Some More Math for Economists: Dynamics

Spring 2023

Dynamics

Consider the consumption savings problem

$$\max_{s,c\in\mathbb{R}_+^{\mathbb{N}}} \sum_{t=0}^{\infty} \delta^t u(c_t)$$
$$c_t + s_t \leq (1+r)s_{t-1} + w$$

with $s_{-1} = 0$. This seems like an especially difficult version of our standard maximization problem. We now need to pick a infinite vector of consumption/saving choices.

But the dynamics actually make this problem in some sense easier.

Dynamics

To see this, consider the finite horizon version of this problem

$$\max_{s,c\in\mathbb{R}_+^T} \sum_{t=0}^{\infty} \delta^t u(c_t)$$
$$c_t + s_t \leq (1+r)s_{t-1} + w$$

where $T \in \mathbb{N}$. This is a particularly simple version of our standard constrained optimization problem. Note that:

- The cost function is additively separable
- s_t shows up in at most 2 constraints, c_t shows up in at most 1.
- We could try to solve this using multipliers (more on this in a bit).
- Let's try a different angle

Bellman Equations

Define a family of value functions

$$V_T(s') = \max_{s,c \in \mathbb{R}_+^T} \sum_{t=0}^{T} \delta^t u(c_t)$$

$$\text{s.t. } c_t + s_t \leq (1+r)s_{t-1} + w$$

$$s_{-1} = s'$$

We can essentially solve this problem backwards. Clearly

$$V_T(s') = \max_{s,c \in \mathbb{R}_+} u(c) + \delta V_{T-1}(s)$$

$$\text{s.t. } c + s \leq (1+r)s' + w$$

If we knew V_T this is an easy problem for babies.

Bellman Equations

The same should hold in our infinite horizon problem.

$$V(s') = \max_{s,c \in \mathbb{R}_+} u(c) + \delta V(s)$$
$$\text{s.t. } c + s \leq (1+r)s' + w$$

where $V(s)$ is the value function for the infinite horizon problem. These sorts of equations are called Bellman Equations.

Principle of Optimality

While this recursion is intuitive, it's requires proof.

Theorem

If $V(s_{-1})$ is the value function for

$$\sup_{s,c\in\mathbb{R}_+^T} \sum_{t=0}^{\infty} \delta^t u(c_t)$$

$$c_t + s_t \leq (1+r)s_{t-1} + w$$

then it is a bounded solution to the Bellman Equation

$$V(s') = \sup_{s,c\in\mathbb{R}_+} u(c) + \delta V(s)$$

$$s.t.\ \ c + s \leq (1+r)s' + w$$

Principle of Optimality

Note first that for we can define a function U that maps from any fixed policy $x = (c_t, s_t)_{t=0}^{\infty}$ to payoffs and satisfies the equation

$$U(x) = \sum_{t=0}^{\infty} \delta^t u(c_t) = u(c_0) + \delta U(x')$$

where $x' = (c_t, s_t)_{t=1}^{\infty}$. For any $\epsilon > 0$ we can find a policy x such that $U(x') \geq V(s_0) - \epsilon$, so

$$V(s_{-1}) \geq u(c_0) + \delta U(x') \geq u(c_0) + \delta V(s_0) - \delta\epsilon$$

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$$V(s_{-1}) \geq u(c_0) + \delta U(x') \geq u(c_0) + \delta V(s_0) - \delta\epsilon$$

So for all feasible s_0, c_0, s_{-1} we get

$$V(s_{-1}) \geq \sup u(c_0) + \delta V(s_0)$$

Since V is a supremum, we also get for any $\epsilon > 0$ there is some for some feasible policy x s.t.

$$V(s_{-1}) \leq u(c_0) + \delta V(s_0) + \epsilon \leq \sup u(c_0) + \delta V(s_0) + \epsilon$$

Therefore, the V satisfies the Bellman Equation

Bellman Equations

So know we know the value function satisfies the equation

$$V(s') = \max_{c,s} u(c) + \delta V(s) \text{ s.t. } c + s \leq (1+r)s' + w$$

We'll now show that:

- V is the unique (continuous, bounded) solution to this problem.
- How to solve for V.

Sequences

Part of our maximization exercise is now solving for a function. It would be helpful to have some analogues to tools we've developed for real numbers here.

