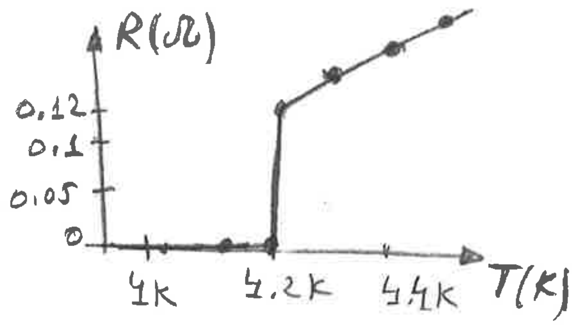
Superconductivity

1911 - Heike Kamerlingh Onnes

Electrical resistance of Hg (metal!) dropped to $< 10^{-5}\Omega$ at $T_c = 4.2K$



Other typical metals become superconductors:

$$T_c = 1.2K \text{ for Al}$$
$$T_c = 7.2K \text{ for Pb}$$
$$T_c = 9.2K \text{ for Nb}$$

1986 – discovery of high-T_c compounds by J.G.Bednorz and K.A.Müller

$$T_c = 95K \text{ for } YBa_2Cu_3O_{7-\delta}$$
$$T_c = 125K \text{ for } Tl_2Ba_2Ca_2Cu_3O_{10}$$
$$T_c = 133K \text{ for } HgBa_2Ca_2Cu_3O_{8+\delta}$$

These are not metals, they are ceramic materials at room temperature!

Meissner effect

- In the beginnings of superconductivity research it was hoped that the electromagnetic properties could be derived from the property of infinite conductivity.

$$\sigma = \infty \qquad \left.\begin{array}{l}\vec{J} = \sigma \cdot \vec{E} \\ \vec{J} = \text{finite}\end{array}\right| \Rightarrow \vec{E} = 0 \Rightarrow \vec{\nabla} \times \vec{E} = 0$$

$$\text{Maxwell: } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Bigg\} \Rightarrow \frac{\partial \vec{B}}{\partial t} = 0$$

So $\vec{B}$ = constant inside a superconductor
and also we expect it to be dependent on the way it was cooled down (e.g. either in the presence or absence of the magnetic field)

BUT in 1933 Meissner and Ochsenfeld discovered that $\vec{B} = 0$.
The magnetic field inside the superconductor is not just constant, but it is exactly zero. Magnetic field lines are expelled.
A superconductor is a perfect diamagnet.

Theory development

- 1935 – phenomenological theory developed by F. & H. London (two brothers!)
- 1957 – BCS (Bardeen – Cooper – Schrieffer) theory
- high-T_c superconductivity – maybe you?

Elements of London theory

Consider a particle of mass m^* and charge q^*. It will turn out that $m^* = 2m_e$ and $q^* = -2e$; these particles are Cooper pairs, and a complete understanding of what they are is provided by the BCS theory.

$\vec{B} = \vec{\nabla} \times \vec{A}$ $\vec{A}$ = magnetic vector potential V = electric potential

Schrödinger equation: $$i\hbar \frac{\partial}{\partial t} \psi(\vec{r},t) = \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)^2 \psi(\vec{r},t) + q^* V(\vec{r},t)\psi(\vec{r},t)$$

- Recall also that $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$

Note: the Hamiltonian of a free particle in a magnetic field is

$$H = \frac{\vec{\Pi}^2}{2m^*} \quad \text{where } \vec{\Pi}(\vec{r}) = -i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})$$

canonical momentum

Probability density

$$P(\vec{r},t) = |\psi(\vec{r},t)|^2$$

$$\frac{\partial P(\vec{r},t)}{\partial t} = \frac{\partial \psi^*(\vec{r},t)}{\partial t}\psi(\vec{r},t) + \psi^*(\vec{r},t)\frac{\partial \psi(\vec{r},t)}{\partial t}$$

$$= \frac{i}{\hbar}\Big\{\left[\frac{1}{2m^*}\left(i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)^2 \psi^*(\vec{r},t)\right]\psi(\vec{r},t) - \psi^*(\vec{r},t)\left[\frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)^2\psi(\vec{r},t)\right]\Big\}$$

$$= -\vec{\nabla}\cdot\vec{J}(\vec{r},t)$$

So $$\frac{\partial P(\vec{r},t)}{\partial t} = -\vec{\nabla}\cdot\vec{J}(\vec{r},t)$$

where $$\vec{J}(\vec{r},t) = \frac{1}{2m^*}\left[\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)\psi(\vec{r},t)\right]^*\psi(\vec{r},t) + \frac{1}{2m^*}\psi^*(\vec{r},t)\cdot\left[\left(-i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\right)\psi(\vec{r},t)\right]$$

Key point: the wavefunction $\psi(\vec r, t)$ for a superconductor can be regarded as an order parameter (a macroscopic wavefunction!)

