Solutions to Exploratory Exercises

# Problem 1

Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ that is

a) Injective but not surjective $f(x) = 2x$

$\star$ Injective: $\forall x, y \in \mathbb{Z} : f(x) = f(y) \implies 2x = 2y \implies x = y$ hence $f$ is injective.

$\star$ Surjective: Let's try to find a number that maps onto $3 \in \mathbb{Z}$. Since $f(x) = 3 \implies 2x = 3$ and $2x = 3$ does not have a solution in $\mathbb{Z}$, there is no $x$ such that $f(x) = 3$. Hence, $f$ is not surjective.

b) Surjective but not injective. Let $f(x) = \lfloor \frac{x}{2} \rfloor$, where $\lfloor \cdot \rfloor$ is the *floor function.*

$\star$ Injective: $f(0) = f(1) = 0$ but $0 \neq 1$, hence $f$ is not injective.

$\star$ Surjective: $\forall x \in \mathbb{Z} : f(2x) = x$. Hence, $f$ is surjective.

c) Both injective and surjective. $f(x) = x$. This is clear. Other simple examples are $f(x) = -x$ and $f(x) = x + 7$.

d) Neither injective nor surjective. $f(x) = x^2$

$\star$ Injective: $f(1) = f(-1)$ but $-1 \neq 1$ hence, $f$ is not injective.

$\star$ Surjective: $-3 \in \mathbb{Z}$ and $f(x) = x^2 = -3$ does not have a solution in $\mathbb{Z}$. Hence, $f$ is not surjective. Even if the codomain is changed to $\mathbb{N}$ the function is still not surjective because $f(x) = x^2 = 2$ does not have an integer solution.

# Problem 2

a) No, there exist no injective map because the number of elements in the domain exceeds those of the codomain.

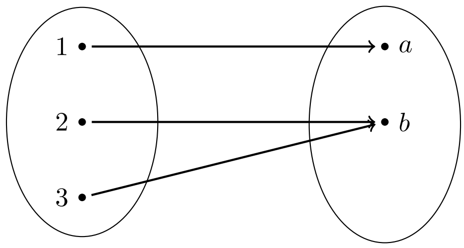


Figure 1: Example of surjective map.

(b) No, there does not exist a surjective map.

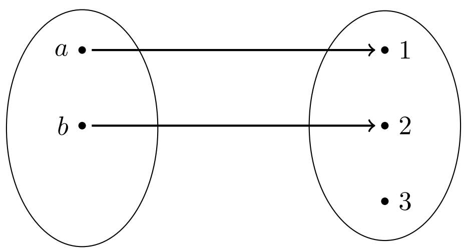


Figure 2: Example of injective map.

(c) If there is an injection then $|A| \leq |B|$. And if the map is surjective, $|B| \leq |A|$.

# Problem 3

a) For the four examples above, determine if they are reflexive, transitive, and/or symmetric.

i) $x$ and $y$ are siblings: Obviously this is symmetric, not reflexive, and it is transitive if we are only considering "full siblings". But if we consider half-siblings, then it is not transitive(for example if $A$ and $B$ have the same mother but different fathers, and B and C have the same father but different mothers).

ii) $x$ divides $y$: For all $x$, $x = x.1 \implies x|x$ so the relation is reflexive. It is not symetric since because, for example, $1|2$ but $2 \nmid 1$. Lastly, $a|b \Rightarrow \exists k \in \mathbb{Z} : b = ak$ and $b|c \Rightarrow \exists l \in \mathbb{Z} : c = bl = akl = am$ where $m = kl \in \mathbb{Z}$. Therefore $a|c$, and so the relation is transitive.

iii) $x < y$ in $\mathbb{R}$: Clearly transitive. But not reflexive since $1 \nless 1$ and not symmetric since $0 < 1$ but $1 \nless 0$.

iv) $x = y$. Clearly reflexive, transitive and symmetric.

b) i) The relation $x = y$.

ii) The relation $R = \{(x, x), (y, y), (z, z), (x, y), (y, z)\}$

iii) $x \neq y$

iv) $x < y$

# Homework

## Problem 1

If $n \in \mathbb{Z}$ then $n^2 + 3n + 4$ is even.

