Lecture 6

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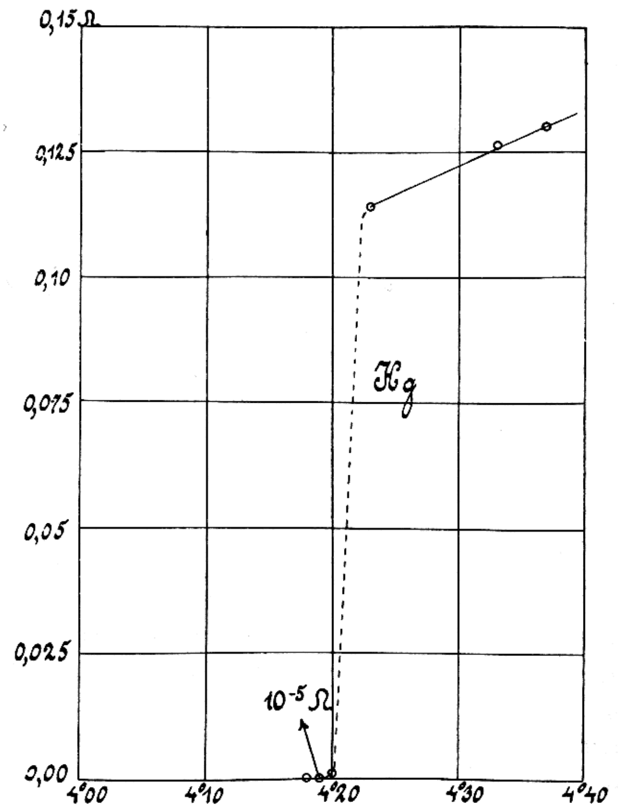
I. SUPERCONDUCTIVITY

- 1911 – Heike Kamerlingh Onnes

  Electrical resistance of Hg (metal!) dropped to $< 10^{-5}\ \Omega$ at $T_c = 4.2$ K.



  Other metals become superconductors:

  $T_c = 1.2$ K for Al

  $T_c = 7.2$ K for Pb

  $T_c = 9.2$ K for Nb.

- 1986 – Discovery of high T_c compounds by J.G. Bednorz and K.A. Müller.

  $T_c = 95$ K for $YBa_2Cu_3O_{7-\delta}$

  $T_c = 125$ K for $Tl_2Ba_2Ca_2Cu_3O_{10}$

  $T_c = 9.2$ K for $HgBa_2Ca_2Cu_3O_{8+\delta}$.

  These are not metals! They are ceramic materials at room temperature!

A. Meissner Effect

– In the beginning of superconductivity research it was hoped that the electromagnetic properties could be derived from the property of infinite conductivity.

$$\sigma = \infty, \qquad \left.\begin{aligned} \vec{\mathcal{J}} &= \sigma \cdot \vec{E} \\ \vec{\mathcal{J}} &= \text{finite} \end{aligned}\right\} \Longrightarrow \vec{E} = 0 \Longrightarrow \vec{\nabla} \times \vec{E} = 0 \tag{1}$$

$$\text{Maxwell:} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Longrightarrow \frac{\partial \vec{B}}{\partial t} = 0\ . \tag{2}$$

So $\vec{B}$ = constant inside a superconductor and also we expect it to be dependent on the way it was cooled down (e.g. either in the presence or absence of the magnetic field).

But in 1933 Meisner and Ochsenfeld discovered that $\vec{B} = 0$. The magnetic field inside the superconductor is not just constant, but it is exactly zero. Magnetic field lines are expelled. A superconductor is a perfect diamagnet.

Theory Development

- 1935 – Phenomenological theory developed by F. & H. London (two brothers!)
- 1957 – BCS (Bardeen-Cooper-Schrieffer) theory.
- high-T_C superconductivity – maybe YOU?

Elements of London Theory:

Consider a particle of mass m^* and charge q^*. It will turn out that $m^* = 2m_e$ and $q^* = -2e$; these particles are Cooper pairs, and a complete understanding of what they are is provided by the BCS theory.

Recall:

$\vec{B} = \vec{\nabla} \times \vec{A}$, $\qquad \vec{A}$ = magnetic vector potential, $\qquad V$ = electric potential.

Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \frac{1}{2m^*}\Big(- i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\Big)^2\psi(\vec{r},t) + q^*V(\vec{r},t)\psi(\vec{r},t)\ , \tag{3}$$

Recall also that: $\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V$.