Recall in $\mathbb{R}^n$ we measure distance according to a norm $||x||$. This satisfies

- $||0|| = 0$.
- $||ax|| = |a|||x||$ for any $a \in \mathbb{R}$
- $||x + y|| \leq ||x|| + ||y||$

Sequences

This norm gives us concepts like convergent sequences.

Definition

A sequence $x_n \in \mathbb{R}^k$ converges to x if for any $\epsilon > 0$, there exists an N s.t. if $n \geq N$ then $||x_n - x|| < \epsilon$.

Definition

A sequence $x_n \in \mathbb{R}^k$ is Cauchy if for any $\epsilon > 0$ there exists a N s.t. if $n, m \geq N$ then $||x_n - x_m|| < \epsilon$.

Theorem

A real valued sequence converges if and only if it is Cauchy.

A Function Space

What if we tried to do the same thing for functions.

- Let $C(X)$ be the set of bounded, continuous functions with domain X.
- Define a norm
$$||f|| = \sup_{x \in X} |f(x)|$$
Why do we call this a norm? It's easy to verify that it satisfies three important properties:
  - $||0|| = 0$.
  - $||af|| = |a|||f||$ for any $a \in \mathbb{R}$
  - $||f+g|| \leq ||f|| + ||g||$

Sequences of Functions

We can define convergence of functions exactly like we did for sequences

Definition

A sequence $f_n \in C(X)$ converges to f if for any $\epsilon > 0$, there exists an N s.t. if $n \geq N$ then $||f_n - f|| < \epsilon$.

This is a concept called uniform convergence. Note that it is asking for more then just $f_n(x) \to f(x)$ at every $x \in X$.

Cauchy Sequences of Functions

Theorem

$C(X)$ with the norm $||f|| = \sup |f(x)|$ is complete, i.e. every Cauchy sequence converges to a function in $C(X)$.

Take a Cauchy sequence $(f_n)_{n=1}^{\infty}$. We know that for any $\epsilon > 0$ there exists an N s.t. if $n, m \geq N$ then $||f_n - f_m|| < \epsilon$.

Take $f(x) = \lim f_n(x)$, which exists since $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$. This is a continuous bounded function (try proving this yourself).

Proof-Continued

We want to show that f is also the limit of f_n under the sup norm.

- Now take any $\epsilon > 0$. We want to show that there exists an N s.t. if $n \geq N$ then $||f - f_n|| < \epsilon$.

Proof-Continued

We want to show that f is also the limit of f_n under the sup norm.

- Now take any $\epsilon > 0$. We want to show that there exists an N s.t. if $n \geq N$ then $||f - f_n|| < \epsilon$.
- Consider $\sup_{x \in X} |f_n(x) - f(x)|$. We know for any N and for any $m > N$ and for any $x \in X$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f(x) - f_m(x)|$$

Proof-Continued

We want to show that f is also the limit of f_n under the sup norm.

- Now take any $\epsilon > 0$. We want to show that there exists an N s.t. if $n \geq N$ then $||f - f_n|| < \epsilon$.
- Consider $\sup_{x \in X} |f_n(x) - f(x)|$. We know for any N and for any $m > N$ and for any $x \in X$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f(x) - f_m(x)|$$

- Choose N so that for $n, m \geq N$, $||f_n - f_m|| < \epsilon/2$. Take any $x \in X$ and choose m so that $|f(x) - f_m(x)| < \epsilon/2$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f(x) - f_m(x)| \leq \epsilon$$

Note that our choice of N was independent of x, so we're done.

Bellman Equations

We've shown that the value function satisfies the Bellman Equation. Is any function that satisfies the Bellman equation a value function?

- Can we find a function that satisfies this recursion.
- Recall(?) when you took differential equations, you showed that a solution to a differential equation existed by writing down an "algorithm" to find it.
- Can we do the same thing here?

We need one more concept

Definition

Let $k \in (0, 1)$. A function $T : X \to X$ is a contraction of modulus k if $||Tx - Ty|| \leq k||x - y||$.

Bellman Equations

Theorem (Contraction Mapping Theorem)

If $T : C(X) \to C(X)$ is a contraction, then it has a unique fixed point, a function $f \in C(X)$ s.t. $Tf = f$.

Moreover, this fixed point is the limit of $T^n g$ (where T^n denotes the n-fold composition of T) for any $g \in C(X)$, i.e. for any $\epsilon > 0$ there is an N s.t. if $n \geq N$ then $||T^n g - f|| < \epsilon$.

