdot \delta\vec{u} dA + \int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}_{\mathrm{c}}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}}$$

and after dividing the boundary into disjoint parts $A = A_u \cup A_t$ and $A_u \cap A_t = \varnothing$ and assuming that $\delta\vec{u} = 0$ on A_u

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}}) \cdot \delta\vec{u} dV + \int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}) dV - \int_{A_t} (\vec{n} \cdot \vec{\vec{\sigma}} - \underline{\vec{t}}) \cdot \delta\vec{u} dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ .$$

The purpose of the manipulation above was just to obtain a representation that allows the use of fundamental lemma of variation calculus. Selection $\delta\vec{u}=0$ on A_t vanishing and $\delta\vec{\vec{\phi}}=0$, and thereafter $\delta\vec{u}\equiv 0$ (everywhere) imply

$$\nabla\cdot\vec{\vec{\sigma}}+\underline{\vec{f}}=0 \quad \text{in} \quad V$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=0 \quad \text{in} \quad V\,.$$

It is noteworthy that the latter selections implies that $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$ due to the restriction $\delta\vec{\vec{\phi}}=-\delta\vec{\vec{\phi}}_{\mathrm{c}}$. Knowing the conditions above, virtual work expression simplifies to

$$\delta W=-\int_{A_t}(\vec{n}\cdot\vec{\vec{\sigma}}-\underline{\vec{t}})\cdot\delta\vec{u}dA=0 \quad \forall\,\delta\vec{u},\delta\vec{\vec{\phi}}$$

which implies $\vec{n}\cdot\vec{\vec{\sigma}}-\underline{\vec{t}}=0$ on A_t. Taking into account the restriction on $\delta\vec{u}$ on A_u, boundary value problem takes the form

$$\left.\begin{aligned}
&\nabla\cdot\vec{\vec{\sigma}}+\underline{\vec{f}}=0 \quad \text{in} \quad V\,,\\
&\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=0 \quad \text{in} \quad V\,,\\
&\vec{n}\cdot\vec{\vec{\sigma}}-\underline{\vec{t}}=0 \quad \text{on} \quad A_t\,,\\
&\vec{u}-\underline{\vec{u}}=0 \quad \text{on} \quad A_u\,.
\end{aligned}\right\}$$

The first three equations are the local balance laws for momentum and moment of momentum.

Derive the component forms of the thin slab equilibrium equation $\nabla\cdot\vec{\vec{N}}+\vec{b}=0$ in the polar coordinate system.

Solution

The component forms of stress, external force, and gradient operator of the polar coordinate system are

$$\vec{\vec{N}}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{rr} & N_{r\phi}\\ N_{\phi r} & N_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix},\ \vec{b}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}b_r\\ b_\phi\end{Bmatrix},\ \ \nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix},\ \ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix}.$$

Let us start with the terms of stress resultant divergence

$$\nabla\cdot\vec{\vec{N}}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)\,.$$

First term of the gradient simplifies to

$$\vec{e}_r\frac{\partial}{\partial r}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\vec{e}_r\cdot(\frac{\partial N_{rr}}{\partial r}\vec{e}_r\vec{e}_r+\frac{\partial N_{r\phi}}{\partial r}\vec{e}_r\vec{e}_\phi+\frac{\partial N_{\phi r}}{\partial r}\vec{e}_\phi\vec{e}_r+\frac{\partial N_{\phi\phi}}{\partial r}\vec{e}_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\vec{e}_r\frac{\partial}{\partial r}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial N_{rr}}{\partial r}\\ \dfrac{\partial N_{r\phi}}{\partial r}\end{Bmatrix}.$$

Then the same manipulation for the second term of the displacement gradient

$$\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\vec{e}_\phi\frac{1}{r}\cdot(\frac{\partial N_{rr}}{\partial\phi}\vec{e}_r\vec{e}_r+N_{rr}\vec{e}_\phi\vec{e}_r+N_{rr}\vec{e}_r\vec{e}_\phi+\frac{\partial N_{r\phi}}{\partial\phi}\vec{e}_r\vec{e}_\phi+N_{r\phi}\vec{e}_\phi\vec{e}_\phi-N_{r\phi}\vec{e}_r\vec{e}_r+$$

$$\frac{\partial N_{\phi r}}{\partial\phi}\vec{e}_\phi\vec{e}_r-N_{\phi r}\vec{e}_r\vec{e}_r+N_{\phi r}\vec{e}_\phi\vec{e}_\phi+\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi\vec{e}_\phi-N_{\phi\phi}\vec{e}_r\vec{e}_\phi-N_{\phi\phi}\vec{e}_\phi\vec{e}_r)\qquad\Rightarrow$$

$$\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial\phi})\\ \dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial\phi})\end{Bmatrix}.$$