(i) Case 1: Let $n \in \mathbb{Z}$ be even. By the definition of "even" $\exists k \in \mathbb{Z} : n = 2k$, and so

$$n^2 + 3n + 4 = (2k)^2 + 3(2k) + 4 = 4k^2 + 3(2k) + 4 = 2(2k^2 + 3k + 2).$$

Since $2k^2 + 3k + 2 \in \mathbb{Z}$ the result is even.

(ii) Case 2: Let $n \in \mathbb{Z}$ be odd. By the definition of "odd" $\exists k \in \mathbb{Z} : n = 2k + 1$, and so

$$\begin{aligned} n^2 + 3n + 4 =& (2k + 1)^2 + 3(2k + 1) + 4 \\ =& (4k^2 + 4k + 1) + 3(2k + 1) + 4 \\ =& (4k^2 + 4k + 1) + 6k + 3 + 4 \\ =& 4k^2 + 10k + 8 = 2(2k^2 + 5k + 8). \end{aligned}$$

Since $2k^2 + 5k + 8 \in \mathbb{Z}$ the result is even.

## Problem 2

(a) $\neg(P \implies Q) \iff \neg(\neg P \vee Q) \overset{\text{(De Morgan)}}{\iff} \neg(\neg P) \wedge \neg Q \iff P \wedge \neg Q$.

(b) Towards contradiction, let us assume that the statement we want to prove is false. Notice that,

$$\begin{aligned} & \neg \forall a, b \in \mathbb{Z}(a + b \geq 39 \implies (a \geq 20 \vee b \geq 20)) \\ \iff & \exists a, b \in \mathbb{Z}(\neg(a + b \geq 39 \implies (a \geq 20 \vee b \geq 20))) && \text{(negation of quantifiers)} \\ \iff & \exists a, b \in \mathbb{Z}(a + b \geq 39 \wedge \neg(a \geq 20 \vee b \geq 20)) && \text{(by our result in (a))} \\ \iff & \exists a, b \in \mathbb{Z}(a + b \geq 39 \wedge (a < 20 \wedge b < 20)). && \text{(De Morgan)} \end{aligned}$$

In the last step we obtain a contradiction, since $(a < 20 \wedge b < 20) \implies a + b \leq 19 + 19 = 38 < 39$.

## Problem 3

We want to prove

$$\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n (n + 1)n}{2}. \tag{1}$$

via induction.

1. Base case: Let $n = 1$. Then

$$\text{LHS} = \sum_{k=1}^{1} (-1)^k k^2 = (-1)1^2 = -1, \text{and}$$

$$\text{RHS} = \frac{(-1)^1(1 + 1)1}{2} = \frac{(-1)(2)}{2} = -1.$$

2. Inductive step:

We show that assuming (1) is true for $n = \ell$, the statement holds for $n = \ell + 1$ as well. To prove this, notice that by expanding the LHS of (1) when $n = \ell + 1$, we obtain

$$\begin{aligned}
\sum_{k=1}^{\ell+1}(-1)^k k^2 &= \left(\sum_{k=1}^{\ell}(-1)^k k^2\right) + (-1)^{\ell+1}(\ell+1)^2 \\
&= \left(\frac{(-1)^{\ell}(\ell+1)\ell}{2}\right) + (-1)^{\ell+1}(\ell+1)^2 \qquad \text{(by assumption)} \\
&= \frac{(-1)^{\ell}(\ell+1)\ell + 2(-1)^{\ell+1}(\ell+1)^2}{2} \\
&= \frac{(-1)^{\ell}(\ell+1)(\ell + 2(-1)(\ell+1))}{2} \\
&= \frac{(-1)^{\ell}(\ell+1)(-\ell-2)}{2} \\
&= \frac{(-1)^{\ell}(\ell+1)(-1)(\ell+2)}{2} = \frac{(-1)^{\ell+1}(\ell+2)(\ell+1)}{2}.
\end{aligned}$$

which is simply the RHS of (1) when $n = \ell + 1$. So by the principle of induction we have shown that (1) holds for all $\mathbb{N}_{>0}$.

# Problem 4

Relation $\sim$ on $\mathbb{R}$ defined as $a \sim b \iff a \leq b$.

(i) $\sim$ is reflexive. $\forall x \in \mathbb{R}(x \sim x) \iff \forall x \in \mathbb{R}(x \leq x)$.

(ii) $\sim$ is *not* symmetric.