Proof

This is a theorem holds more broadly in complete metric spaces, i.e. spaces where Cauchy sequences converge. The statement of the theorem already gives a big hint for how to prove this.

Proof

- Consider the sequence $f_n = T^n f_0$ for some arbitrary f_0.
- This is a Cauchy sequence.
- So it has a limit f^*. Since T is a contraction, it is continuous and thus this is a fixed point ($Tf^* = \lim_{n\to\infty} T(T^n f_0) = \lim_{n\to\infty} T^{n+1} f_0 = f^*$).

Proof

- Consider the sequence $f_n = T^n f_0$ for some arbitrary f_0.
- This is a Cauchy sequence.
- So it has a limit f^*. Since T is a contraction, it is continuous and thus this is a fixed point ($Tf^* = \lim_{n\to\infty} T(T^n f_0) = \lim_{n\to\infty} T^{n+1} f_0 = f^*$).
- If there were two fixed points f^* and g^* then $||f^* - g^*|| = ||Tf^* - Tg^*|| < k||f^* - g^*||$.

Bellman Equations

Theorem (Blackwell's Theorem)

Let $T : C(X) \to C(X)$ be an operator that satisfies

- *Monotonicity: if $f, g, \in C(X)$ and $f(x) \leq g(x)$ for all $x \in X$ then $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*
- *Discounting: There exists some $\delta \in (0, 1)$ such that*

  $$(T(f + a))(x) \leq (Tf)(x) + \delta a$$

  for all $f \in C(X)$, $a \geq 0$ and $x \in X$.

then T is a contraction with modulus δ.

Blackwell's Theorem

To see this, note that for any f, g and any $x \in X$

$$f(x) \leq g(x) + ||f(x) - g(x)||$$

and then just apply T to both sides to get

$$Tf \leq Tg + \delta||f - g||.$$

The same argument with f and g flipped gives

$$Tg \leq Tf + \delta||f - g||.$$

Bellman Equation

In our consumption/savings problem, let

$$(TV)(s') = \max_{s,c \in \mathbb{R}_+} u(c) + \delta V(s)$$
$$\text{s.t. } c + s \leq (1+r)s' + w$$

This clearly satisfies both properties of Blackwell's theorem, and thus it has a fixed point.

Dynamic Programming

The key trick here was that the dynamic structure let us break this up into a bunch of recursive subproblems.

- This is called the principal of optimality
- Uses the natural recursive structure implied by optimization over time to simplify the maximization problem.
- In finite horizon problems we can work backwards.
- In infinite horizon problems, we have a difference equation that $V(\cdot)$ must satisfy, and a numerical tool to solve for it.

Euler Equation

We can also use the Bellman equation to get a equation for consumption:

We have

$$V(s_t) = \max u(c) + \delta V(s) \text{ s.t. } c + s \leq (1+r)s_t + w$$

If V is differentiable, then the first order condition is

$$u'(c_{t+1}) = \delta V'(s_{t+1})$$

The envelope theorem gives us

$$V'(s_t) = (1+r)u'(c_{t+1})$$

Since this holds at all t, we get

$$u'(c_{t+1}) = \delta(1+r)u'(c_t)$$

Euler Equation

Hoping for closed form solutions to these is mostly a waste of time. A notable exception are the class of CRRA utility functions:

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

which make the Euler equations satisfy a linear difference equation

$$c_{t+1} = (\delta(1 + r))^{\frac{1}{\gamma}} c_t.$$

So consumption grows/shrinks by a constant factor each period until we hit the non-negativity constraint. We could in principle use these consumption paths to then solve for the optimal c_0.

Guess and Verify

An alternative approach would be to solve for V, usually by guessing a functional form and verifying it.

- For instance $w = 0$ and $u(c) = \ln c$, one could conjecture that $V(s') = A \ln s' + K$ and solve for an A and K that make V satisfy the Bellman equation.
- Sometimes (often) closed form solutions are just going to be hopeless.
- But, even without a closed form solution, we have a lot to work with.
  - e.g. we can see from the Euler equation whether consumption is growing or shrinking over time

Search

A worker is searching for a job.

