Aalto University
School of Science

S-Shaped Growth

Linear vs non-linear systems

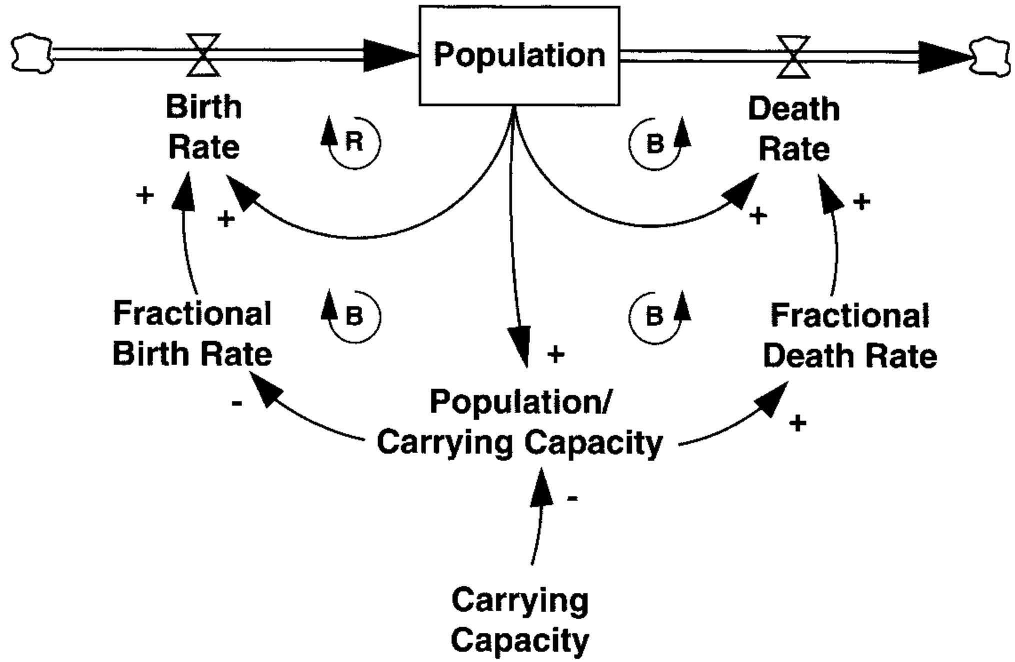
- Linear first order systems can only show
  - exponential growth in case of positive feedback loop
  - exponential decay to goal in case of negative feedback
  - stay constant when net feedback is zero
- Linear first order systems don’t oscillate (if time is treated continuously)
  - discrete time can cause oscillation, e.g. too large time step in integration
- Non-linear first order systems can show more interesting behaviour:
  - Oscillation, S-shaped growth, oscillating overshoot, overshoot and collapse

S-shaped growth

- Nothing in the real world can grow without limits
- S-shaped growth is common model for diffusion of innovations, spread of infectious diseases and computer viruses, growth of market for new products…
- If the loop has delays, the system may oscillate
- If the population uses up resources, the response may be overshoot and collapse

General S-Shaped Growth

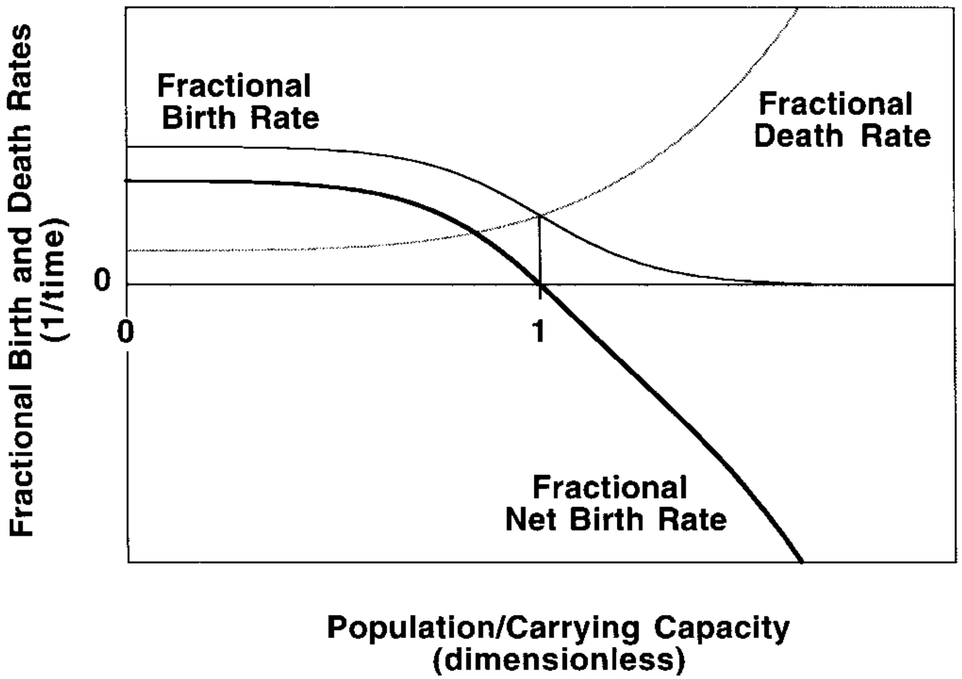
- S-shaped growth is common in real life
  - Initially exponential growth of population **P**
  - Later slower growth and goal seeking towards ***carrying capacity* C**
  - Nonlinear dependency between positive and negative loop