Finally, by combining the terms of the divergence and external loading

$$\nabla\cdot\overset{\leftrightarrow}{N}+\vec{b}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial N_{rr}}{\partial r}+\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial\phi})+b_r\\ \dfrac{\partial N_{r\phi}}{\partial r}+\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial\phi})+b_\phi\end{Bmatrix}=0\,.$$

Derive the component form of the thin slab model constitutive equation $f(\vec{N},\vec{u})=0$ in the polar coordinate system starting from the stress resultant definition, stress-strain relationship, and elasticity tensor of the plane stress

$$\vec{\vec{N}}=\int\vec{\vec{\sigma}}dn\,,\quad \vec{\vec{\sigma}}=\vec{\vec{\vec{\vec{E}}}}:\nabla\vec{u}\,,\quad \vec{\vec{\vec{\vec{E}}}}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}.$$

Solution

Polar coordinate system representations of the gradient expression, planar displacement, and the basis vector derivatives are

$$\nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix},\quad \vec{u}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_r\\ u_\phi\end{Bmatrix},\ \text{and}\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix}.$$

Substitution into the displacement gradient gives

$$\nabla\vec{u}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})(u_r\vec{e}_r+u_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\nabla\vec{u}=\vec{e}_r\vec{e}_r\frac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\phi\frac{\partial u_\phi}{\partial r}+\vec{e}_\phi\vec{e}_r\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\vec{e}_\phi\vec{e}_\phi\frac{1}{r}u_r+\vec{e}_\phi\vec{e}_\phi\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}-\vec{e}_\phi\vec{e}_r\frac{1}{r}u_\phi\quad\Rightarrow$$

$$\nabla\vec{u}=\vec{e}_r\vec{e}_r\frac{\partial u_r}{\partial r}+\vec{e}_\phi\vec{e}_\phi\frac{1}{r}(\frac{\partial u_\phi}{\partial\phi}+u_r)+\vec{e}_r\vec{e}_\phi\frac{\partial u_\phi}{\partial r}+\vec{e}_\phi\vec{e}_r\frac{1}{r}(\frac{\partial u_r}{\partial\phi}-u_\phi)\,.$$

The double inner product with the basis vector combinations of the elasticity tensor gives the stress expression

$$\vec{\vec{\sigma}}=\vec{\vec{\vec{\vec{E}}}}:\nabla\vec{u}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u_r}{\partial r}\\ \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial\phi}+u_r)\\ \dfrac{\partial u_\phi}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u_r}{\partial\phi}-u_\phi)\end{Bmatrix}.$$

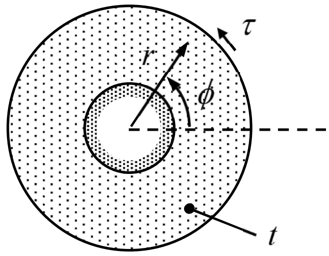
According to the definition, stress resultant is integral of stress over the thickness

$$\vec{\vec{N}}=\int\vec{\vec{\sigma}}dn=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}\frac{tE}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u_r}{\partial r}\\ \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial\phi}+u_r)\\ \dfrac{\partial u_\phi}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u_r}{\partial\phi}-u_\phi)\end{Bmatrix}$$

or in the component form

$$\begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix} = \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_r}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial \phi} + u_r) \\ \dfrac{\partial u_\phi}{\partial r} + \dfrac{1}{r}(\dfrac{\partial u_r}{\partial \phi} - u_\phi) \end{Bmatrix} \text{ and } N_{\phi r} = N_{r\phi}\,.$$