- At each time t, if the worker is unemployed they draw a new job with wage w, drawn from pdf $f(w)$ with support W contained in $\mathbb{R}_+$.
- Once they accept a job, they collect w from then on.
- The worker collects 0 in each period they are unemployed.
- There are T periods, discount rate δ.
- Bellman equations make this problem very easy to formulate and solve

Search

If a worker accepts a wage of w at time t, they receive

$$\sum_{0}^{T-t} \delta^t w = w \left(\frac{1-\delta^{T-t}}{1-\delta} \right)$$

so this problem has Bellman equation

$$V_t(w) = \max \left\{ w \left(\frac{1-\delta^{T-t}}{1-\delta} \right), \delta \int_W V_{t+1}(x) f(x)\, dx \right\}.$$

From this, it's easy to see

- The optimal strategy is a cutoff strategy.
- The cutoffs are decreasing.

Search

What if we solved the infinite horizon version of this problem. This has Bellman equation

$$V(w) = \max\left\{\frac{w}{(1-\delta)}, \delta \int_W V(x)f(x)\,dx\right\}$$

What does equilibrium look like?

Search

Clearly the accept/reject decision is monotone. There must be an w^* where you accept if $w > w^*$.

Above the cutoff

$$V(w) = w/(1-\delta)$$

below the cutoff

$$V(w) = \delta E(V(x))$$

If the value function is continuous then

$$V(w^*) = \frac{w^*}{1-\delta} = \delta E(V(x))$$

Search

So at the optimal w^* it must be that

$$\begin{aligned}
\frac{w^*}{1-\delta} &= \delta E(V(x)) \\
&= \delta(\int_0^{w^*} V(w^*) f(x)\,dx + \int_{w^*}^{\infty} \frac{x}{1-\delta} f(x)\,dx) \\
&= \delta F(w^*) \frac{w^*}{1-\delta} + \frac{\delta}{1-\delta} \int_{w^*}^{\infty} x f(x)\,dx
\end{aligned}$$

So w^* solves

$$(1-\delta F(w^*))w^* - \delta \int_{w^*}^{\infty} x f(x)\,dx = 0$$

The LHS is strictly increasing over the support of the wage distribution, continuous, and negative at 0, positive at ∞. So this implicitly defines w^*.

Continuous Time

When you start looking at dynamic models, you often see them formulated in continuous time. Your intuition may be that this seems like a terrible idea; we've gone from solving for a sequence of numbers to a function.

But, thinking about the Bellman equation makes it clear why this can be a convenient modeling too.

Consumption/Savings

Consider the problem

$$\max \int_0^\infty e^{-\rho t} u(c_t)\, dt$$
$$\text{s.t. } \dot{s}_t = w + rs_t - c_t$$

We can write the Bellman equation

$$V(s_t) = \max \int_t^{t+\Delta} e^{-\rho(x-t)} u(c_x)\, dx + e^{-\rho\Delta} V(s_{t+\Delta})$$

HJB Equations

$$V(s_t) = \max \int_t^{t+\Delta} e^{-\rho(x-t)} u(c_x)\, dx + e^{-\rho\Delta} V(s_{t+\Delta}).$$

Assuming things are well behaved

$$V(s_t) = \max \int_t^{t+\Delta} e^{-\rho(x-t)} u(c_x)\, dx + e^{-\rho\Delta}(V(s_t) + V'(s_t)\dot{s}_t\Delta + o(\Delta)$$

$$(1 - e^{-\rho\Delta})V(s_t) = \max \int_t^{t+\Delta} e^{-\rho(x-t)} u(c_x) dx + e^{-\rho\Delta} V'(s_t)\dot{s}_t\Delta + o(\Delta$$

$$\rho V(s_t) = \max u(c_t) + V'(s_t)\dot{s}_t$$

$$\rho V(s_t) = \max u(c_t) + V'(s_t)(w + rs_t - c_t)$$

HJB Equation

So we are left with the differential equation

$$\rho V(s) = u(c) + V'(s)(w + rs - c)$$

where c solves $u'(c) = V'(s)$. This is called the Hamilton-Jacobi-Bellman Equation.

We can learn a bit more from this, by the envelope theorem

$$\rho V'(s) = V''(s)(w + rs - c) + rV'(s)$$

Combining this with the FOC, we get

$$(\rho - r)u'(c) = u''(c)\frac{dc}{ds}\frac{ds}{dt}$$

so c_t solves the functional equation

$$\frac{\dot{c}_t/c_t}{r - \rho} = -\frac{u'(c_t)}{c_t u''(c_t)}$$

This the continuous time version of our Euler equation. This tells us the "elasticity of intertemporal substitution."