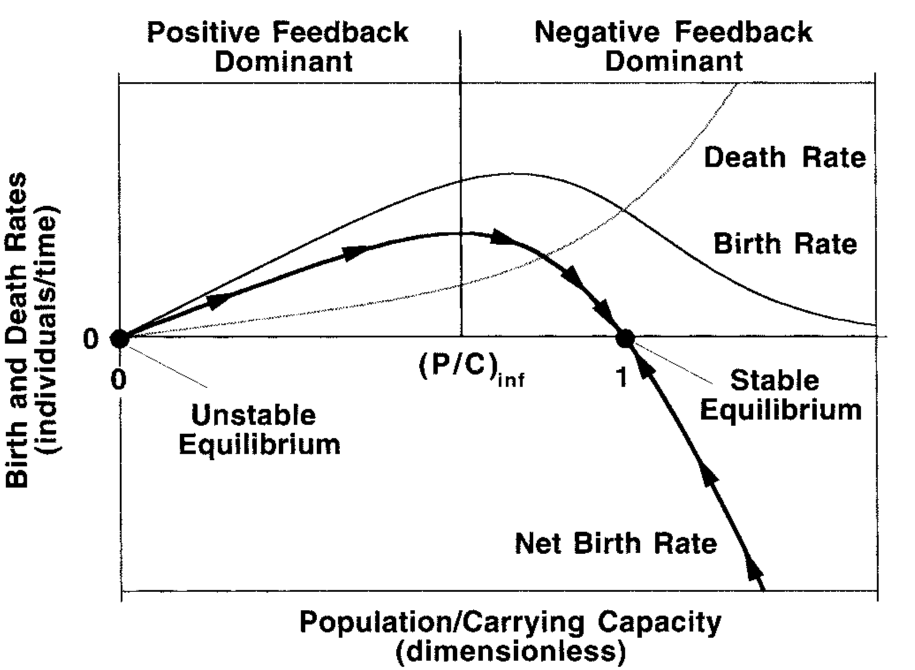


  - Net birth rate = BR – DR = b(P/C) · P – d(P/C) · P

S-shaped growth

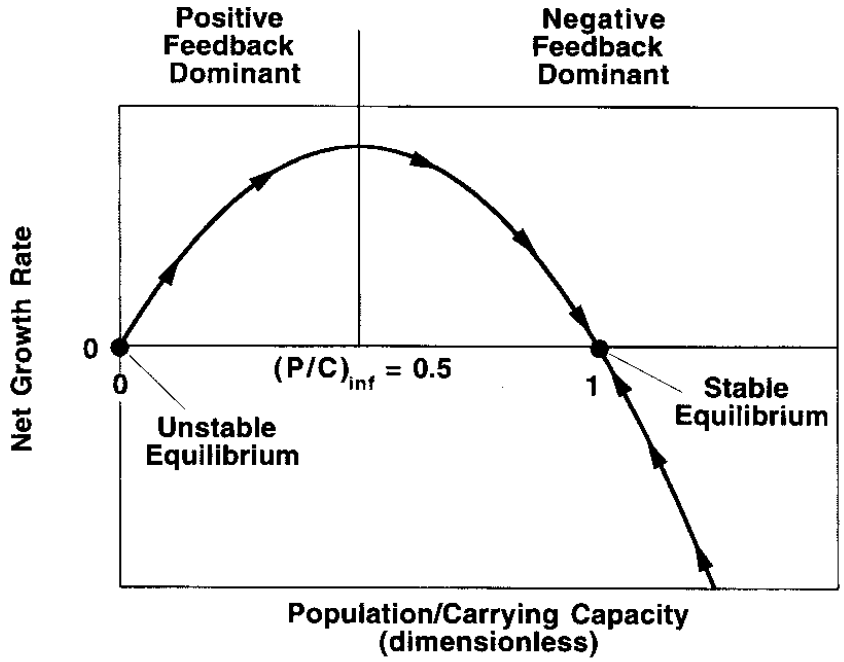
- Phase plot of typical relationship between
  - fractional birth and death rates in relation to P/C
  - birth and death rates in relation to P/C





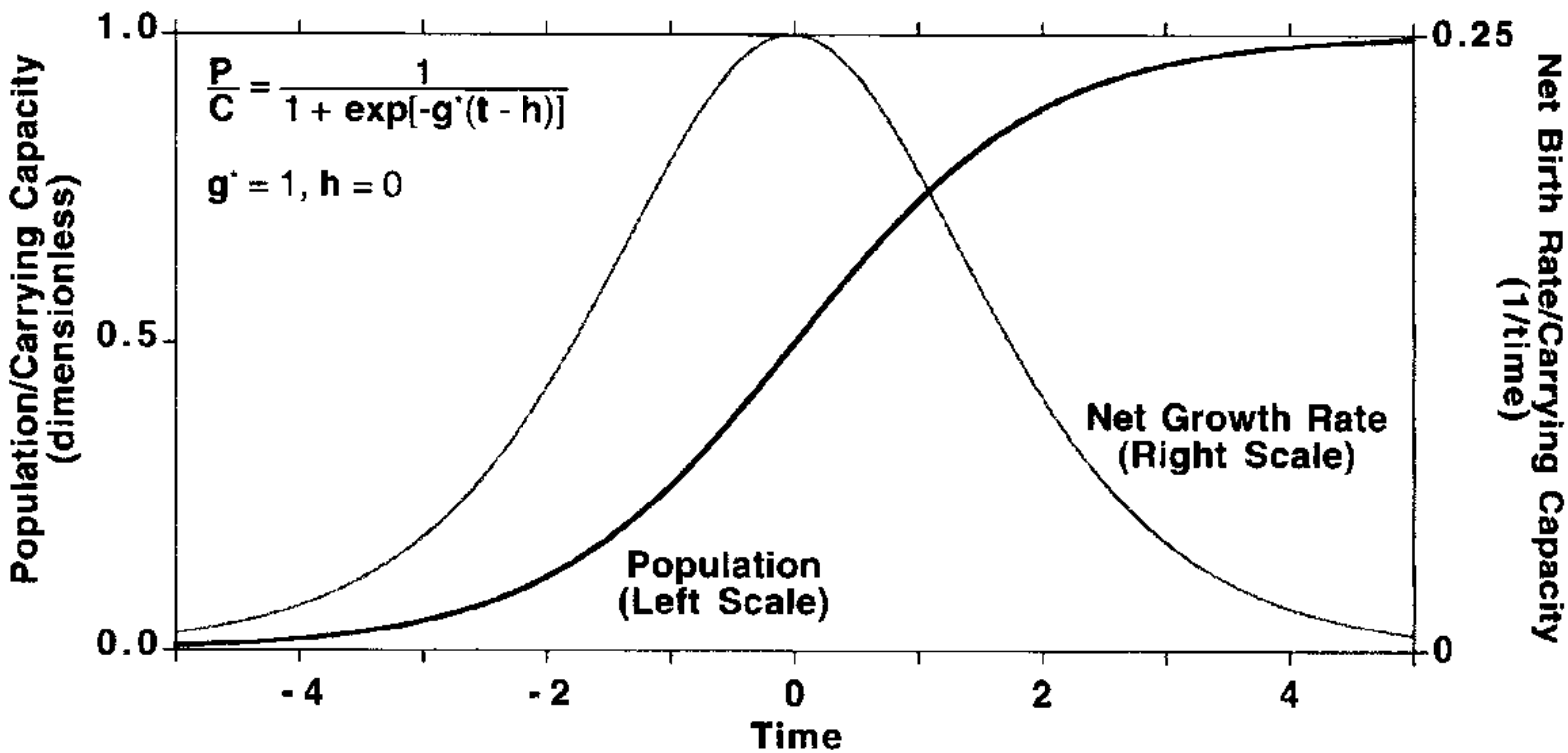
S-Shaped growth – logistic model

- Developed by Belgian mathematician Pierre Verhulst (1838)
  - Represented originally fractional population birth rate as downward sloping linear function of population $g(P,C) = g^{*} \cdot (1-P/C)$
- Net birth rate N(P) was then an inverted parabola
  - $N(P)=g^{*} \cdot (1-P/C) \cdot P=g^{*}P-g^{*}P^{2}$
  - Growth rate = 0 for P/C=0 and P/C=1
  - Max growth at P/C=0.5



