ELEC-C9430 Electromagnetism (week 4)

Dynamic fields.

Faraday's law. Displacement current

Wave equation. Retarded potentials.

Time-harmonic fields. Complex vectors

Plane waves

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

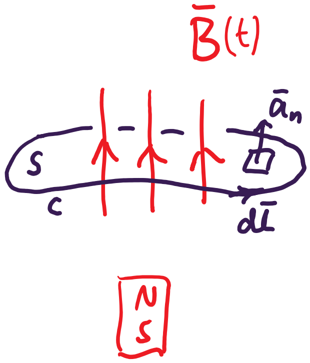
$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E}$$

Faraday's law

$$\nabla \times \bar{E}(\bar{r}, t) = -\frac{\partial \bar{B}(\bar{r}, t)}{\partial t}$$



STOKES

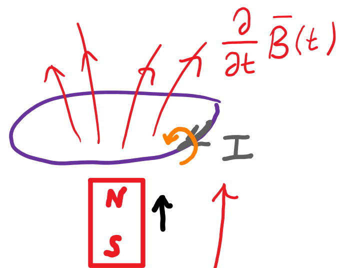
$$\int_S \nabla\times\bar{E}\cdot d\bar{S} = \oint_C \bar{E}\cdot d\bar{l} = emf$$

$$= \mathcal{V}$$

$$= -\frac{\partial}{\partial t}\underbrace{\int \bar{B}\cdot d\bar{S}}_{\Phi \text{ magnetic flux}} \qquad \underbrace{\mathcal{V} = -\frac{\partial\Phi}{\partial t}}$$

Stokes' law

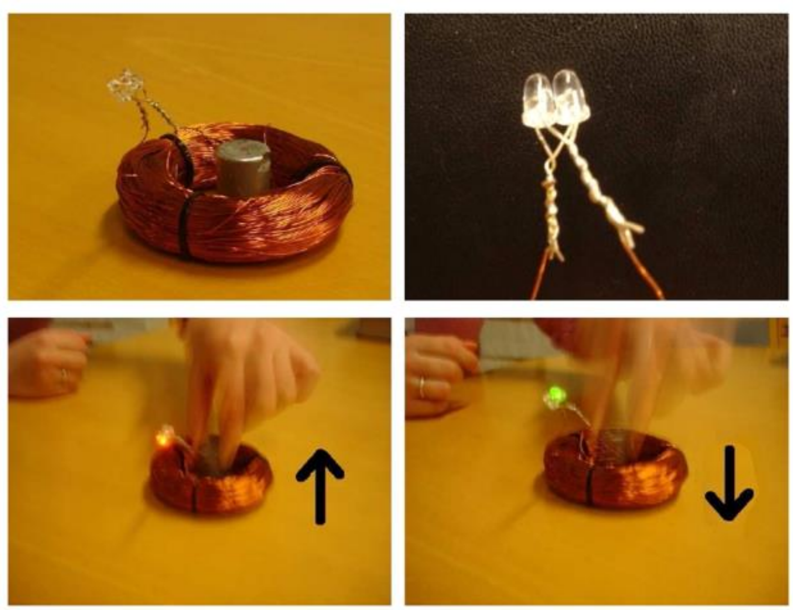
$$\int_S \nabla\times\bar{E}\cdot d\bar{S} = \oint_C \bar{E}\cdot d\bar{l}$$



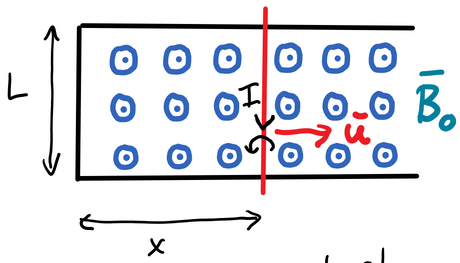
Electromotive force (emv) $\mathcal{V} = -\frac{\partial\Phi}{\partial t}$

Lenz's law

(induced current opposes the change of the magnetic flux)