Recall that $\sim$ is symmetric $\iff \forall (x, y) \in \mathbb{R}, x \sim y \implies x \sim y$.

Consider any $x \in \mathbb{R}$ and $y = x + 1$. While $x \leq y$ is true, $y \leq x \iff x + 1 \leq x$ is not. The counter-example $a = 3$ and $b = 5$ is also sufficient.

(iii) $\sim$ is transitive.

Recall that $\sim$ is transitive $\iff \forall a, b, c \in \mathbb{R}((a \sim b \wedge b \sim c) \implies a \sim c)$. This is true because if $a \leq b$ and $b \leq c$ then $a \leq c$. To add more detail the definition of $\leq$ should be used. First we really need to define positive numbers and show that the sum of two positive numbers is positive. But let's take these as given facts. We define $x \leq y$ if and only if $\exists c \geq 0$ such that $x + c = y$. Now if $a \leq b$ and $b \leq c$ then $a + r = b$ and $b + s = c$ for some $r \geq 0$ and $s \geq 0$. Then $a + (r + s) = c$, and since $r + s \geq 0$ we have by definition that $a \leq c$.

(iv) The statement $\forall a, b, c \in \mathbb{R}(ac \leq bc \implies a \leq b)$ is not true. One way to see this is to consider the counterexample $a = 1, b = -1, c = -1$. Notice that $ac = (1)(-1) = -1 \leq bc = (-1)(-1) = 1$ is true, but $1 = a < b = -1$ is false.

# Additional Exercises

# Problem 1

Prove that, if $A$, $B$ and $C$ are sets, then $(A \cup B) \times C = (A \times C) \cup (B \times C)$

- $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C))$: We see that $(x, y) \in (A \cup B) \times C \Rightarrow a \in A \cup B$ and $y \in C \Rightarrow x \in A$ and $y \in C$ or $x \in B$ and $y \in C$. Therefore $(x, y) \in A \times C$ or $(x, y) \in B \times C$. Hence $(x, y) \in (A \times C) \cup (B \times C))$.
- $(A \times C) \cup (B \times C)) \subseteq (A \cup B) \times C$ : We see that $(x, y) \in (A \times C) \cup (B \times C)) \Rightarrow (x, y) \in (A \times C)$ or $(x, y) \in (B \times C) \Rightarrow x \in A \cup B$ and $y \in C$.

# Problem 2

Prove that $(\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$

- $(\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) \subseteq \mathbb{Z} \times \mathbb{Z}$ : It follows that $(x, y) \in (\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) \Rightarrow (x, y) \in (\mathbb{Z} \times \mathbb{R})$ and $(x, y) \in (\mathbb{R} \times \mathbb{Z}) \Rightarrow (x, y) \in \mathbb{Z} \times \mathbb{Z}$ since $x$ is an integer from the first product and $y$ is an integer from the second.
- $\mathbb{Z} \times \mathbb{Z} \subseteq (\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z})$ : It follows that $(x, y) \in \mathbb{Z} \times \mathbb{Z} \Rightarrow x \in \mathbb{Z} \subset \mathbb{R} \Rightarrow (x, y) \in \mathbb{R} \times \mathbb{Z}$ and similarly, $y \in \mathbb{R} \Rightarrow (x, y) \in \mathbb{Z} \times \mathbb{R}$. Hence $(x, y) \in (\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z})$

## Problem 3

Define $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $f(x) = x^2$ and $g(y) = \sqrt{y}$. Then $f(g(y)) = (\sqrt{y})^2 = y$, but $g(f(-1)) = \sqrt{(-1)^2} = \sqrt{1} = 1$.

## Problem 4

a) Clearly $\sim$ is reflexive since $x^2 - x^2 = 0 \in \mathbb{Z}$. Also it is symmetric since the negative of an integer is also an integer. Lastly it is transitive for if $x^2 - y^2 \in \mathbb{Z}$ and $y^2 - z^2 \in \mathbb{Z}$, then $x^2 - z^2 = (x^2 - y^2) + (y^2 - z^2)) \in \mathbb{Z}$. Hence $\sim$ is indeed an Equivalence relation.