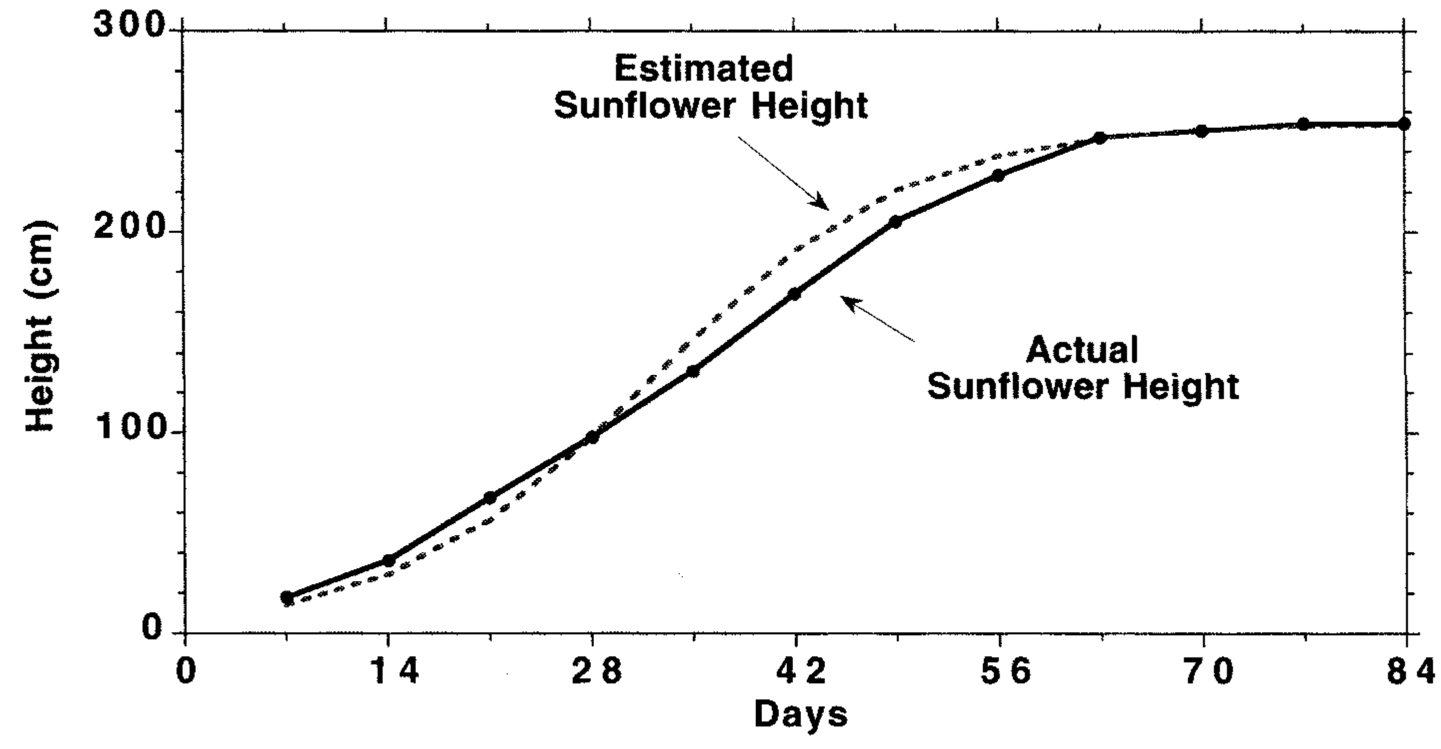
S-Shaped growth – logistic model

- Logistic model $\frac{dP(t)}{dt} = g\left(1 - \frac{P(t)}{C}\right)P(t)$
- Analytical solution $P(t) = \frac{C}{1+\left(\frac{C}{P(0)}-1\right)e^{-gt}}$



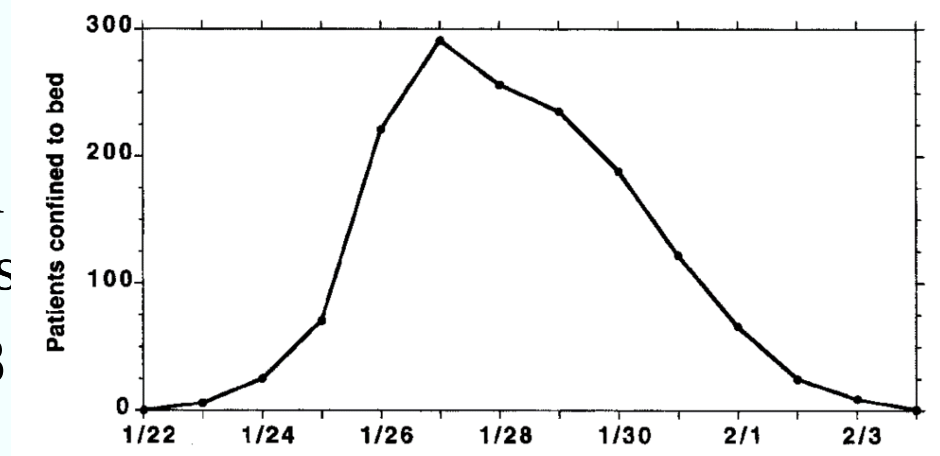
Testing the logistic model

- Logistic model fits reasonably well to sunflower growth but another model may be more suitable
  - If only initial part of model is available, it may be difficult to predict remaining part accurately

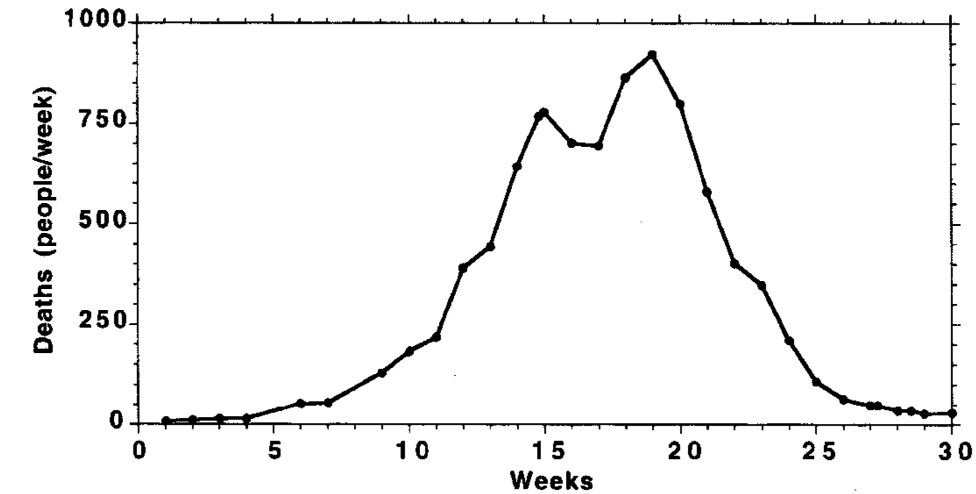


Dynamics of disease

- Epidemics of infectious diseases follow often S-shaped growth
  - Rate of new cases rises exponentially, peaks and falls when epidemic ends
  - Influenza epidemic 1978