Moving circuit in static magnetic field



$$\bar{F}_m = q\,\bar{u} \times \bar{B}_o$$

$$\bar{F}_e = q\,\bar{E}$$

$$|\bar{E}| = u B_o$$

$$|V| = EL = \underset{\frac{dx}{dt}}{u} B_o L = \frac{d}{dt}\underbrace{\left(B_o\, \overset{S}{\overbrace{x L}}\right)}_{\Phi} = \frac{d\bar{\Phi}}{dt}$$

Displacement current

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$\underbrace{\nabla \cdot \nabla \times \bar{H}}_{0} = \nabla \cdot \bar{J} + \nabla \cdot \frac{\partial \bar{D}}{\partial t} \Rightarrow \nabla \cdot \bar{J} = -\frac{\partial}{\partial t} \overbrace{\nabla \cdot \bar{D}}^{\rho_v}$$

$$\underbrace{\int_V \nabla \cdot \bar{J}\, dV}_{I_{out} = \oint \bar{J} \cdot d\bar{S}} = -\frac{\partial}{\partial t} \underbrace{\int_V \rho_v\, dV}_{Q_{IN}}$$

$d\bar{S} = \bar{a}_n dS$

Conservation of charge
(equation of continuity)

$$I_{out} = -\frac{dQ_{IN}}{dt}$$

Wave equation

FREE SPACE: ε_o, μ_o

NO SOURCES: $\rho_v = 0,\ \bar{J} = 0$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\mu_o \frac{\partial \bar{H}}{\partial t}$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} = \varepsilon_o \frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times (\nabla \times \bar{E}) = -\mu_o \frac{\partial}{\partial t} \nabla \times \bar{H} = -\mu_o \varepsilon_o \frac{\partial^2}{\partial t^2} \bar{E}$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = -\mu_0\frac{\partial\bar{H}}{\partial t}$$

$$\nabla\times\bar{H} = \frac{\partial\bar{D}}{\partial t} = \varepsilon_0\frac{\partial\bar{E}}{\partial t}$$

FREE SPACE: ε_0, μ_0

NO SOURCES: $\rho_v = 0,\ \bar{J} = 0$

$$\underbrace{\nabla\times(\nabla\times\bar{E})}_{\nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E}} = -\mu_0\frac{\partial}{\partial t}\nabla\times\bar{H} = -\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\bar{E}$$

$$\nabla\cdot\bar{D} = \nabla\cdot(\varepsilon_0\bar{E}) = \varepsilon_0(\nabla\cdot\bar{E}) = \rho_v = 0$$

$$\Rightarrow \nabla^2\bar{E}(\bar{R},t) - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\bar{E}(\bar{R},t) = 0$$

$$\bar{E}(\bar{R},t) = \bar{a}_x E(z,t)$$

$$\nabla^2 = \cancel{\frac{\partial^2}{\partial x^2}} + \cancel{\frac{\partial^2}{\partial y^2}} + \frac{\partial^2}{\partial z^2}$$

$$\Rightarrow \frac{\partial^2}{\partial z^2}E(z,t) - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}E(z,t) = 0$$

What about the following arbitrary function? $f(z-ut)$

$$\frac{\partial^2}{\partial z^2}f(z\mp ut) = f''(z\mp ut)$$

$$\frac{\partial^2}{\partial t^2}f(z\mp ut) = (\mp u)^2 f''(z\mp ut) = u^2 f''(z\mp ut)$$

$$\frac{\partial^2}{\partial z^2}f(z\mp ut) - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}f(z\mp ut) = 0$$

$\mu_0\varepsilon_0$

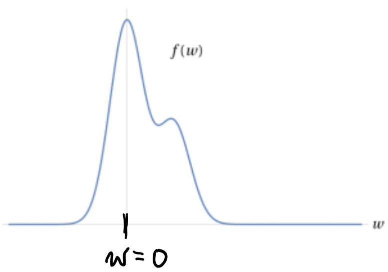
$$\Rightarrow E(z-ut) = E\left(z - \frac{t}{\sqrt{\mu_0\varepsilon_0}}\right)$$ (velocity u to $+z$ direction)