HJB Equations

The HJB equation gives us a differential equation to solve. In some cases, we can guess a solution. Assume $u(c) = \ln c$, $r > \rho$ and guess $V(s) = b \ln(s + a) + k$ for some b, a, k. Then

$$\frac{1}{c} = \frac{b}{s+a}$$

And want to find b, a, k s.t.

$$\rho V(s) = \ln \frac{s+a}{b} + V'(s)(w + rs + \frac{s+a}{b})$$
$$\rho(b \ln(s+a) + k) = \ln(s+a) - \ln b + \frac{b(w+rs)}{s+a} - 1$$

HJB Equation

$$(b \ln(s+a) + k) = \frac{1}{\rho}\ln(s+a) - \frac{1}{\rho}\ln b + \frac{b(w+rs)}{\rho(s+a)} - \frac{1}{\rho}$$

So $b = \frac{1}{\rho}$. We need to set a so that

$$\frac{1}{\rho}\ln \rho + \frac{1}{\rho^2}\frac{w+rs}{s+a} - \frac{1}{\rho}$$

is independent of s, so $a = w/r$. Thus

$$V(s) = \frac{1}{\rho}\ln(s + w/r) + \frac{1}{\rho}\ln \rho + \frac{r}{\rho^2} - \frac{1}{\rho}$$

and

$$c = \rho\left(s + \frac{w}{r}\right)$$

so you consume a constant share ρ/r of your flow income.

Hamiltonians

Let's go back to our discrete time consumption/savings problem (with some modified time indices)

$$\max_{s,c \in \mathbb{R}_+^T} \sum_{t=0}^{T} \delta^t u(c_t)$$

$$s_{t+1} - s_t = rs_t + w - c_t$$

What if we tried solving this with multipliers? We get FOCs

$$\delta^t u'(c_t) = -\lambda_t$$

and

$$\lambda_t - \lambda_{t-1} = -r\lambda_t.$$

We can see how the recursive structure comes into play here.

Hamiltonians

Note that c_t only appears in one constraint, while s_t links consecutive constraints together. Maybe we can reformulate this as a sequence of static problems?

Consider the following static problem

$$H(c, s, \lambda, t) = \max_{c} \delta^t u(c_t) - \lambda_t (rs_t + w - c_t)$$

When we optimize with respect to c we get

$$u'(c_t) = -\lambda_t.$$

Hamiltonians

We have the function

$$H^*(s_t, \lambda_t, t) = \max_c \delta^t u(c_t) - \lambda_t(rs_t + w - c_t).$$

The objective is called a Hamiltonian. Observe that, by the envelope theorem

$$H_{s_t} = -r\lambda_t$$

and

$$H_{\lambda_t} = -(s_{t+1} - s_t)$$

Finally we know that, across problems

$$\lambda_t - \lambda_{t-1} = -r\lambda_t$$

The Maximum Principle

This gives us the following maximum principle

Theorem (The Maximum Principle)

Consider a maximization problem,

$$\max \sum_{t=0}^{T} F(x_t, y_t) \qquad s.t. \ y_{t+1} - y_t = G(x_t, y_t)$$

With initial and terminal conditions $y_T = b, y_0 = a$. Let x_t^* be the solution to this. Define the Hamiltonian

$$H(x_t, y_t, \lambda_t, t) = F(x_t, y_t, t) - \lambda_t G(x_t, y_t, t)$$

For each t, x_t^* maximizes H when λ and y satisfy

- $\lambda_t - \lambda_{t-1} = F_y(x_t^*, y_t, t) - \lambda_t G_y(x_t^*, y_t, t)$
- $y_{t+1} - y_t = G(x_t^*, y_t, t)$
- $y_T = b, y_0 = a$

The Maximum Principle

Even though c_t only appears in a single period, it impacts all periods.

The Hamiltonian uses the multiplier to turn this into a static problem, adjusting the objective using the shadow price of consumption today. The shadow prices are exactly equal to the rate of return from holding y_t, so the condition on λ's shuts down "arbitrage"

Hamiltonians

Back to consumption/savings. We have

$$
\begin{aligned}
\delta^t u'(c_t) &= -\lambda_t \\
\lambda_t - \lambda_{t-1} &= -r\lambda_t \\
s_{t+1} - s_t &= rs_t + w - c_t
\end{aligned}
$$

Note that

$$u'(c_{t-1}) = \delta(1+r)u'(c_t),$$

which is a discrete time Euler equation.

Unsurprisingly, there's a connection between this approach and the Bellman equation approach, the multiplier here is indirectly capturing the marginal benefit of savings.