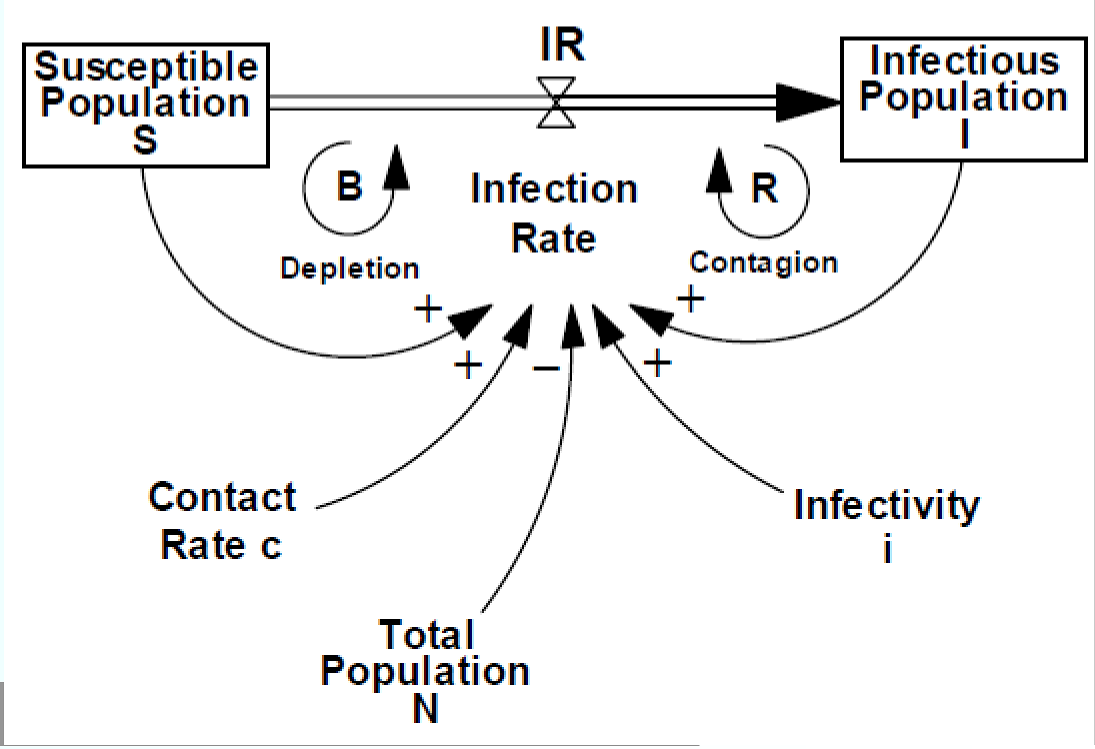


  - Plague, Bombay, India 1905-06



Susceptible, Infectious (SI) model

- Population is divided into two categories:
  - Those susceptible and those infectious
  - As people are infected, they move from susceptible category to infectious
  - People meet c·S times per time unit
  - Probability to meet infectious person is I/N
  - Infection probability is i
  - N=S+I



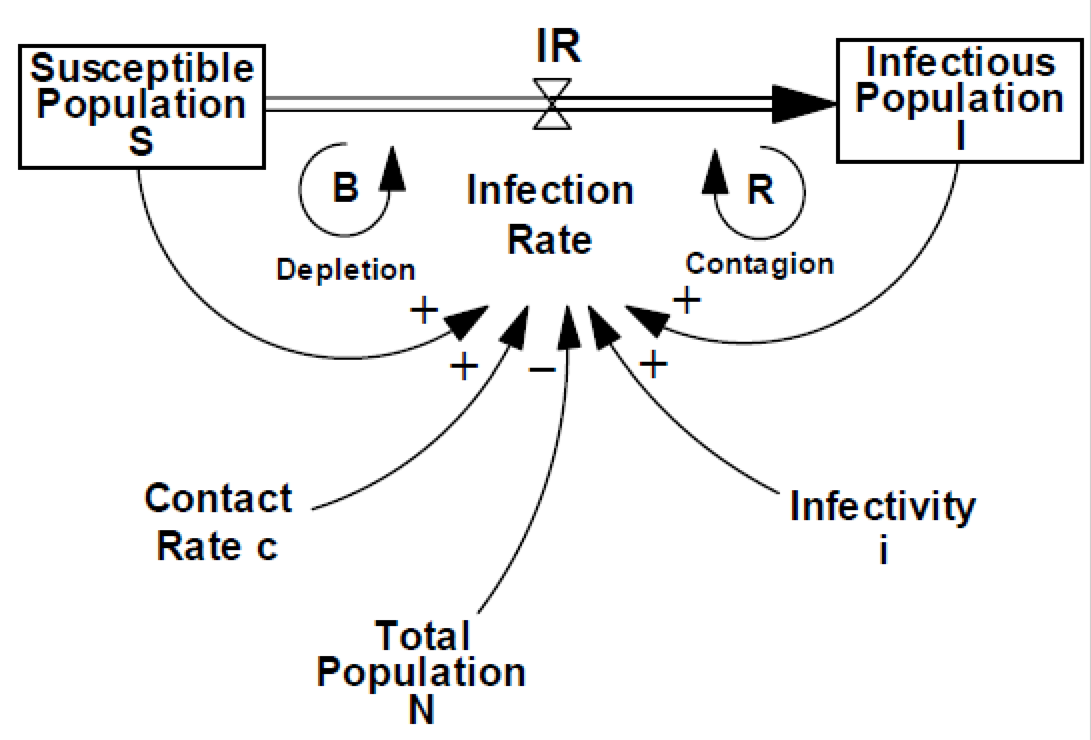
SI-model

- Functions

$$IR = i \cdot (c \cdot S) \cdot (I/N) = (c \cdot i \cdot I) \cdot (1-I/N)$$

$$I(t) = I_0 + \int IR(t)\, dt$$

$$S(t) = N - I_0 + \int -IR(t)\, dt$$



SI-model restrictions

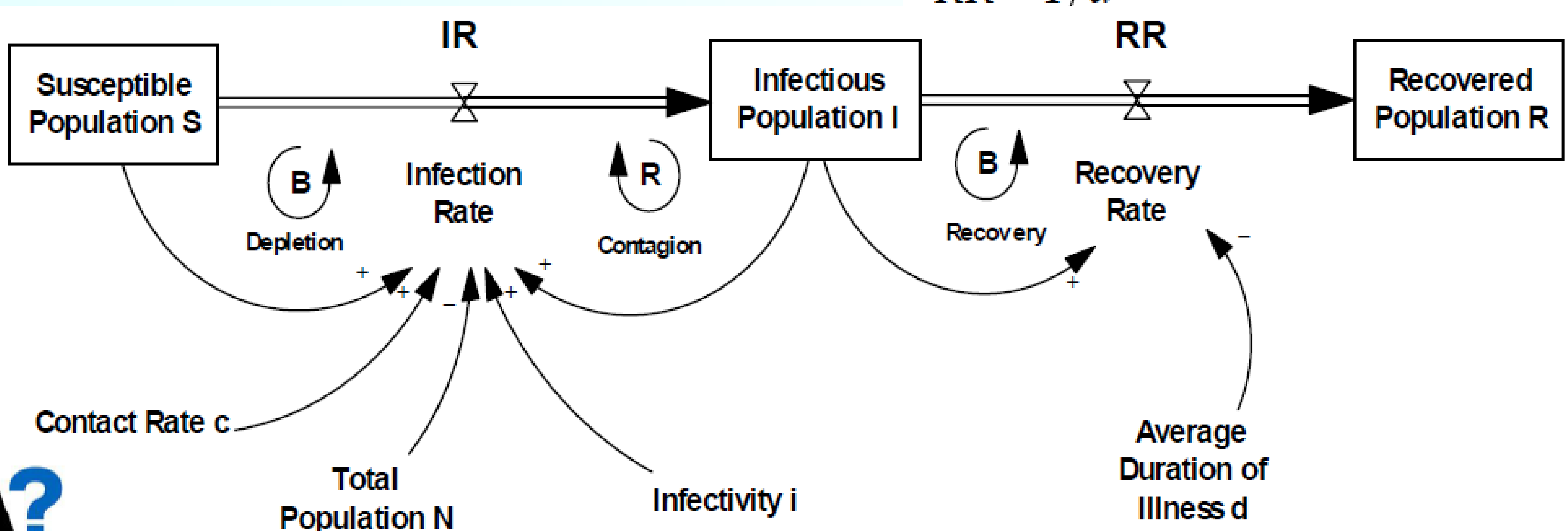
- Many simplifications:
  - Births and deaths are ignored
  - Spatial distribution is ignored
  - Population is assumed to be homogenous
  - Infection does not affect behaviour of infected person
  - No medical treatment
- Model can be extended to cover these issues
- No healing – entire population will ultimately get infected

