ELEC-C9430 Electromagnetism (week 3)

Conductivity. Ohm's law

Magnetostatics: Ampère's law

Lorentz force. Biot–Savart

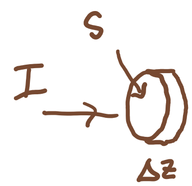
Magnetic dipole

Vector potential

.

Steady electric currents
(Cheng: Ch. 4)

$\bar{J}$



$q \cdot \to \bar{v}$ $\quad \bar{E}$

$\bar{v} = \mu \bar{E}$

$I = \frac{dQ}{\Delta t} = \frac{\rho_v S \Delta z}{\Delta t} =$

Charge carrier mobility μ

$J = \frac{I}{S} = \rho_v \frac{\Delta z}{\Delta t} = \rho_v v$

Current density $\bar{J} = \rho_v \bar{v} = \underbrace{\rho_v \mu}_{\sigma} \bar{E}$

$[\sigma] = \frac{A}{m^2} \frac{m}{V} = \frac{A}{Vm} = \frac{S}{m}$ ↙ SIEMENS

Conductivity σ

Ohm's law $\bar{J} = \sigma \bar{E}$

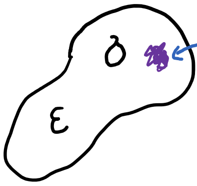
$\left(\frac{1}{S} = \Omega \right)$ OHM

Cu Ag — Conductor — Semiconductors, snow — rubber — Insulator

$\sigma = 10^{8} \frac{S}{m}$ ⟷ $10^{-15} \frac{S}{m}$

(See Cheng, Appendix B-4)

Relaxation time τ



$\rho_v(t)$

$$\nabla\cdot\left(\nabla\times\bar{H} = \bar{J} + \frac{\partial}{\partial t}\bar{D}\right)$$

$$\nabla\cdot\bar{D} = \rho_v$$

$$\nabla\cdot\nabla\times\bar{H} = 0$$

$$\nabla\cdot\bar{J} = -\frac{\partial}{\partial t}\underbrace{\nabla\cdot\bar{D}}_{\rho_v}$$

$$\frac{\sigma}{\varepsilon}\bar{D}$$

$$\bar{J} = \sigma\bar{E}$$
$$\bar{D} = \varepsilon\bar{E}$$

$$\frac{\partial}{\partial t}\rho_v(t) = -\frac{\sigma}{\varepsilon}\rho_v(t)$$

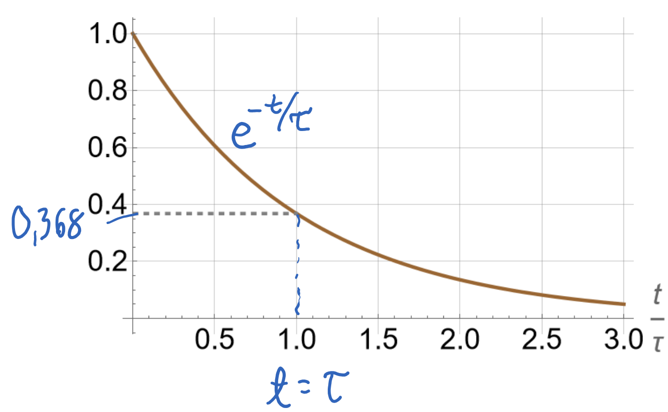
$$\frac{\partial}{\partial t}e^{-\frac{\sigma}{\varepsilon}t} = -\frac{\sigma}{\varepsilon}e^{-\underbrace{\frac{\sigma}{\varepsilon}t}_{t/\tau}}$$

$$\rho_v(t) = \rho_v(0)\,e^{-t/\tau} \qquad \tau = \frac{\varepsilon}{\sigma}$$

Copper

$$\tau = \frac{\varepsilon}{\sigma} = \frac{10^{-11}}{10^{8}}\,\overbrace{\frac{As}{Vm}\,\frac{Vm}{A}}^{s} \approx 10^{-19}\,s$$

$$\varepsilon_0 = 8.854\cdot 10^{-12}\,\frac{As}{Vm}$$



Rubber $\tau = \frac{10^{-11}}{10^{-15}}\,s = 10^{4}\,s$

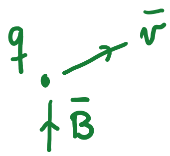
Static magnetic fields
(Cheng, Ch. 5)

$$\frac{\partial}{\partial t} = 0$$

Magnetic field (intensity) $\bar{H}$ $\qquad [\bar{H}] = \frac{A}{m}$

Magnetic flux density $\bar{B}$ $\qquad [\bar{B}] = \frac{Vs}{m^2} = T$ (tesla)