Hamiltonians

Consider for instance the simple version of our problem

$$
\begin{aligned}
\max \sum_{t=0}^{T} \delta^t \ln(c_t) \\
\text{s.t. } s_{t+1} - s_t = -c_t \\
s_0 = Y
\end{aligned}
$$

How do I optimally distribute Y units of consumption?

Hamiltonians

We have the Hamiltonian

$$\delta^t \ln(c_t) + \lambda_t c_t$$

Which gives us

$$\begin{aligned} \delta^t / c_t &= -\lambda_t \\ \lambda_t - \lambda_{t-1} &= 0 \\ s_{t+1} - s_t &= -c_t \\ s_{T+1} &= 0 \end{aligned}$$

So

$$\delta c_{t-1} = c_t$$

and this problem becomes

$$\sum_{t=0}^{T} \delta^t c_0 = Y,$$

so $c_0 = (1-\delta)Y/(1-\delta^T)$.

Hamiltonians

What if we wanted to use this to solve an infinite horizon problem?

The obvious thing to do is to replace $s_T = 0$ with $\lim_{t\to\infty} s_t = 0$.

- This is called a transversality condition.
- We could incorporate the constraint that savings just have to end above 0 by modifying this to $\lim s_t \lambda_t = 0$.
- In general, figuring out these terminal conditions is beyond the scope of this course.

In our previous problem, this gives us exactly what we'd get if we took $T \to \infty$.

Hamiltonians

We have a bunch of difference equations. Again, continuous time is going to make our lives much easier. Consider

$$\max \int_{t=0}^{T} e^{-\rho t} \ln(c_t)$$
$$\text{s.t. } \dot{s}_t = -c_t$$
$$s_0 = Y, S_T = 0$$

and the continuous time analogues of our conditions are

$$e^{-\rho t}/c_t = -\lambda_t$$
$$\dot{\lambda}_t = 0$$
$$\dot{s}_t = -c_t.$$

The Maximum Principle

This gives us the following maximum principle

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Consider a maximization problem,

$$\max \int_{t=0}^{T} F(x_t, y_t, t) \qquad \text{s.t. } \dot{y}_t = G(x_t, y_t, t)$$

With initial and terminal conditions $y_T = b, y_0 = a$*. Let* x_t^* *be the solution to this. Define the Hamiltonian*

$$H(x_t, y_t, \lambda_t, t) = F(x_t, y_t, t) - \lambda_t G(x_t, y_t, t)$$

For each t*,* x_t^* *maximizes* H *when* λ *and* y *satisfy*

- $\dot{\lambda}_t = F_y(x_t^*, y_t, t) + \lambda_t G_y(x_t^*, y_t, t)$
- $\dot{y}_t = G(x_t^*, y_t, t)$
- $y_T = b, y_0 = a$

Hamiltonians

$$e^{-\rho t}/c_t = -\lambda_t$$
$$\dot{\lambda}_t = 0$$
$$\dot{s}_t = -c_t$$

We can differentiate the first order condition to get

$$-\rho e^{-\rho_t} = \dot{\lambda}_t c_t + \dot{c}_t \lambda_t$$

which gives

$$-\rho = \dot{c}_t / c_t$$

so $c_t = c_0 e^{-\rho t}$ and $c_0 \int_0^T e^{-\rho t}\, dt = Y$

Hamiltonians

Back to our old problem.

$$\max \int_{t=0}^{\infty} e^{-\rho t} u(c_t)$$
$$\text{s.t. } \dot{s}_t = w + rs_t - c_t$$
$$s_0 = 0$$

we get

$$e^{-\rho t} u'(c_t) = -\lambda_t$$
$$\dot{\lambda}_t = -r\lambda_t$$
$$\dot{s}_t = w + rs_t - c_t$$
$$s_0 = 0, \lim_{t \to \infty} \lambda_t s_t = 0;$$

Hamiltonians

From some algebra

$$
\begin{aligned}
-\rho e^{-\rho t}u'(c_t) + e^{-\rho t}u''(c_t)\dot{c}_t &= -\dot{\lambda}_t \\
-\rho e^{-\rho t}u'(c_t) + e^{-\rho t}u''(c_t)\dot{c}_t &= re^{-\rho t}u'(c_t) \\
\frac{u''(c_t)}{u'(c_t)}\dot{c}_t &= (\rho - r)
\end{aligned}
$$

which looks familiar.