Let us call this "solution" ψ_s, and let's normalise it to the number of particles.

Ginzburg-Landau order parameter

$$\vec{\psi}_s(\vec r, t) = \sqrt{n_s(\vec r, t)}\, e^{i\theta(\vec r, t)}$$

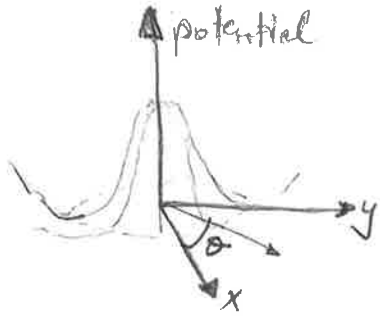
$$\int \psi_s^*(\vec r, t)\psi_s(\vec r, t)\, d\vec r = \text{total number of superconducting particles ("superelectrons")}$$

$n_s(\vec r, t)$ = density of superconducting particles

$\theta(\vec r, t)$ = superconducting phase

Appears as a result of a broken symmetry.

- From now on we will assume $n_s(\vec r, t) \equiv n_s = \text{const.}$



So $\vec{j}_s(\vec r, t) = \frac{\hbar n_s}{m^*}\left[\vec\nabla\theta(\vec r, t) - \frac{e^*}{\hbar}\vec A(\vec r, t)\right]$

Electrical current: $\vec J = e^* \vec j$

$$\vec J_s(\vec r, t) = \frac{\hbar e^* n_s}{m^*}\left[\vec\nabla\theta(\vec r, t) - \frac{e^*}{\hbar}\vec A(\vec r, t)\right] = \text{superconducting current density}$$

supercurrent

Gauge-invariant phase gradient

$$\begin{cases}\theta \to \theta + \frac{e^*}{\hbar}\chi \\ \vec A \to \vec A + \vec\nabla \chi\end{cases}$$

CONSEQUENCES:

PERFECT CONDUCTIVITY

- Here we consider θ = constant in $\vec r$; $\vec\nabla\theta = 0$

$$\vec J_s = -\frac{e^{*2}}{m^*} n_s \vec A \Rightarrow$$

$$\frac{d\vec J_s(\vec r, t)}{dt} = -\frac{e^{*2}}{m^*} n_s \frac{d\vec A(\vec r, t)}{dt} \quad \text{or} \quad \boxed{\frac{d\vec J_s(\vec r, t)}{dt} = +\frac{e^{*2} n_s}{m^*}\vec E(\vec r, t)}$$

Recall Maxwell: $\begin{cases}\vec\nabla \times \vec E = -\frac{\partial \vec B}{\partial t} \\ \vec B = \vec\nabla \times \vec A\end{cases}$

What does it mean? Take a ballistic superelectron (no collisions with atoms, impurities etc.)

$$\left.\begin{aligned} m^*\frac{d\vec v_s}{dt} &= e^*\vec E \\ \vec J_s &= n_s \cdot e^* \cdot \vec v_s \end{aligned}\right\} \Rightarrow \boxed{\frac{d\vec J_s}{dt} = \frac{e^{*2} n_s}{m^*}\vec E}$$

Note the difference w.r.t $\vec J = \sigma\vec E$ (Ohm's law)!

Meissner effect

Let us look at Maxwell's equations: $\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_s \end{cases}$

Now $\vec{B} = \vec{\nabla}\times\vec{A}$ so

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{\substack{\text{we can use Coulomb gauge} \\ \vec{\nabla}\cdot\vec{A}=0}} - \vec{\nabla}^2\vec{A} = -\vec{\nabla}^2\vec{A}$$

So
$$\left.\begin{aligned} \vec{\nabla}^2\vec{A} &= -\mu_0 \vec{J}_s \\ \vec{J}_s &= -\frac{q^{*2}}{m^*} n_s \vec{A} \end{aligned}\right\} \Rightarrow \boxed{\vec{\nabla}^2\vec{A} = \frac{\mu_0 q^{*2} n_s}{m^*}\vec{A}}$$

<u>Notation</u>: $\lambda_L = \sqrt{\dfrac{m^*}{\mu_0 n_s q^{*2}}}$ = London penetration length.