A thin slab of inner radius $r=\varepsilon R$ and outer radius $r=R$ is loaded by tangential traction $\vec{t}=\tau\vec{e}_\phi$ on the outer edge $r=R$ (shear stress τ is constant). Assuming rotation symmetry i.e. that stress and displacement components depend only on the distance r from the center point and $u_r\equiv u=0$, solve for the stress and displacement. Material is linearly elastic and isotropic with material parameters E and ν. External distributed forces vanish. Use the component forms



$$\begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r}+\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial \phi})+b_r \\ \dfrac{\partial N_{r\phi}}{\partial r}+\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial \phi})+b_\phi \end{Bmatrix}=0,\quad \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial v}{\partial \phi}+u) \\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u}{\partial \phi}-v) \end{Bmatrix}.$$

Solution

As stress resultants and displacement components depend only on r, equilibrium equations and the constitutive equations simplify to (notice that the derivatives are ordinary ones as the quantities are known to depend on r only)

$$\frac{dN_{rr}}{dr}+\frac{1}{r}N_{rr}-\frac{1}{r}N_{\phi\phi}=0,\quad \frac{dN_{r\phi}}{dr}+\frac{2}{r}N_{r\phi}=\frac{1}{r^2}\frac{d}{dr}(r^2N_{r\phi})=0 \quad \text{and}$$

$$N_{rr}=\frac{tE}{1-\nu^2}(\frac{du}{dr}+\nu\frac{u}{r}),\quad N_{\phi\phi}=\frac{tE}{1-\nu^2}(\frac{u}{r}+\nu\frac{du}{dr}),\quad N_{r\phi}=tG(\frac{dv}{dr}-\frac{v}{r})=tGr\frac{d}{dr}(\frac{v}{r})$$

Solution to the shear stress resultant $N_{r\phi}$ follows from boundary value problem composed of the equilibrium equation and boundary condition on the outer edge

$$\frac{1}{r^2}\frac{d}{dr}(r^2N_{r\phi})=0 \quad r\in(\varepsilon R,R) \quad \text{and} \quad N_{r\phi}=\tau t \quad \text{at} \quad r=R \quad \Rightarrow \quad N_{r\phi}=\tau t\frac{R^2}{r^2}.$$

After that, solution to displacement component $u_\phi=v(r)$ follows from boundary value problem composed of the constitutive equation and boundary condition on the inner edge

$$tGr\frac{d}{dr}(\frac{1}{r}v)=\tau t\frac{R^2}{r^2} \quad r\in(\varepsilon R,R) \quad \text{and} \quad v=0 \quad \text{at} \quad r=\varepsilon R \quad \Rightarrow \quad v=-\frac{\tau}{2G}(\frac{R^2}{r}-\frac{r}{\varepsilon^2}).$$

Displacement component $u_r=u(r)=0$ by assumption which implies that

$$N_{rr}=N_{\phi\phi}=0.$$

The same solution follows also without the assumption $u_r=u(r)=0$. Eliminating the stress resultants from the second equilibrium equation gives the boundary value problem (on the outer edge $N_{rr}=0$)

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=\frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(ru)]=0 \quad r\in(\varepsilon R,R),$$

$u=0$ at $r=\varepsilon R$ and $\frac{du}{dr}+\nu\frac{u}{r}=0$ at $r=R$ $\Rightarrow$ $u_r=u(r)=0$.

Virtual work expression of a linearly elastic bar supported by a spring at the right end $x=L$ ($n=1$) is given by

$$\delta W=-\int_0^L \left(\frac{d\delta u}{dx}EA\frac{du}{dx}\right)dx+\int_0^L (\delta u b)dx-(\delta u k u)_{x=L} \,,$$

in which $EA=EA(x)$ and $k\,,b$ are constants. Displacement vanishes at the left end $x=0$ ($n=-1$) of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W=0\ \forall \delta u\in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x=0$.