Hamiltonians

We have an initial and terminal condition.
Let $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. We have Euler equation

$$\dot{c}_t = \frac{1}{\sigma}(r-\rho)c_t$$

giving $c_t = c_0 e^{\frac{r-\rho}{\sigma}t}$. In addition we have

$$\dot{\lambda}_t = -r\lambda_t \dot{s}_t = w + rs_t - c_0 e^{\frac{r-\rho}{\sigma}t}$$

Hamiltonians

So now we have differential equation

$$\dot{s}_t = w + rs_t - c_0 e^{\frac{(r-\rho)}{\sigma}t}.$$

If you remember enough differential equation tricks, this is easy to solve

$$\frac{d}{dt}[s_t e^{-rt}] = (\dot{s}_t - rs_t)e^{-rt} = e^{-rt}(w - c_0 e^{\frac{(r-\rho)}{\sigma}t})$$

Which gives us

$$s_t e^{-rt} = \frac{1}{r}w[1 - e^{-rt}] + c_0 \frac{\sigma}{r(1-\sigma)-\rho}[1 - e^{\frac{r(1-\sigma)-\rho}{\sigma}t}].$$

We can then plug in $\lim \lambda_t s_t = \lim e^{-rt} s_t = 0$ and solve for c_0. We get

$$c_0 = \left(\frac{\rho}{r\sigma} - \frac{1-\sigma}{\sigma}\right) w.$$

Growth

We can reframe this problem in terms of the growth of an economy.

$$\max \int_0^\infty e^{-\rho t} u(c_t)\, dt$$
$$\text{s.t. } \dot{k}_t = F(k_t) - \delta k_t - c_t$$
$$c_t, k_t \geq 0.$$

where k is capital, F is capital production function, strictly concave and strictly increasing, $F'(0) > \rho + \delta$.

Hamiltonians

As before

$$u'(c_t) = -e^{\rho t}\lambda_t$$
$$\dot{\lambda}_t = -\lambda_t[F'(k_t) - \delta]$$

and through substitution we get

$$-\frac{u''(c)}{u'(c)}\dot{c} = F'(k) - (\rho + \delta)$$

Phase Diagrams

So we have two differential equations

$$\dot{c} = \frac{F'(k) - (\rho + \delta)}{-\frac{u''(c)}{u'(c)}}$$
$$\dot{k} = F(k) - \delta k - c$$

we can just these to graphically trace trajectories in (k, c) space.

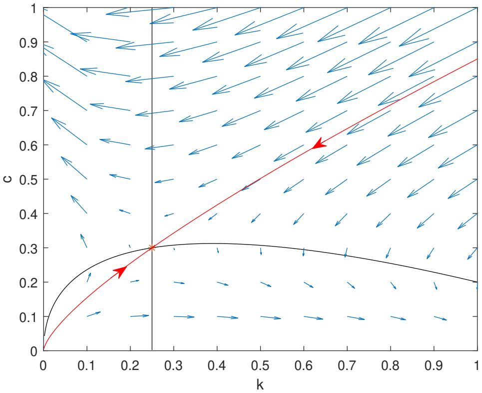
Phase Diagrams

We can divide (k, c) space into regions.

- k is increasing iff $c < F(k) - \delta k$.
- c is increasing iff $F'(k) < \rho + \delta$

Let c^*, k^* solve the equations

$$\begin{aligned} c^* &= F(k^*) - \delta k^* \\ F'(k^*) &= \rho + \delta \end{aligned}$$



- Qualitatively, three possible paths:
  - $c \to 0$ and $k \to \infty$.
  - $k \to 0$ and $c \to 0$ (due to non-negativity constraints).
  - $c \to c^*$ and $k \to k^*$.
- Intuitively, $c(0)$ is optimal if it puts us on the third path.
- This path satisfies the transversality condition $\lambda_t k_t \to 0$.

Dynamics

We've seen two different approaches to solving dynamic programs. Which is more useful depends a bit on the problem.

- Some of the early literature (e.g. the textbook) is a bit dismissive of the HJB approach, which requires us to characterize the entire value function as opposed to the optimal trajectory.
- But, the HJB approach is, at the moment, more prominent in economics.
  - Numerical tools have made solving for the value function less costly.
  - Stochastic models are in general a bit more straightforward to analyze with Bellman equations and have become prominent in modern economics
  - Bellman equations seem a bit easier to work with in things like dynamic games.