Note: the Hamiltonian of a free particle in a magnetic field is $H = \frac{\vec{\Pi}^2}{2m^*}$, where $\vec{\Pi}(\vec{r}) = -i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})$ is the canonical momentum.

The probability density: $\quad P(\vec{r},t) = |\psi(\vec{r},t)|^2$

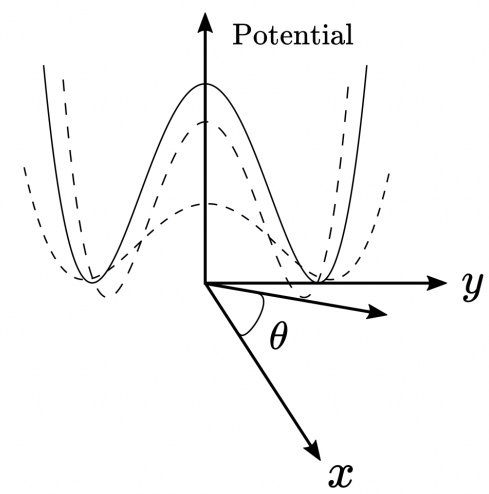
$\therefore \quad \frac{\partial P(\vec{r},t)}{\partial t} = \frac{\partial\psi^*(\vec{r},t)}{\partial t}\psi(\vec{r},t) + \psi^*(\vec{r},t)\frac{\partial\psi(\vec{r},t)}{\partial t} = \frac{i}{\hbar}\Big\{\Big[\frac{1}{2m^*}\Big(i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\Big)^2\psi^*(\vec{r},t)\Big]\psi(\vec{r},t) - \psi^*(\vec{r},t)\Big[\frac{1}{2m^*}\Big(- i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\Big)^2\Big]\psi(\vec{r},t)\Big\} = -\vec{\nabla}\cdot\vec{j}(\vec{r},t)$.

$$\therefore \qquad \frac{\partial P(\vec{r},t)}{\partial t} = -\vec{\nabla}\cdot\vec{j}(\vec{r},t)\ , \tag{4}$$

where $\vec{j}(\vec{r},t) = \frac{1}{2m^*}\Big[\Big(- i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\Big)\psi(\vec{r},t)\Big]^*\psi(\vec{r},t) + \frac{1}{2m^*}\psi^*(\vec{r},t)\cdot\Big[\Big(- i\hbar\vec{\nabla} - q^*\vec{A}(\vec{r})\Big)\psi(\vec{r},t)\Big]$.

Key point: The wavefunction $\psi(\vec{r},t)$ for a superconductor can be regarded as an order parameter (a macroscopic wavefunction!). Let us call this "solution" ψ_s. It is also convenient to normalize it to the number of particles rather than 1 (as usual for a wavefunction), and we will call this object Ψ_s (the superconductor Ginzburg-Landau order parameter).

The Ginzburg-Landau Order Parameter: $\quad \Psi_s(\vec{r},t) = \sqrt{n_s(\vec{r},t)}e^{i\theta(\vec{r},t)}$, where $n_s(\vec{r},t) =$ density of superconducting particles, and $\theta(\vec{r},t) =$ superconducting phase. Appears as a result of a broken symmetry.



The dynamical equation of the Ginzburg-Landau order parameter looks the same as the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi_s = \frac{1}{2m^*}\Big(-i\hbar\vec{\nabla} - q^*\vec{A}\Big)^2\Psi_s + q^*V\Psi_s. \tag{5}$$

Note that indeed with this definition $\int d\vec{r}\Psi_s^*(\vec{r},t)\Psi_s(\vec{r},t)$ = total number of superconducting particles.

- From now on we will assume $n_s(\vec{r},t) \equiv n_s = \text{const}$.

$\therefore$ $\vec{J}_s(\vec{r},t) = \frac{\hbar n_s}{m^*}\Big[\vec{\nabla}\theta(\vec{r},t) - \frac{q^*}{\hbar}\vec{A}(\vec{r},t)\Big]$ is the particle current associated with Ψ_s. This leads to an electrical current density $\vec{\mathcal{J}}_s = q^*\vec{J}_s$.