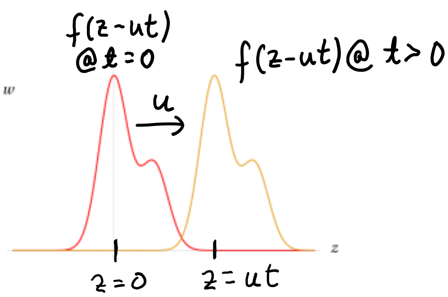
Speed of light C

Speed of light C

$$u = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299792458 \ \frac{m}{s} \quad = c$$

Another solution: $E(z + ut)$ (same velocity, $-z$-direction)





Solution to wave equation with sources $(\rho_v(\bar{R},t), \bar{J}(\bar{R},t))$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{A}$$

(free space)

$\rho_v \ \bar{J}$ $\quad \varepsilon_0 \quad \mu_0$

$\bar{E}(\bar{R},t) = ?$

$\bar{H}(\bar{R},t) = ?$

$$\begin{pmatrix} \rho_v \\ \bar{J} \end{pmatrix} \rightarrow \begin{pmatrix} V \\ \bar{A} \end{pmatrix} \rightarrow \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix}$$

Electrostatics

$$\nabla \times \bar{E} = 0$$

$$\bar{E} = -\nabla V$$

Magnetostatics

$$\nabla \cdot \bar{B} = 0$$

$$\bar{B} = \nabla \times \bar{A}$$

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t} \nabla \times \bar{A}$$

$$\nabla \times \Big(\underbrace{\bar{E} + \frac{\partial \bar{A}}{\partial t}}_{-\nabla V} \Big) = 0$$

$$-\nabla V - \frac{\partial \bar{A}}{\partial t}$$

$$\nabla \times \mu_0 \bar{H} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times \underbrace{\mu_0 \bar{H}}_{\nabla\times\bar{A}} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

$$\nabla(\underbrace{\nabla\cdot\bar{A}}_{-\mu_0\varepsilon_0 \frac{\partial V}{\partial t}}) - \nabla^2 \bar{A} = \mu_0 \bar{J} - \underbrace{\mu_0 \varepsilon_0 \nabla \frac{\partial V}{\partial t}} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \bar{A}$$

Lorenz gauge $\nabla\cdot\bar{A} = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t}$

$$\nabla^2 \bar{A} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \bar{A} = -\mu_0 \bar{J}$$

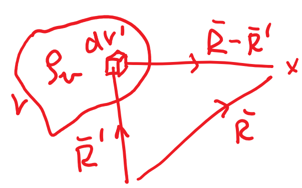
$$\nabla\cdot\bar{D} = \rho_v \Rightarrow \nabla\cdot\bar{E} = \frac{\rho_v}{\varepsilon_0} = \nabla\cdot\left(-\nabla V - \frac{\partial \bar{A}}{\partial t}\right)$$

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu_0\varepsilon_0 \frac{\partial V}{\partial t}\right) = -\frac{\rho_v}{\varepsilon_0}$$

$$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} V = -\frac{\rho_v}{\varepsilon_0}$$

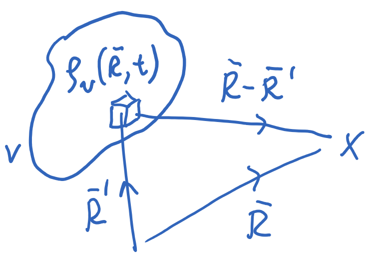
$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$

EL. STATICS



$$V(\bar{R}) = \int_V \frac{\rho_v(\bar{R}')\,dv'}{4\pi\varepsilon_0 |\bar{R}-\bar{R}'|}$$

EL. DYNAMICS



$$V(\bar{R},t) = \int_V \frac{\rho_v\left(\bar{R}', t - \frac{|\bar{R}-\bar{R}'|}{c}\right) dv'}{4\pi\varepsilon_0 |\bar{R}-\bar{R}'|}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\bar{A}(\bar{R},t) = \int_V \frac{\mu_0 \bar{J}\left(\bar{R}, t - \frac{|\bar{R}-\bar{R}'|}{c}\right) dv'}{4\pi |\bar{R}-\bar{R}'|}$$