Free-space permeability $\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}$



$$\bar{F} = q\,\bar{v} \times \bar{B}$$

Lorentz force

$$\bar{F} = q\left(\bar{E} + \bar{v} \times \bar{B}\right)$$

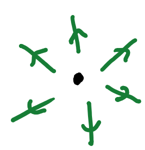
$$\nabla \times \bar{E} = \cancel{-\frac{\partial \bar{B}}{\partial t}}$$
$$\nabla \times \bar{H} = \bar{J} + \cancel{\frac{\partial \bar{D}}{\partial t}}$$
$$\nabla \cdot \bar{D} = \rho_v$$
$$\nabla \cdot \bar{B} = 0$$

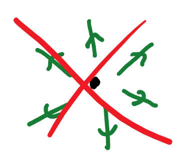
ELECTROSTATICS

$$\nabla \times \bar{E} = 0$$
$$\nabla \cdot \bar{D} = \rho_v$$

MAGNETOSTATICS

$$\nabla \times \bar{H} = \bar{J}$$
$$\nabla \cdot \bar{B} = 0$$

OK: 



OK: 

Hans Christian Ørsted (1820)

EXPERIMENTA

Hans Christian Ørsted (1820)

EXPERIMENTA

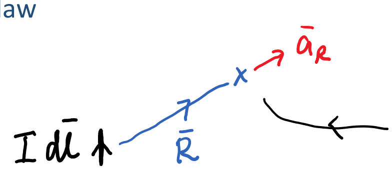
CIRCA EFFECTUM

CONFLICTUS ELECTRICI IN ACUM

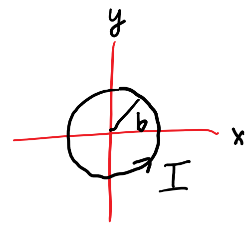
MAGNETICAM.

Ampère's law $\nabla \times \bar{H} = \bar{J}$

Biot–Savart law

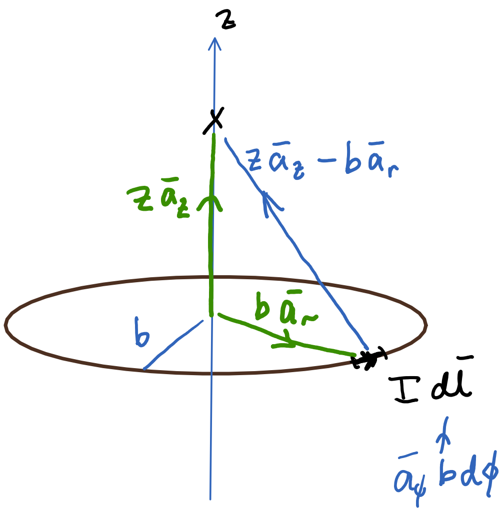


$$d\bar{H}(\bar{R}) = \frac{I\, d\bar{l} \times \bar{a}_R}{4\pi R^2}$$



Circular loop carrying steady current

(r, ϕ, z)



$$\bar{H}(z) = \oint \frac{I\, d\bar{l} \times (z\bar{a}_z - b\bar{a}_r)}{4\pi (z^2+b^2)^{3/2}}$$

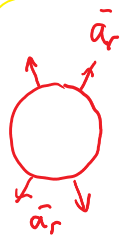
$$= \frac{I}{4\pi (z^2+b^2)^{3/2}} \int_0^{2\pi} \Big(z\, b\, \underbrace{\bar{a}_\phi \times \bar{a}_z}_{\bar{a}_r} - b^2 \underbrace{\bar{a}_\phi \times \bar{a}_r}_{-\bar{a}_z}\Big)\, d\phi$$

$$\int_0^{2\pi} \bar{a}_z\, d\phi = \bar{a}_z\, 2\pi$$

$$\int_0^{2\pi} \bar{a}_r\, d\phi = \int_0^{2\pi} (\bar{a}_x \cos\phi + \bar{a}_y \sin\phi)\, d\phi$$