Therefore, we have the

$$\vec{\mathcal{J}}_s(\vec{r},t) = \frac{\hbar q^* n_s}{m^*}\Big[\vec{\nabla}\theta(\vec{r},t) - \frac{q^*}{\hbar}\vec{A}(\vec{r},t)\Big] = \text{superconducting current density}\ , \tag{6}$$

where $\Big[\vec{\nabla}\theta(\vec{r},t) - \frac{e^*}{\hbar}\vec{A}(\vec{r},t)\Big]$ is a gauge-invariant phase:

$$\begin{cases} \theta \to \theta + \frac{q^*}{\hbar}\chi \\ \vec{A} \to \vec{A} + \vec{\nabla}\chi\ . \end{cases} \tag{7}$$

<u>Consequences:</u>

- Let us consider θ = constant in $\vec{r}$, $\vec{\nabla}\theta = 0$.

- Perfect Conductivity

$\vec{\mathcal{J}}_s = -\frac{q^{*2}}{m^*} n_s \vec{A} \implies \frac{d\mathcal{J}_s(\vec{r},t)}{dt} = -\frac{q^{*2}}{m^*} n_s \frac{d\vec{A}(\vec{r},t)}{dt}$ (Recall Maxwell: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ & $\vec{B} = \vec{\nabla} \times \vec{A}$), or

$$\frac{d\vec{\mathcal{J}}_s(\vec{r},t)}{dt} = +\frac{q^{*2} n_s}{m^*} \vec{E}(\vec{r},t) \; . \tag{8}$$

What does it mean? A constant-in-time current can flow through a superconductor even if the electric field is zero (resulting in a zero voltage drop - therefore zero electrical resistance).

Take a ballistic superelectron (no collision with atoms, impurities, etc.)

$$\left.\begin{array}{l} m^* \frac{d\vec{v}_s}{dt} = q^* \cdot \vec{E} \\ \vec{\mathcal{J}}_s = \rho_s \cdot q^* \cdot \vec{v}_s \end{array}\right\} \implies \frac{d\vec{\mathcal{J}}_s}{dt} = \frac{q^{*2} \rho_s}{m^*} \vec{E} \tag{9}$$

Note the difference with respect to $\vec{\mathcal{J}} = \sigma \vec{E}$ (Ohm's law)!

- Meissner Effect

Let us look at Maxwell's equations:

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \; , \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{\mathcal{J}}_s \; . \end{cases}$$

Now $\vec{B} = \vec{\nabla} \times \vec{A}$ so $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \cdot \vec{A} = -\vec{\nabla}^2 \vec{A}$, where we can use the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$.

$$\therefore \quad \begin{cases} \vec{\nabla}^2 \vec{A} = -\mu_0 \vec{\mathcal{J}}_s \; , \\ \vec{\mathcal{J}}_s = -\frac{q^{*2}}{m^*} n_s \vec{A} \; , \end{cases} \implies \vec{\nabla}^2 \vec{A} = \frac{\mu_0 q^{*2} n_s}{m^*} \vec{A} \; . \tag{10}$$

<u>Notation:</u>

$\lambda_L = \sqrt{\frac{m^*}{\mu_0 n_s q^{*2}}}$ = London penetration length.

Since $\vec{\mathcal{J}}_s = -\frac{q^{*2} n_s}{m^*}\vec{A}$, we have

$$\begin{cases} \vec{J}_s = -\frac{1}{\mu_0 \lambda_L^2}\vec{A}\,, \\ \vec{\nabla}^2 \vec{A} = \frac{1}{\lambda_L^2}\vec{A}\,. \end{cases} \tag{11}$$

$$\therefore \quad \mu_0 \lambda_L^2 \vec{\mathcal{J}}_s = -\vec{A} \implies \mu_0 \lambda_L^2 \vec{\nabla}\times\vec{\mathcal{J}}_s = -\vec{\nabla}\times\vec{A} = -\vec{B},$$

or $\quad \mu_0 \lambda_L^2 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\mathcal{J}}_s) = -\vec{\nabla}\times\left(\frac{\partial \vec{A}}{\partial t} \equiv -\vec{E}\right)$, since voltage is zero and $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}V$.