Solution

Fundamental theorem of calculus (integration by parts) and the fundamental lemma of variation calculus are the tools for deriving a boundary value problem starting from a virtual work expression. In the one-dimensional case, for any continuous functions a and b (or values at some point), it holds

$$\int_\Omega a\frac{db}{dx}dx=\sum_{\partial\Omega}(nab)-\int_\Omega \frac{da}{dx}bdx \quad \text{(where } n=\pm 1\text{)},$$

$$a,b\in\mathbb{R}: \qquad ab=0 \quad \forall b \quad \Leftrightarrow \quad a=0\,,$$

$$a,b\in C^0(\Omega): \ \int_\Omega abdx=0 \quad \forall b \quad \Leftrightarrow \quad a=0 \quad \text{in} \ \ \Omega\,.$$

In the present case $\Omega=(0,L)$ and $\partial\Omega=\{0,L\}$. Displacement has continuous derivatives up to and including second order i.e. $u\in C^2(\Omega)$. The constraint on the function set $u=0$ at $x=0$ implies that $\delta u=0$ at $x=0$. Integration by parts gives equivalent forms (the aim is to remove the derivatives from variations in the integral over the domain)

$$\delta W=-\int_0^L \left(\frac{d\delta u}{dx}EA\frac{du}{dx}\right)dx+\int_0^L (\delta u b)dx-(\delta u k u)_{x=L} \quad \Leftrightarrow$$

$$\delta W=\int_0^L \left[\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b\right]\delta u dx-\left[\left(EA\frac{du}{dx}+ku\right)\delta u\right]_{x=L}. \quad (\text{as } \delta u=0 \text{ at } x=0)$$

The purpose of the manipulation above was to obtain a representation which allows the use of fundamental lemma of variation calculus. According to principle of virtual work, $\delta W=0\ \ \forall \delta u\in U$. Let us consider first a subset $U_0\subset U$ for which $\delta u=0$ at $x=L$ so that the boundary term vanishes. Then

$$\delta W=\int_0^L \left[\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b\right]\delta u dx=0 \quad \delta u\in U_0\subset U$$

and the fundamental lemma of variation calculus implies that

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+b=0 \quad \text{in } (0,L).$$

Knowing this and considering the full set U, the variational equation simplifies into

$$\delta W = -[(EA\frac{du}{dx} + ku)\delta u]_{x=L} = 0 .$$

Then, the fundamental lemma of variation calculus implies that

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L .$$

Finally combining the equations to form a boundary value problem (notice that the definition of the function set implies also a boundary condition):

$$\frac{d}{dx}(EA\frac{du}{dx}) + b = 0 \quad \text{in} \ (0, L) ,$$

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L ,$$

$$u = 0 \quad \text{at} \quad x = 0 .$$

Virtual work expression of a torsion bar is given by

$$\delta W = \int_0^L \left(-\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} - \delta\phi k\phi + \delta\phi c\right)dx + (T\delta\phi)_{x=L}$$

in which $c(x)$ and T represent the given loading. Deduce *in detail* the differential equation for the rotation $\phi(x)$ and boundary conditions implied by principle of virtual work and the fundamental lemma of variation calculus. The unknown $\phi(x)$ and the given $GJ(x)$, $k(x)$ and $c(x)$ are assumed to have continuous derivatives of all orders. In addition, $\delta\phi$ and ϕ are assumed to vanish at $x = 0$.

Solution

Here $\Omega = (0, L)$ and $\partial\Omega = \{0, L\}$. Rotation has continuous derivatives up to and including second order, i.e., $\phi \in U \subset C^2(\Omega)$. Function set U is constrained by $\phi = 0$ at $x = 0$, which implies that $\delta\phi = 0$ at $x = 0$. Integration by parts gives equivalent forms

$$\delta W = \int_0^L \left(-\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} - \delta\phi k\phi + \delta\phi c\right)dx + (T\delta\phi)_{x=L} \quad \Leftrightarrow$$

$$\delta W = \int_0^L \left[\frac{d}{dx}\left(GJ \frac{d\phi}{dx}\right) - k\phi + c\right]\delta\phi dx - \sum_{\{0,L\}}\left[n\left(GJ \frac{d\phi}{dx}\right)\delta\phi\right] + (T\delta\phi)_{x=L} \quad \Rightarrow$$

$$\delta W = \int_0^L \left[\frac{d}{dx}\left(GJ \frac{d\phi}{dx}\right) - k\phi + c\right]\delta\phi dx - \left[\left(GJ \frac{d\phi}{dx} - T\right)\delta\phi\right]_{x=L} \quad \text{as} \quad \delta\phi = 0 \text{ at } x = 0.$$

The purpose of the manipulation is to obtain a representation that allows the use of fundamental lemma of variation calculus.