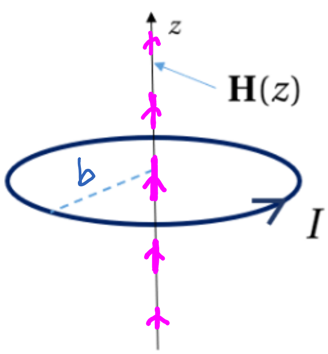
$$= \bar{a}_x \underbrace{\int_0^{2\pi} \cos\phi\, d\phi}_{0} + \bar{a}_y \underbrace{\int_0^{2\pi} \sin\phi\, d\phi}_{0}$$

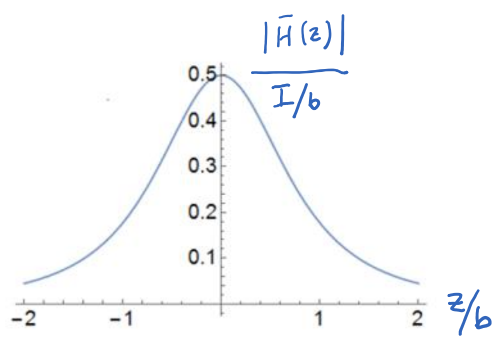
$$\underbrace{\qquad}_{0} \qquad \underbrace{\qquad}_{0}$$



$$\Rightarrow \int_0^{2\pi} \bar{a}_r \, d\phi = 0$$

$$\Rightarrow \quad \bar{H}(z) = \bar{a}_z \frac{2 I \pi b^2}{4\pi \sqrt{z^2+b^2}^{\,3}} = \bar{a}_z \frac{I b^2}{2\,(z^2+b^2)^{3/2}}$$



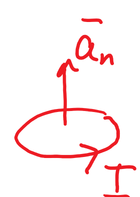


$$|z| >> b \qquad \sqrt{z^2+b^2} \simeq |z|$$

$$\bar{H}(z) \simeq \bar{a}_z \frac{I b^2}{2 z^3} = 2\bar{a}_z \frac{I \pi b^2}{4\pi z^3} = 2\bar{a}_z \frac{\overbrace{\mu_0 I \pi b^2}^{p_m}}{4\pi \mu_0 z^3}$$

ELECTRIC DIPOLE p_e $\quad (\theta = 0, R = z)$

$$\bar{E}(z) = 2\bar{a}_z \frac{p_e}{4\pi \varepsilon_0 z^3}$$



Magnetic dipole $\quad \bar{p}_m = \bar{a}_z \mu_0 \underbrace{I \overbrace{\pi b^2}^{S}}_{m}$

$$[m] = A m^2$$

m

$$[m] = A\,m^2$$

$$[p_m] = [\mu_0][m] = \frac{Vs}{Am} A m^2 = Vsm$$

$$[p_e] = [Q][d] = Asm$$

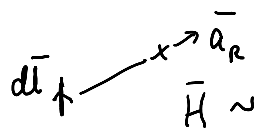
Geomagnetism

$B \approx 52\,\mu T$ (Otaniemi, now)

Declination $+7^\circ$ (East)

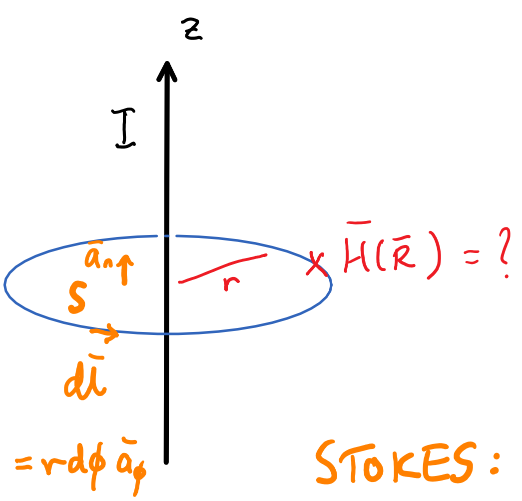
Inclination

homework ☺



$$\bar{H} \sim d\bar{l} \times \bar{a}_R$$

Magnetic field of a straight current wire



$$\bar{H}(\bar{R}) = \bar{a}_r \cancel{H_r(r,\phi,z)} + \bar{a}_\phi H_\phi(r,\cancel{\phi},\cancel{z}) + \bar{a}_z \cancel{H_z(r,\phi,z)}$$