Since $\vec{J}_s = -\dfrac{q^{*2} n_s}{m^*}\vec{A}$ we have $\boxed{\vec{J}_s = -\dfrac{1}{\mu_0 \lambda_L^2}\vec{A}}$

and $\boxed{\vec{\nabla}^2\vec{A} = \dfrac{1}{\lambda_L^2}\vec{A}}$

So $\mu_0 \lambda_L^2 \vec{J}_s = -\vec{A} \Rightarrow \mu_0 \lambda_L^2\, \vec{\nabla}\times\vec{J}_s = -\vec{\nabla}\times\vec{A} = -\vec{B}$

$$\Downarrow$$

$$\mu_0 \lambda_L^2 \frac{\partial}{\partial t}\underbrace{(\nabla\times\vec{J}_s)}_{\text{but } \vec{J}_s = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}} = -\vec{\nabla}\times\underbrace{\frac{\partial\vec{A}}{\partial t}}_{=-\vec{E}}$$

because the voltage is zero

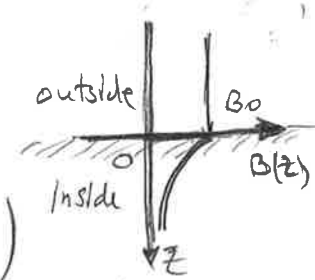
see $\vec{E} = -\dfrac{\partial\vec{A}}{\partial t} - \vec{\nabla}V$

$$\Downarrow$$

$$\lambda_L^2 \frac{\partial}{\partial t}\cdot\underbrace{\left(\vec{\nabla}\times(\vec{\nabla}\times\vec{B})\right)}_{=\vec{\nabla}\cdot(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0}) - \vec{\nabla}^2 B} = \underbrace{\vec{\nabla}\times\vec{E}}_{= -\frac{\partial\vec{B}}{\partial t}}$$

$$\Rightarrow \lambda_L^2\, \vec{\nabla}^2\vec{B} = +\vec{B}$$

or $\left[\dfrac{1}{\lambda_L^2} - \vec{\nabla}^2\right]\vec{B}(\vec{r}) = 0$



Take $\vec{B}(\vec{r}) = (0, B(z), 0) \Rightarrow B(z) = B_0 \exp(-z/\lambda_L)$

This is the Meissner effect. The field decays exponentially in the superconductor.

To review: we found $\boxed{\vec{J}_s = -\frac{1}{\mu_0 \lambda_L^2}\vec{A}}$

and $\boxed{\nabla^2 \vec{A} = \frac{1}{\lambda_L^2}\vec{A}}$

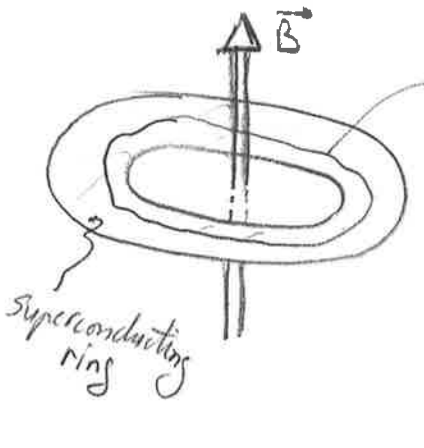
or $\boxed{\frac{d\vec{J}_s}{dt} = \frac{1}{\mu_0 \lambda_L^2}\vec{E}}$ — called 1st London equation

$\boxed{\vec{B} = -\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{J}_s}$ — called 2nd London equation

So the magnetic field can penetrate at most to depths $\simeq \lambda_L$. Currents can flow in this region, but deep in the bulk they will be zero.