But $\vec{\mathcal{J}}_s = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}$,

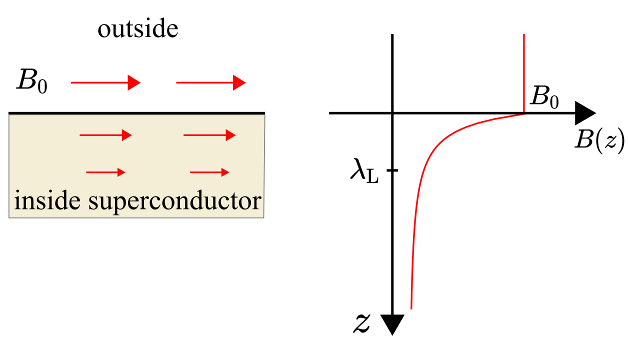
$$\therefore \quad \lambda_L^2 \frac{\partial}{\partial t}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{B})) = \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

Note $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{B}) - \vec{\nabla}^2\vec{B}$, and that $\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{B}) = 0$.

This implies $\lambda_L^2 \vec{\nabla}^2 \vec{B} = +\vec{B}$, or

$$\left[\frac{1}{\lambda_L^2} - \vec{\nabla}^2\right]\vec{B}(\vec{r}) = 0\,. \tag{12}$$

Take $\vec{B}(\vec{r}) = (0, B(z), 0)$ (the field is parallel to the surface of the superconductor, let's say in the y direction). The solution of the equation above is $B(z) = B_0 \exp(-z/\lambda_L)$. This is the Meissner effect. The field decays exponentially in the superconductor.



To review: we found

$$\vec{\mathcal{J}}_s = -\frac{1}{\mu_0 \lambda_L^2} \vec{A}\,, \tag{13}$$

and

$$\nabla^2 \vec{A} = \frac{1}{\lambda_L^2} \vec{A}\,, \tag{14}$$

or

$$\frac{d\vec{\mathcal{J}}_s}{dt} = \frac{1}{\mu_0 \lambda_L^2} \vec{E} \quad \text{— called } 1^{st} \text{ London equation,} \tag{15}$$

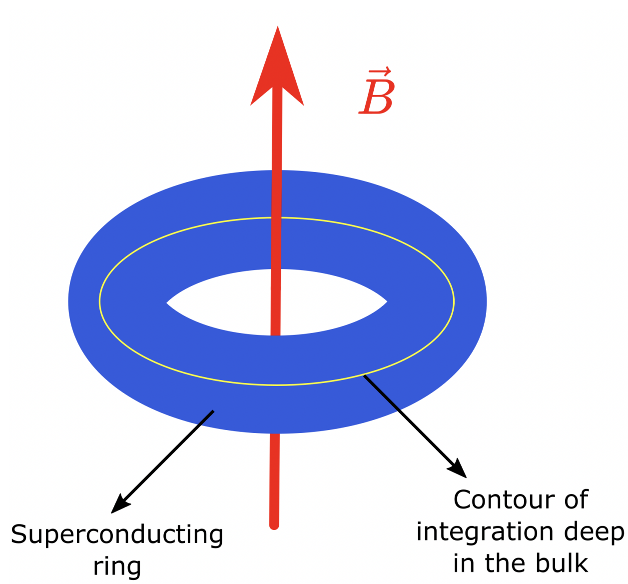
$$\vec{B} = -\mu_0 \lambda_L^2 \vec{\nabla} \times \vec{\mathcal{J}}_s \quad \text{— called } 2^{nd} \text{ London equation.} \tag{16}$$

So the magnetic field can penetrate at most to depths of $\simeq \lambda_L$.

Currents can flow in this region, but deep in the bulk they will be zero.

II. QUANTIZATION OF FLUX

So far we have not discussed the phase θ from the general expression of the current. Now it's time ... with a spectacular example!

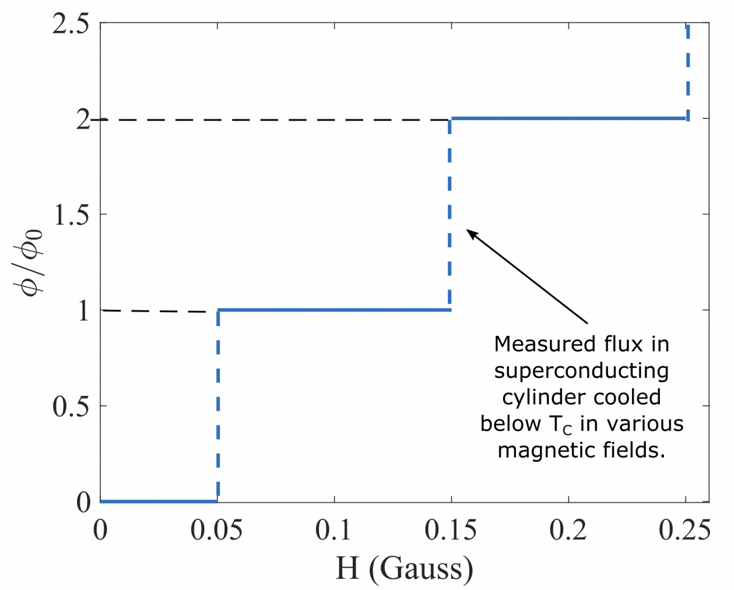


We consider a superconducting ring and choose a contour of integration deep in the bulk where $\vec{\mathcal{J}}_s(\vec{r}, t) = 0$. This implies