SIR-model: Susceptible, Infectious, Recovered

Most epidemics end due to recovery speed exceeding infection speed

$$\begin{cases} I(t) = I_0 + \int_{t_0}^{t} \left(IR(\tau) - RR(\tau) \right) d\tau \\ R(t) = R_0 + \int_{t_0}^{t} RR(\tau)\, d\tau \end{cases}$$

$$RR = I/d$$

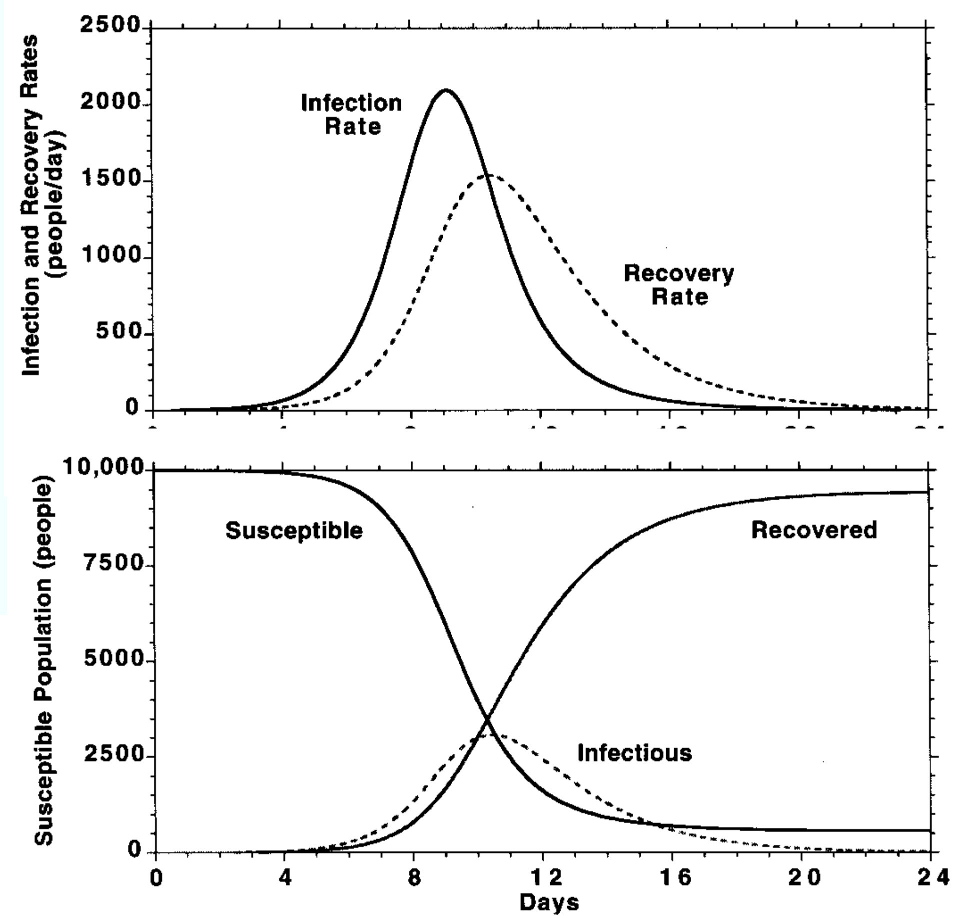


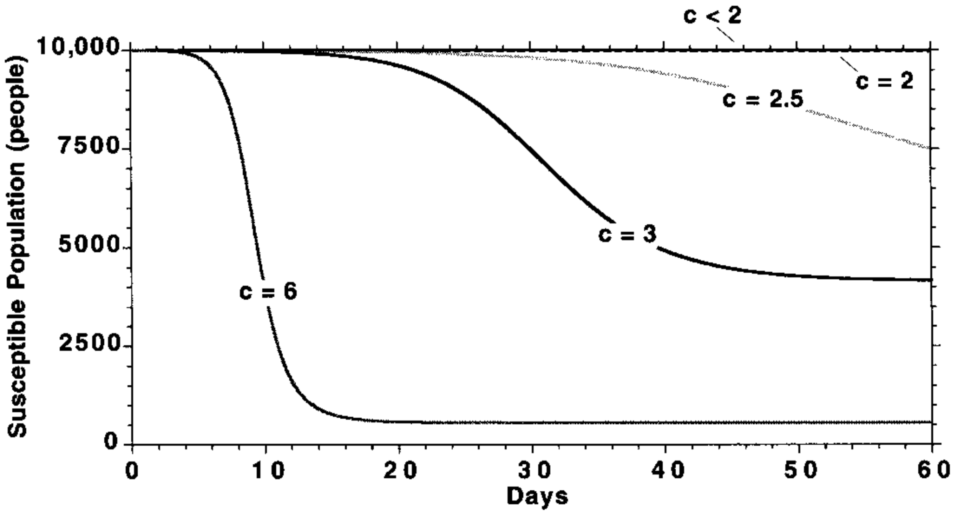
SIR-model: Tipping point

- SIR-model is second order
- Epidemic may end before all get infected – or not
  - Epidemic spreads if infected people infect more than 1
  - Epidemic spreads if infection rate is greater than recovery rate
- The *tipping point* is a critical combination of contact frequency, infectivity and disease duration where the positive loop dominates the negative loops
  - IR > RR $\Rightarrow$ ciS(I/N) > I/d or cid(S/N) > 1