$$\bar{H}(\bar{R}) = \bar{a}_\phi H(r)$$

STOKES: $\displaystyle\int_S \underbrace{\nabla\times\bar{H}}_{\bar{J}} \cdot d\bar{S} = \oint_C \bar{H}\cdot d\bar{l}$

$$= \int_0^{2\pi} \underbrace{\bar{a}_\phi H(r) \cdot \bar{a}_\phi}_{1} r\,d\phi$$

$$= I$$

$$= \int_0^{2\pi} r\,H(r)\,d\phi$$

$$= r\,H(r) \int^{2\pi} d\phi = 2\pi r\,H(r)$$

$$\bar{H}(\bar{R}) = \bar{a}_\phi \frac{I}{2\pi r}$$

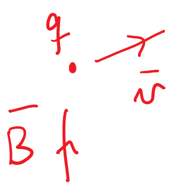
$\bar{S}$ $\bar{J}$ $= I$

$$= \int_0^{2\pi} \bar{a}_\phi H(\bar{r}) \cdot \bar{a}_\phi \, r \, d\phi$$

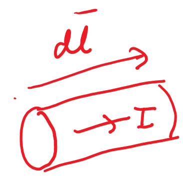
$$= \int_0^{2\pi} r \, H(r) \, d\phi$$

$$= r \, H(r) \int_0^{2\pi} d\phi = 2\pi r \, H(r)$$

$$\bar{H}(\bar{R}) = \bar{a}_\phi \frac{I}{2\pi r}$$



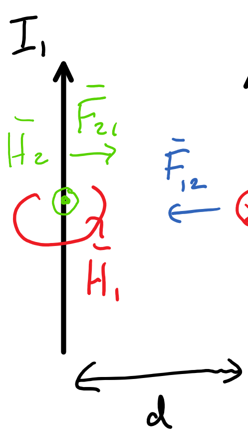
$$\bar{F} = q \, \bar{v} \times \bar{B}$$



$$\bar{F} = dQ \underbrace{\bar{v}}_{\frac{d\bar{l}}{dt}} \times \bar{B} = \underbrace{\frac{dQ}{dt}}_{I} \, d\bar{l} \times \bar{B}$$

$$[F] = A \, m \, \frac{Vs}{m^2} = \frac{VAs}{m} = \frac{\underbrace{Ws}_{J = Nm}}{m} = N$$

Force between two wires



$$H_1 = \frac{I_1}{2\pi d}$$

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

$$\Rightarrow \frac{F_{12}}{L} = I_2 B_1 = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Force per unit length: $F/L = \frac{\mu_0 \, 1A \, 1A}{2\pi \, 1m} = \frac{4\pi \cdot 10^{-7}}{2\pi} N/m = 200 \frac{nN}{m}$

Permeability $\mu = \mu_r \mu_0 = (1 + \chi_m) \mu_0$ $\qquad \mu_0$

Magnetic susceptibility χ_m

Magnetic susceptibility χ_m

Diamagnetic materials	$\lvert\chi_m\rvert \ll 1$,	$\chi_m < 0$	Ag, Cu, water
Paramagnetic materials	$\lvert\chi_m\rvert \ll 1$,	$\chi_m > 0$	Al, Mg
Ferromagnetic materials	$\chi_m \gg 1$	Fe, Co, Ni	

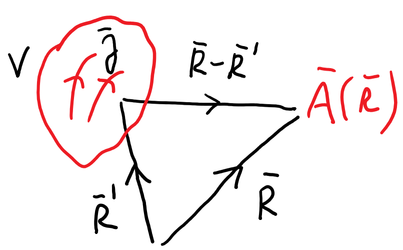
(Magnetic) vector potential $\bar{A}$

$$\nabla \cdot \underset{\uparrow \atop \nabla \times \bar{A}}{\bar{B}} = 0 \qquad\qquad \nabla \times \underbrace{\mu_0 \bar{H}}_{\bar{B} = \nabla \times \bar{A}} = \mu_0 \bar{J}$$

$$\nabla \times (\nabla \times \bar{A}) = \nabla (\underbrace{\nabla \cdot \bar{A}}_{=0 \text{ (Coulomb gauge)}}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