According to principle of virtual work $\delta W = 0 \;\; \forall \delta\phi$. Let us consider first a subset for which $\delta\phi = 0$ at $x = L$ so that the boundary term vanishes. Then

$$\delta W = \int_0^L \left[\frac{d}{dx}\left(GJ \frac{d\phi}{dx}\right) - k\phi + c\right]\delta\phi dx = 0 \quad \forall \delta\phi \text{ satisfying } \delta\phi = 0 \text{ at } x = L$$

and the fundamental lemma of variation calculus implies that

$$\frac{d}{dx}\left(GJ \frac{d\phi}{dx}\right) - k\phi + c = 0 \quad \text{in } (0, L).$$

Knowing this and considering the function set without the additional constraint

$$\delta W = -\left[\left(GJ \frac{d\phi}{dx} - T\right)\delta\phi\right]_{x=L} = 0 \quad \forall \delta\phi.$$

The fundamental lemma of variation calculus implies now

$$GJ \frac{d\phi}{dx} - T = 0 \quad \text{at} \quad x = L.$$

Finally combining the equations to form a boundary value problem (notice that the constraint $\phi = 0$ at $x = 0$ implies a boundary condition) :

$$\frac{d}{dx}(GJ\frac{d\phi}{dx}) - k\phi + c = 0 \quad \text{in } (0, L),$$

$$GJ\frac{d\phi}{dx} - T = 0 \quad \text{at} \quad x = L,$$

$$\phi = 0 \quad \text{at} \quad x = 0.$$

Virtual work expression of a Bernoulli beam, clamped at the left end $x=0$ and loaded by force F and moment R at the right end $x=L$ of solution domain $\Omega=(0,L)$, is given by

$$\delta W=\int_0^L (\frac{d^2\delta w}{dx^2}M+\delta wb)dx+(F\delta w-R\frac{d\delta w}{dx})_{x=L} .$$

Use the principle of virtual work $\delta W=0 \;\forall\delta w\in U$ to derive the beam equilibrium equation in Ω, natural boundary conditions on $x=L$, and essential boundary conditions on $x=0$. Functions of set U have continuous derivatives up to the fourth order in Ω. In addition, a function of U vanishes at $x=0$ as does also its first derivative.

Solution

In MEC-E8003, principle of virtual work is used to derive the equilibrium equation(s) in terms of the stress resultants (like shear forces and bending moments). The constitutive equation, giving the relationship between the stress resultants and kinetic quantities (like displacements and rotations), is a separate story. The mathematical tools needed in the derivation are (one-dimensional case $\Omega\subset\mathbb{R}$) $a,b\in C^0(\Omega)$

$\int_\Omega \frac{d}{dx}(ab)dx=\sum_{\partial\Omega} nab$, where $n=\pm 1$ is the unit outward normal to Ω (on $\partial\Omega$)

$$\int_\Omega abdx=0 \quad \forall b \quad \Leftrightarrow \quad a=0 \;\text{ in }\Omega .$$

Integration by parts once in the first term gives an equivalent form (notice that $\delta w\in U$ and therefore $\delta w=d\delta w/dx=0$ at $x=0$

$$\delta W=\int_0^L (M\frac{d^2\delta w}{dx^2}+b\delta w)dx+(F\delta w-R\frac{d\delta w}{dx})_{x=L} \quad \Leftrightarrow$$

$$\delta W=\int_0^L (-\frac{dM}{dx}\frac{d\delta w}{dx}+b\delta w)dx+[(M-R)\frac{d\delta w}{dx}]_{x=L}+(F\delta w)_{x=L} .$$

Integration by parts second time in the first term gives also an equivalent form

$$\delta W=\int_0^L (\frac{d^2M}{dx^2}+b)\delta wdx+[(-\frac{dM}{dx}+F)\delta w]_{x=L}+[(M-R)\frac{d\delta w}{dx}]_{x=L} .$$

According to the principle of virtual work $\delta W=0 \;\;\forall\delta w\in U$. Let us first consider a subset $U_0\subset U$ for which $\delta w=d\delta w/dx=0$ at $x=L$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W=\int_0^L (\frac{d^2M}{dx^2}+b)\delta wdx=0 \quad \forall\delta w\in U_0 \qquad \Leftrightarrow \qquad \frac{d^2M}{dx^2}+b=0 \quad \text{in} \quad (0,L) .$$