b) $[0] = \{a \in \mathbb{R} | 0Ra\} = \{a \in \mathbb{R} | -a^2 \in \mathbb{Z}\} = \{\dots, -\sqrt{3}, -\sqrt{2}, -1, 0, 1, \sqrt{2}, \sqrt{3} \dots\}$

c) $[\frac{1}{3}] = \{a \in \mathbb{R} | \frac{1}{3}Ra\} = \{a \in \mathbb{R} | \frac{1-(3a)^2}{9} \in \mathbb{Z}\} = \left\{\dots, -\frac{\sqrt{19}}{3}, -\frac{\sqrt{10}}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{\sqrt{10}}{3}, \frac{\sqrt{19}}{3} \dots\right\}$

## Problem 5

Prove by induction that for every $n \in \mathbb{N}$ holds $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$

- Base Case: Let $n = 1$. $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = \frac{1}{2} = 1 - \frac{1}{2!}$ so the statement is true for $n = 1$.
- I.H: Assume true for $n = k$. That is, $\displaystyle\sum_{i=1}^{k} \frac{i}{(i+1)!} = 1 - \frac{1}{(k+1)!}$
- Verify for $n = k + 1$ :

$$\sum_{i=1}^{k+1} \frac{i}{(i+1)!} = \sum_{i=1}^{k} \frac{i}{(i+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

Hence the statement is true for all $n \in \mathbb{N}_{>0}$

## Problem 6

The third point: The reasoning here is valid if $n \geq 2$. For example $P(2)$ does imply $P(3)$ and so on. But let's look at the case $n = 1$. The $n + 1 = 2$ socks is the set $\{s_1, s_2\}$. The two subsets are just $\{s_1\}$ and $\{s_2\}$ and so there is no overlap in this case. So we can not conclude that $P(2)$ is true. So the required chain of implications $P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \cdots$ breaks at the first step. So the induction proof fails.

## Problem 7

A vector $\mathbf{x}$ is in the span of $\mathbf{u}, \mathbf{v}, \mathbf{w} \iff \exists a, b, c \in \mathbb{R}(a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{x})$. In set notation:

$$\text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \{\mathbf{x} \in \mathbb{R}^n \,|\, \exists a, b, c \in \mathbb{R}(a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{x})\}$$

## Problem 8

$$\begin{aligned}
&\neg\forall a, b, c \in \mathbb{R}(a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0 \implies (a = 0) \land (b = 0) \land (c = 0)) \\
\iff &\exists a, b, c \in \mathbb{R}(\neg(a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0 \implies (a = 0) \land (b = 0) \land (c = 0))) && \text{(negation of quantifiers)} \\
\iff &\exists a, b, c \in \mathbb{R}(a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0 \land \neg((a = 0) \land (b = 0) \land (c = 0))) && \text{(by 2(a)).}
\end{aligned}$$

In natural language: Vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly *dependent*, if there exists *non-trivial* $a, b, c \in \mathbb{R}$ such that $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = 0$. Here, non-trivial refers to the solutions excluding the trivial solution $a = b = c = 0$.

## Problem 9

The statement is not true for the base case. i.e for $n = 1$, $1 \neq \frac{9}{8}$

## Problem 10

By Double induction show that the Fibonacci numbers satisfy

$$f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1} \ \forall m, n \geq 0.$$

Let the statement be denoted by $P(m, n)$

- Base Case $P(0, 0) : f_1 = 1 = f_1 f_1$ since $f_0 = 0, f_1 = 1$ So statement true.
- Assume $P(m, n)$ and $P(m, n-1)$ is true. We show that $P(m, n+1)$ is also true.

$$\begin{aligned} f_{m+n+2} = f_{m+n+1} + f_{m+n} &\overset{\text{I.H}}{=} f_m f_n + f_{m+1} f_{n+1} + f_{n-1} f_m + f_n f_{m+1} \\ &= f_m f_{n+2} + f_{m+1} f_{n+2} \end{aligned}$$

- Similarly, assume $P(m, n)$ and $P(m-1, n)$. We show $P(m+1, n)$ is also true.

$$\begin{aligned} f_{m+n+2} = f_{m+n+1} + f_{m+n} &\overset{\text{I.H}}{=} f_m f_n + f_{m+1} f_{n+1} + f_n f_{m-1} + f_n f_{m+1} \\ &= f_n f_{m+2} + f_{n+1} f_{m+2} \end{aligned}$$

This implies $P(m, n)$ is true $\forall m, n \geq 0$