SIR-model: Tipping point

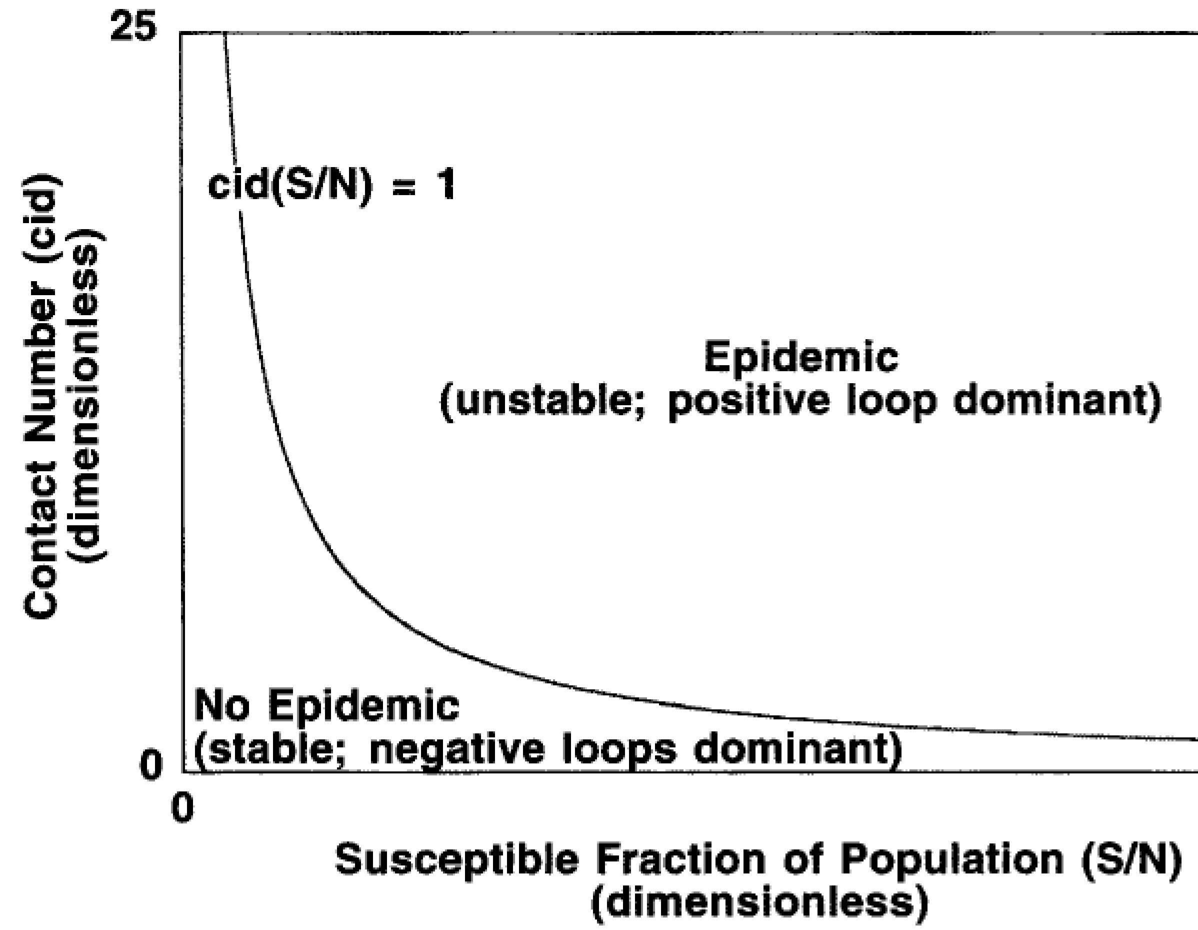
- Population 10 000
- Contact rate 6/day/pers.
- Infection rate 0.25
- Infectivity duration 2 d
- Initially 1 infected
- Varying contact rate:





SIR-model: Tipping point

- Tipping point as function of model parameters



Immunization and eradication of Smallpox

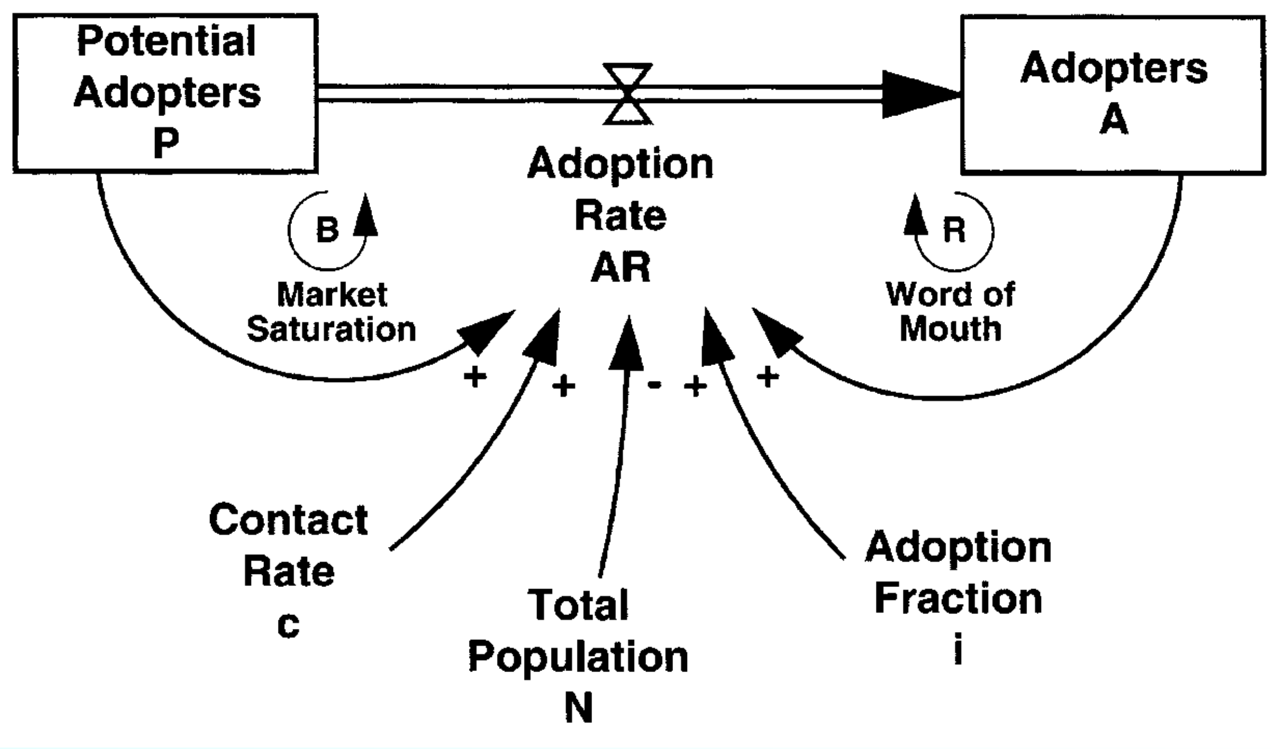
- Diseases may have very different parameters
  - Knowing the parameters means theoretical possibility to eradicate a disease
- Smallpox has high infectivity, but
  - Duration of infection is short
  - Survivors acquired long-lived immunity
  - Virus cannot survive outside human host
  - Development of effective vaccine, deployed sufficiently broadly reduced infection rate below recovery rate
  - First vaccine by Jenner 1796
  - Mass vaccinations by WHO in 1960's
  - Last natural infection in Somalia 1976

Innovation diffusion as infection

- Diffusion and adoption of new ideas and new products often follows S-shaped growth patterns
- Can be modelled using SI and SIR models
- People who adopt new ideas/products spread their experience to others (positive loop)
- Growth has bounds, such as exhausting the population

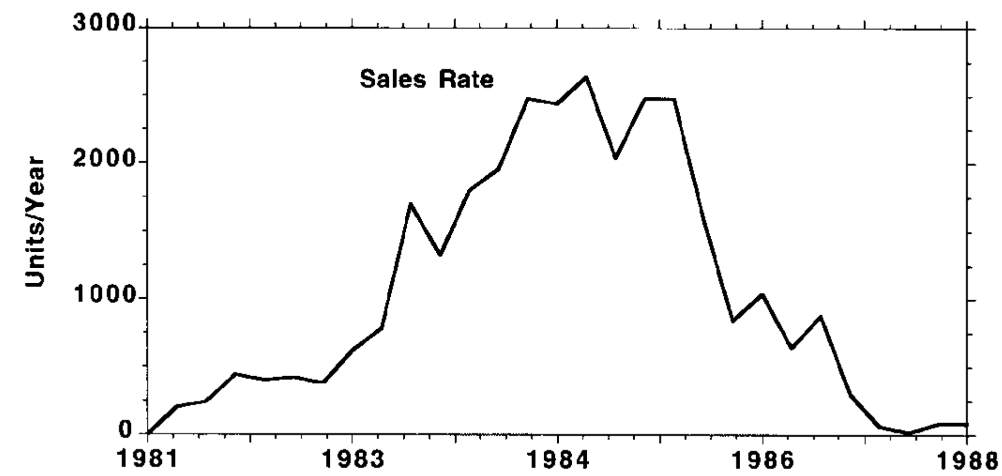
Innovation diffusion as infection

- Unlike diseases, innovations and product news can be distributed by various media: phone, email, internet