$\hbar\vec{\nabla}\theta(\vec{r}) = q^*\vec{A}(\vec{r}) \implies \hbar\overbrace{\oint \vec{\nabla}\cdot\theta(\vec{r})d\vec{\ell}}^{2\pi n} = q^*\Phi$, where Φ = magnetic flux, and n = integer number. With $\phi = \iint \vec{B}d\vec{s}$, we have

$$\phi = \frac{2\pi n\hbar}{q^*} = \frac{h}{q^*}\cdot n\ . \tag{17}$$

The flux quantum is $\phi_0 = \frac{h}{2e} = 2.067\times10^{-15}$ Wb, and $q^* = -2e$.



- Another useful relation: the energy-phase relationship

$$\underbrace{-\hbar\frac{\partial\theta}{\partial t}}_{\text{“change of phase”}} = \underbrace{\frac{1}{2}\frac{\mu_0\lambda_L^2}{n_s}\cdot\vec{\mathcal{J}}_s^{\,2}}_{\text{“kinetic energy”}} + \underbrace{q^*V}_{\text{“potential energy”}} \tag{18}$$

Proof:

From the Schrödinger-like equation for the order parameter $i\hbar\frac{\partial}{\partial t}\Psi_s = \frac{1}{2m^*}\Big(-i\hbar\vec{\nabla} - q^*\vec{A}\Big)^2\Psi_s + q^*V\Psi_s$ we replace $\Psi_s = \sqrt{n_s}e^{i\theta}$ where n_s = const.

$\implies \quad -\hbar\frac{\partial\theta}{\partial t}\cdot\sqrt{n_s} = \frac{1}{2m^*}\Big(+\hbar\vec{\nabla}\theta - q^*\vec{A}\Big)^2\cdot\sqrt{n_s} + q^*V\sqrt{n_s}$,
but $\vec{\mathcal{J}}_s^{\,2} = \frac{q^{*2}n_s^2}{m^{*2}}\Big(\hbar\vec{\nabla}\theta - q^*\vec{A}\Big)^2$

$$\Longrightarrow \qquad -\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2}\underbrace{\frac{m^*}{n_s^2 q^{*2}}}_{\equiv\frac{\mu_0\lambda_L^2}{n_s}}\mathcal{J}_s^2 + q^{\,*}V.$$

Let us recap a bit:

Electrodynamics of superconductors is described by

$$\begin{cases} 1^{st} \text{ London equation} & \frac{d\vec{\mathcal{J}}_s}{dt} = \frac{1}{\mu_0\lambda_L^2}\vec{E}\,, \\ 2^{nd} \text{ London equation} & \vec{B} = -\mu_0\lambda_L^2\vec{\nabla}\times\vec{\mathcal{J}}_s\,. \end{cases} \tag{19}$$

Here $\vec{\mathcal{J}}_s = \frac{\hbar q^* n_s}{m^*}\left[\vec{\nabla}\theta - \frac{q^*}{\hbar}\vec{A}\right]$ or
$\vec{\mathcal{J}}_s = -\frac{\phi_0}{2\pi\mu_0\lambda_L^2}\left[\vec{\nabla}\theta + \frac{2\pi}{\phi_0}\vec{A}\right]$, where the London penetration length is $\lambda_L^2 = \frac{m^*}{\mu_0 n_s q^{*2}}$ and $\phi_0 = \frac{h}{2e}$ = flux quantum, $q^* = -2e$.

The quantity: $\vec{\nabla}\theta + \frac{2\pi}{\phi_0}\vec{A}$ = gauge-invariant phase gradient.

References

- Terry P. Orlando and Kevin A. Delin — Foundations of Applied Superconductivity.
- D.R. Tilley and J. Tilley — Superfluidity and Superconductivity.
- Antonio Barone and Giafranco Paternò — Physics and Applications of the Josephson Effect.