Let us next consider a subset $U_0 \subset U$ for which only $d\delta w / dx = 0$ at $x = 0$ so that the last boundary term of the virtual work expression vanishes. Also, the first term can be omitted due to the equilibrium equation. The natural boundary condition follows from the fundamental lemma of variation calculus:

$$\delta W = [(-\frac{dM}{dx} + F)\delta w]_{x=L} = 0 \quad \forall \delta w \in U_0 \quad \Leftrightarrow \quad -\frac{dM}{dx} + F = 0 \quad \text{at} \quad x = L .$$

Finally, let us consider a subset $U_0 \subset U$ for which only $\delta w = 0$ at $x = L$ and use the equations already obtained to simplify the virtual work expression. The natural boundary condition follows from the fundamental lemma of variation calculus:

$$\delta W = [(M - R)\frac{d\delta w}{dx}]_{x=L} = 0 \quad \forall \delta w \in U_0 \qquad \Leftrightarrow \quad M - R = 0 \quad \text{at} \quad x = L .$$

As the last step, the essential boundary conditions follow from the problem definition (clamped). They can also partly be deduced from the definition of U . Vanishing of variation $d\delta w / dx$ and δw at $x = 0$ imply that dw / dx and w are given at $x = 0$.

A beam boundary value problem is composed of the equations implied by the principle of virtual work

$$\frac{d^2M}{dx^2} + b = 0 \quad \text{in} \quad (0, L) . \leftarrow$$

$$-\frac{dM}{dx} + F = 0 \quad \text{and} \quad M - R = 0 \quad \text{at} \quad x = L . \leftarrow$$

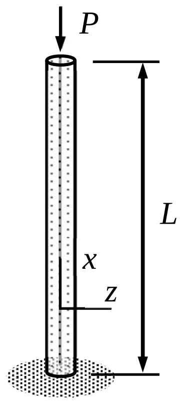
$$w = 0 \text{ and } \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 . \leftarrow$$

Definition of stress resultant, stress-strain relationship, and elasticity tensor for the beam problem gives the constitutive equation

$$M = -EI \frac{d^2w}{dx^2}$$

which is needed for a closed system.

When displacement is confined to the xz – plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx .$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus. Assume that have continuous derivatives up to (and including) fourth order.

Solution

Integration by parts gives an equivalent but a more convenient form (assuming continuity up to and including second derivatives)

$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx \quad \Leftrightarrow \quad (P \text{ is a constant})$$

$$\delta W = -\int_0^L \delta w\left(EI \frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2}\right)dx + \sum_{\{0,L\}} n\delta w\left(EI\frac{d^3 w}{dx^3} + P\frac{dw}{dx}\right) - \sum_{\{0,L\}} n\frac{d\delta w}{dx}\left(EI\frac{d^2 w}{dx^2}\right).$$

According to principle of virtual work $\delta W = 0 \ \forall \delta w$. Let us consider first the subset of variations for which $\delta w = 0$ and $d\delta w / dx = 0$ on $\{0, L\}$. The fundamental lemma of variation calculus implies

$$EI\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2} = 0 \quad \text{in} \quad (0, L).$$

Let us consider then the subset of variations for which $d\delta w / dx = 0$ on $\{0, L\}$. Knowing the condition above, the fundamental lemma of variation calculus implies

$$EI\frac{d^3 w}{dx^3} + P\frac{dw}{dx} = 0 \quad \text{or} \quad w - \underline{w} = 0 \quad \text{on} \quad \{0, L\}.$$

Finally, let us consider the subset of variations for which $\delta w = 0$ on $\{0, L\}$. Knowing the previous results, the fundamental lemma of variation calculus implies

$$EI\frac{d^2 w}{dx^2} = 0 \quad \text{or} \quad \frac{dw}{dx} - \underline{\theta} = 0 \quad \text{on} \quad \{0, L\}.$$

For the problem of the figure, one obtains

$$EI\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2} = 0 \quad \text{in} \quad (0, L),$$

$$EI\frac{d^3w}{dx^3}+P\frac{dw}{dx}=0 \quad \text{and} \quad EI\frac{d^2w}{dx^2}=0 \quad \text{at } x=L,$$

$$w=0 \quad \text{and} \quad \frac{dw}{dx}=0 \quad \text{at } x=0.$$