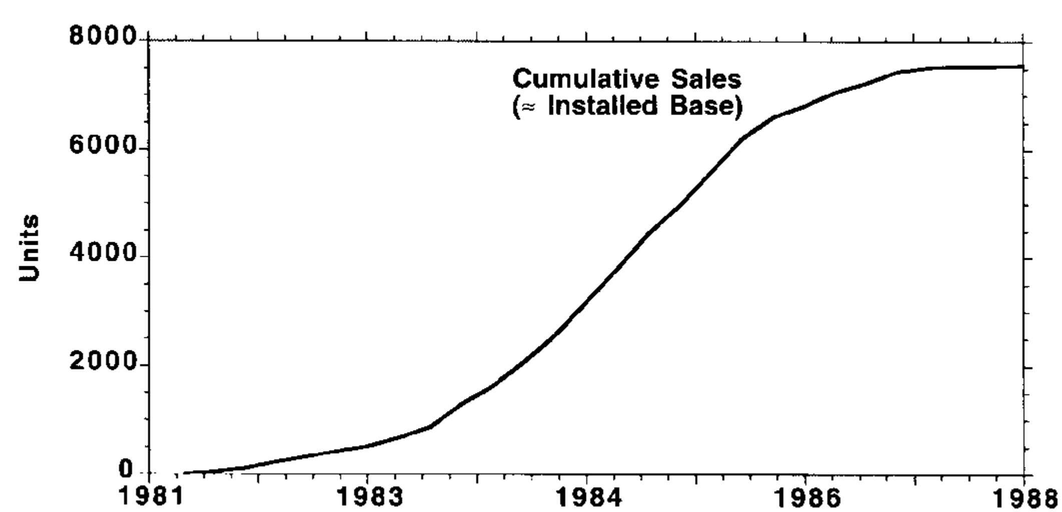


Example: VAX 11/750 computers in Europe

- VAX 11/750 was launced in 1981
- Very successful
- Peak sales in 1984
- Withdrawn 1989