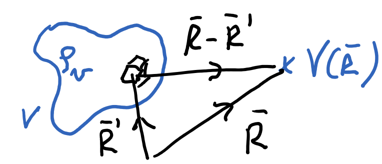
$$\nabla^2 \bar{A}(\bar{R}) = -\mu_0 \bar{J}(\bar{R})$$

$$\bar{A}(\bar{R}) = \int_V \frac{\mu_0 \bar{J}(\bar{R}')\, dv'}{4\pi |\bar{R} - \bar{R}'|}$$



ELECTROSTATICS:

$$\nabla^2 V(\bar{R}) = -\frac{\rho_v(\bar{R})}{\varepsilon_0}$$



$$V(\bar{R}) = \int \frac{\rho_v(\bar{R}')\, dv'}{4\pi\varepsilon_0 |\bar{R} - \bar{R}'|}$$

$$\nabla^2 \bar{A} = \nabla \cdot \nabla \bar{A} \quad \text{(gradient of a vector ???)}$$

$$\nabla^2 \bar{A} = \nabla\cdot\nabla\bar{A} \qquad \text{(gradient of a vector ???)}$$

$$\nabla^2 \bar{A} = \nabla^2\left(\bar{a}_x A_x + \bar{a}_y A_y + \bar{a}_z A_z\right)$$

$$= \bar{a}_x \underbrace{\nabla^2 A_x}_{\nabla\cdot\nabla A_x} + \bar{a}_y \nabla^2 A_y + \bar{a}_z \nabla^2 A_z$$

(= gradient of a scalar ☺)

$$\nabla^2 A_x = -\mu_0 J_x \quad\Rightarrow\quad A_x(\bar{R}) \quad\Rightarrow\quad \bar{A}(\bar{R}) = \bar{a}_x A_x(\bar{R}) + \cdots$$

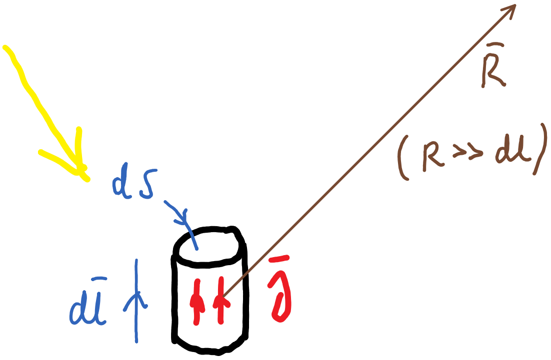
$$\nabla^2 \bar{a}_x = 0$$

$$\nabla^2 \bar{a}_\phi \neq 0$$

$$\nabla^2 \bar{a}_\phi = \nabla(\nabla\cdot\bar{a}_\phi) - \nabla\times(\nabla\times\bar{a}_\phi) = \cdots$$

$$I d\bar{l} \xrightarrow{\bar{R}} \times \quad \bar{H}(\bar{R}) = ?$$

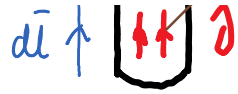
$$\bar{A}(\bar{R}) = \int \frac{\mu_0 \bar{J}(\bar{R}')\, dV'}{4\pi|\bar{R}-\bar{R}'|}$$



$$= \frac{\mu_0}{4\pi R}\int \bar{J}(\bar{R}')\, \underbrace{dV'}_{ds\, dl}$$

$$= \frac{\mu_0 I\, d\bar{l}}{4\pi R}$$

$$I = JS$$



$$I = JS$$

$$= \frac{\mu_0 I \, d\bar{l}}{4\pi R}$$

$$\bar{B} = \nabla\times\bar{A} = \frac{\mu_0 I}{4\pi} \underbrace{\nabla\times\left(\frac{d\bar{l}}{R}\right)}_{\frac{1}{R}\underbrace{\nabla\times d\bar{l}}_{=0} + \underbrace{\left(\nabla\frac{1}{R}\right)}_{-\frac{\bar{a}_R}{R^2}}\times d\bar{l}}$$

$$\Rightarrow \bar{B} = \frac{\mu_0 I \, d\bar{l}\times\bar{a}_R}{4\pi R^2}$$

(BIOT-SAVART !)