• QUANTIZATION OF FLUX

So far we did not discuss the phase θ from the general expression of the current. Now it's the time.... with a spectacular example!



contour of integration deep in the bulk, where $\vec{J}_s(\vec{r},t) = 0$. From the expression of $\vec{J}_s$,

$$\Rightarrow \hbar \vec{\nabla}\theta(\vec{r}) = q^* \vec{A}(\vec{r})$$

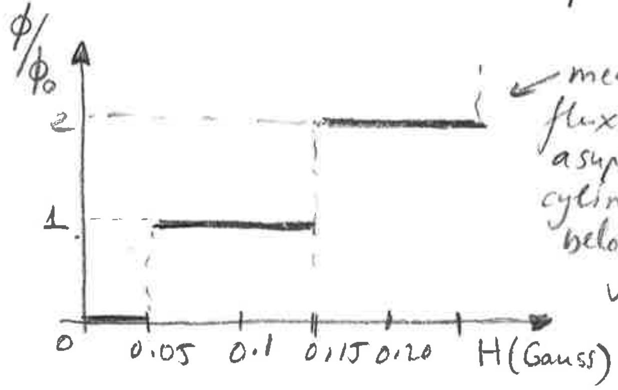
$$\Rightarrow \underbrace{\hbar \oint \vec{\nabla}\cdot\theta(\vec{r})\, d\vec{\ell}}_{= 2\pi n} = q^* \Phi$$

Φ = magnetic flux

$$\Phi = \iint \vec{B}\, d\vec{S}$$

$$\Rightarrow \quad \phi = \frac{2\pi n \hbar}{q^*} = \frac{h}{q^*} \cdot n$$

n = integer no.



measured flux in a superconducting cylinder cooled below T_c in various magnetic fields

$$\Phi_0 = \frac{h}{2e} = \text{flux quantum}$$

$$= 2{,}067 \times 10^{-15}\ \text{Wb}$$

Here I already have to disclose that:

$$q^* = -2e.$$

Another useful relation: the energy-phase relationship

$$\underbrace{-\hbar \frac{\partial \theta}{\partial t}}_{\text{change of phase}} = \underbrace{\frac{1}{2} \frac{\mu_0 \lambda_L^2}{n_s} \cdot J_s^2}_{\text{"kinetic" energy}} + \underbrace{q^* V}_{\text{potential energy}}$$

Proof:

From the Schrödinger equation $i\hbar \frac{\partial}{\partial t} \Psi_s = \frac{1}{2m^*} \left(-i\hbar \vec{\nabla} - q^* \vec{A}\right)^2 \Psi_s + q^* V \Psi_s$

we replace $\Psi_s = \sqrt{n_s}\, e^{i\theta}$, where $n_s = \text{const.}$

$$\Rightarrow -\hbar \frac{\partial \theta}{\partial t} \cancel{\sqrt{n_s}} = \frac{1}{2m^*} \left(+\hbar \vec{\nabla} \theta - q^* \vec{A}\right)^2 \cancel{\sqrt{n_s}} + q^* V \cancel{\sqrt{n_s}}$$

But $\vec{J_s}^2 = \frac{q^{*2} n_s^2}{m^{*2}} \left(\hbar \vec{\nabla} \theta - q^* \vec{A}\right)^2$

$$\Rightarrow -\hbar \frac{\partial \theta}{\partial t} = \frac{1}{2} \underbrace{\frac{m^*}{n_s^2 q^{*2}}}_{= \frac{\mu_0 \lambda_L^2}{n_s}} J_s^2 + q^* V$$

∘ Let us recap a bit:

Electrodynamics of superconductors is described by

1st London equation $\boxed{\dfrac{d\vec{J}_s}{dt} = \dfrac{1}{\mu_0 \lambda_L^2}\vec{E}}$

2nd London equation $\boxed{\vec{B} = -\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{J}_s}$

Here $\vec{J}_s = \dfrac{\hbar q^* n_s}{m^*}\left[\vec{\nabla}\theta - \dfrac{q^*}{\hbar}\vec{A}\right]$

or

$$\vec{J}_s = \frac{-\phi_0}{2\pi\mu_0\lambda_L^2}\left[\vec{\nabla}\theta + \frac{2\pi}{\phi_0}\vec{A}\right]$$

× London penetration length: $\lambda_L^2 = \dfrac{m^*}{\mu_0 n_s q^{*2}}$

$\Phi_0 = \dfrac{h}{2e}$ = flux quantum

$q^* = -2e$

The quantity: $\vec{\nabla}\theta + \dfrac{2\pi}{\phi_0}\vec{A}$ = gauge-invariant phase gradient