- Logistic growth curve fits well with cumulative sales



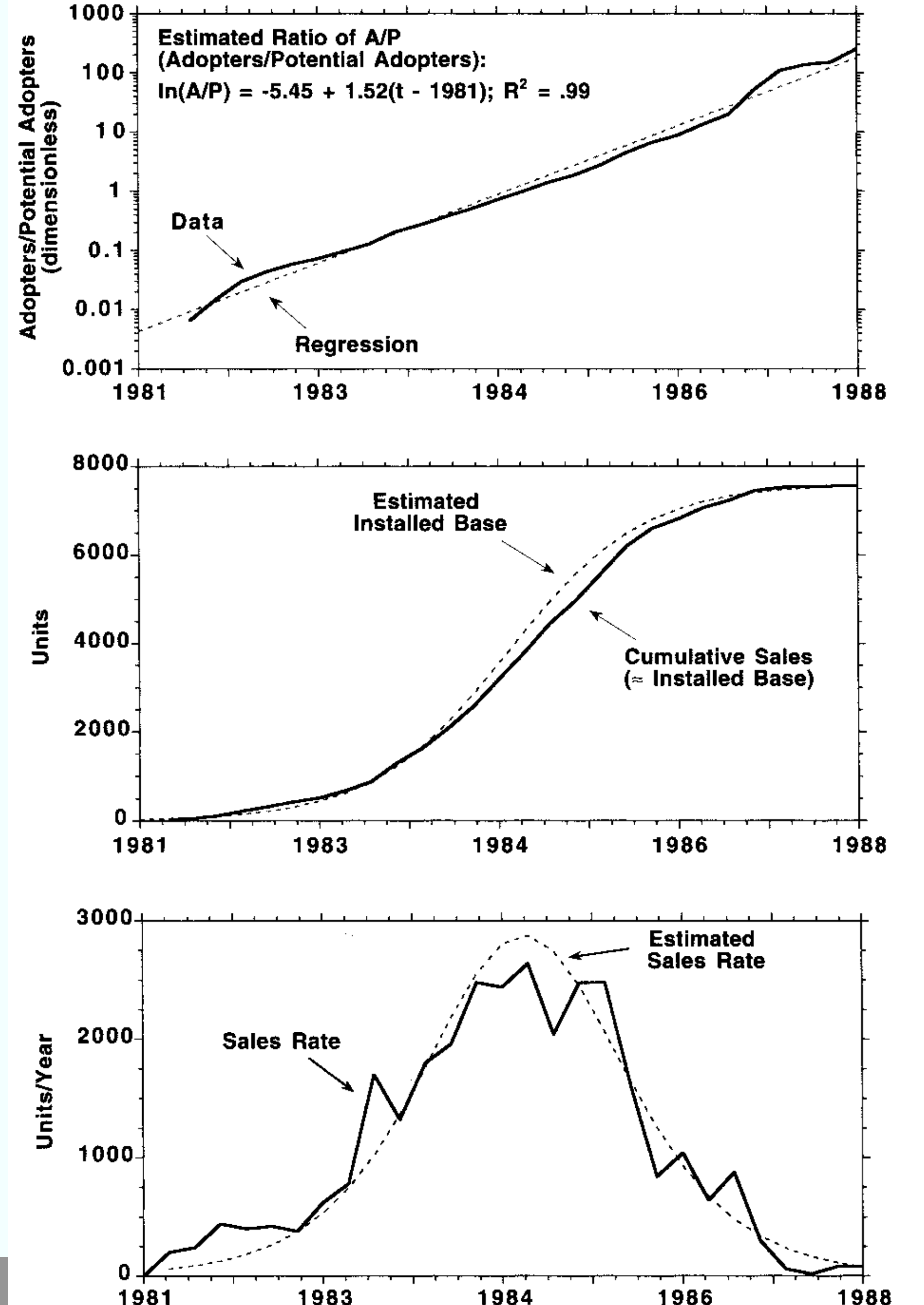
- Solution for logistic equation

$$\frac{A}{N-A}=\frac{A_0}{N-A_0}e^{g_0 t}$$

- Ln() to obtain line

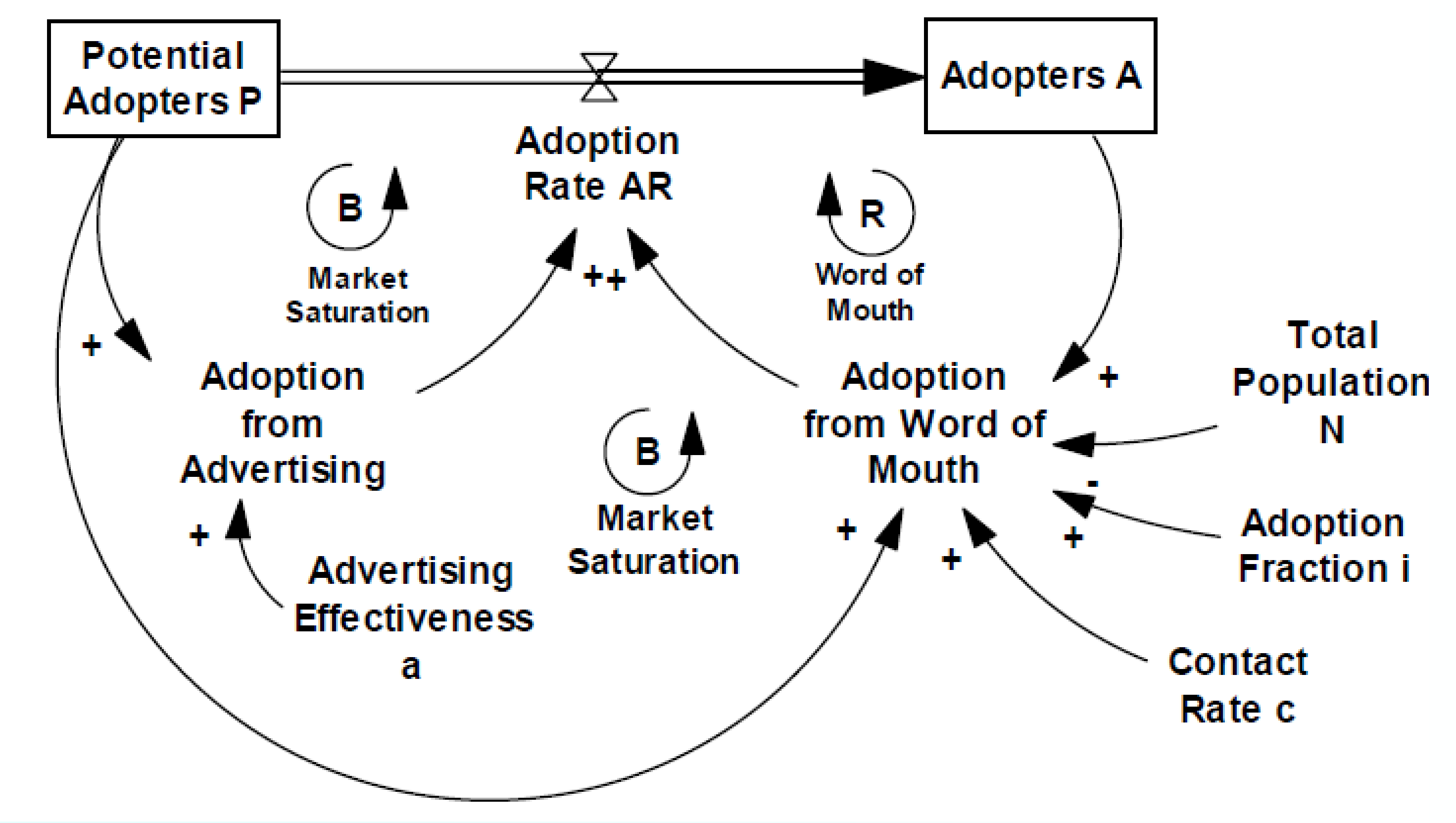
$$\ln\left(\frac{A}{N-A}\right)=\ln\left(\frac{A_0}{N-A_0}\right)+g_0 t$$

- Use Linear regression to find parameters



Bass diffusion model

- Frank Bass 1969 developed model for technology diffusion – works also when initial A is empty

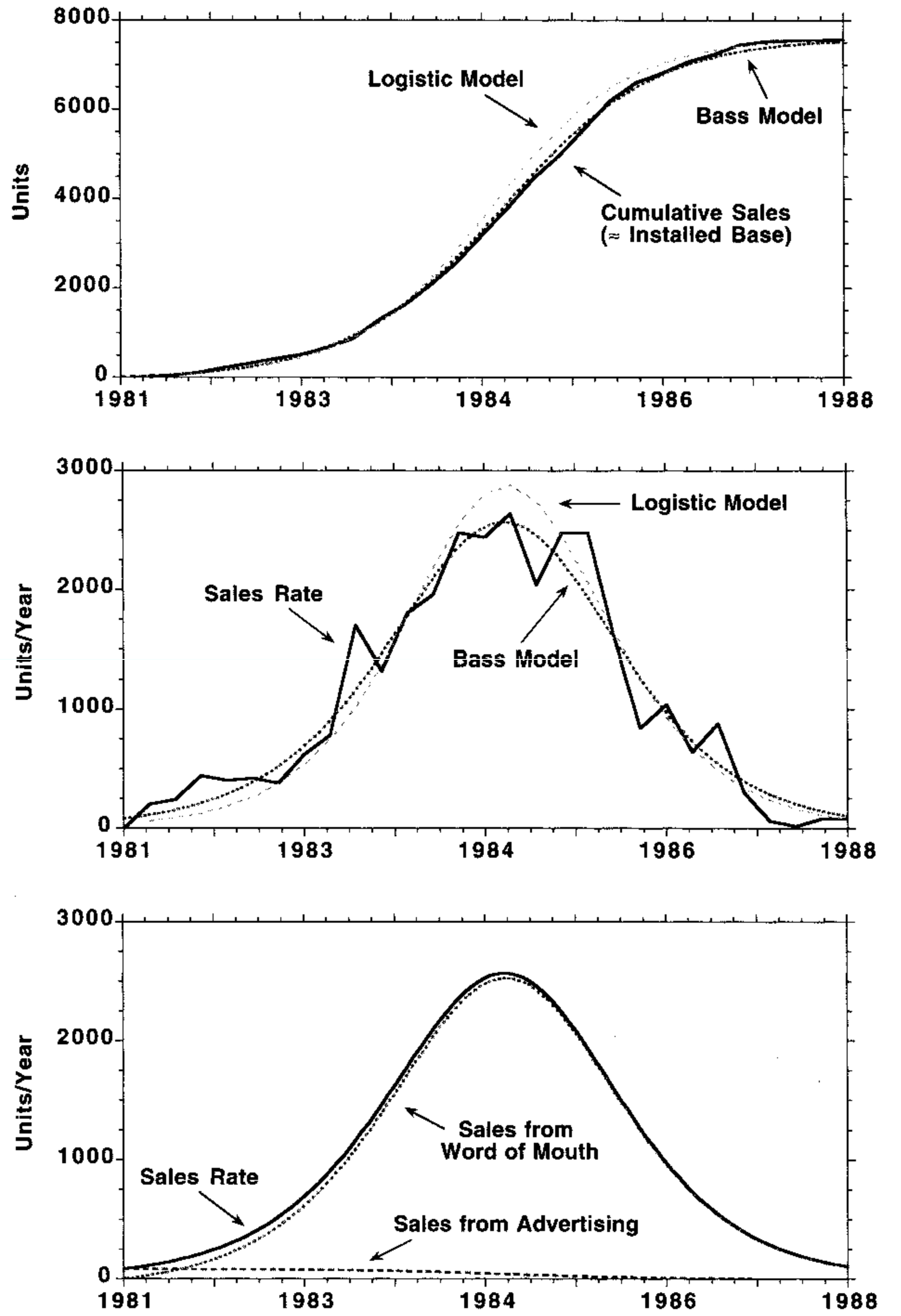


Bass diffusion model

- Differential equation $$\frac{dF}{dt} = p(1-F) + q(1-F)F.$$
- F(t) = fraction of adopters (A/N)
- p = coefficient of innovation
  - p = 0 → logistic distribution
- q = coefficient of imitation
  - q = 0 → exponential distribution

Bass model on VAX computer sales

- Fits better than the logistic model
- Early adoption is caused also by advertising efforts, not just word by mouth



Modelling product discard and replacement purchases

- Bass diffusion model is a first-purchase model
  - The product is not discarded, consumed, or upgraded
- The model can be modified so that Adopters after some time became again Potential adopters

Modelling product discard and replacement purchases