Retarded potentials

Poynting's vector

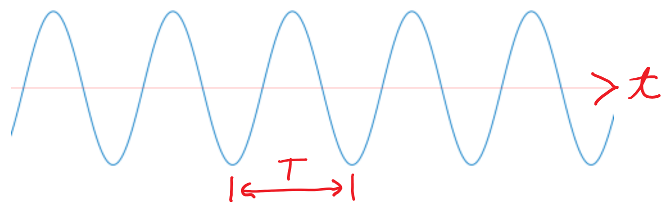
$$\bar{S}(\bar{R},t) = \bar{E}(\bar{R},t) \times \bar{H}(\bar{R},t)$$

Poynting's vector

$$\bar{S}(\bar{R},t) = \bar{E}(\bar{R},t) \times \bar{H}(\bar{R},t)$$

$$[\bar{S}] = [\bar{E}][\bar{H}] = \frac{V}{m}\frac{A}{m} = \frac{W}{m^2}$$ (POWER DENSITY)

Time-harmonic fields



Angular frequency $\omega = 2\pi f$

(Linear) frequency f $[f] = Hz$

$$\omega T = 2\pi$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{f}$$

$$\frac{\partial}{\partial t}\sin(\omega t) = \omega\cos(\omega t)$$

$$\frac{\partial}{\partial t}\cos(\omega t) = -\omega\sin(\omega t)$$

$$\frac{\partial}{\partial t}\cancel{e^{j\omega t}} = j\omega\,\cancel{e^{j\omega t}}$$

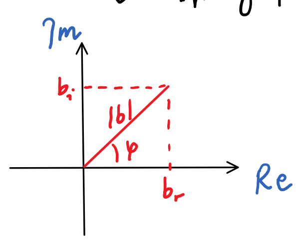
$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

$$j^2 = -1$$

Imaginary unit j

Phasors

$$b = b_r + jb_i = |b|e^{j\varphi} = \underbrace{|b|\cos\varphi}_{b_r} + j|b|\sin\varphi$$



$$|b|^2 = b_r^2 + b_i^2 = bb^*$$

$$= (b_r + jb_i)(b_r - jb_i)$$

$$b^* = (b_r + jb_i)^* = b_r - jb_i$$

$$\left(e^{j\beta}\right)^* = e^{-j\beta^*}$$

$$(cd)^* = c^* d^*$$

$$(c+d)^* = c^* + d^*$$

$$(j\beta) = -j\beta^*$$

Real $v(t)$ Complex $V = V_r + jV_i$

Real time-dependent $v(t)$

Complex (no explicit time dependence) $V = V_r + jV_i$

$$v(t) = \mathrm{Re}\{V e^{j\omega t}\}$$
$$= V_r \cos\omega t - V_i \sin\omega t$$

Complex vectors

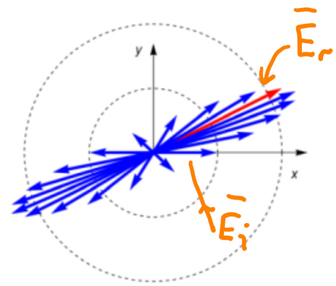
$$\bar{E} = \bar{E}_r + j\bar{E}_i$$

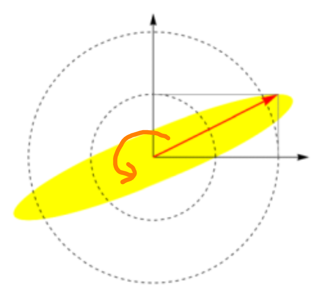
$$\bar{E}(t) = \mathrm{Re}\{\bar{E}e^{j\omega t}\} = \mathrm{Re}\{(\bar{E}_r + j\bar{E}_i)(\cos\omega t + j\sin\omega t)\}$$
$$= \bar{E}_r \cos\omega t - \bar{E}_i \sin\omega t$$

$\bar{E}(0) = \bar{E}_r$ $\quad\bar{E}(\omega t = \pi/2) = -\bar{E}_i$ $\quad\bar{E}(\omega t = \pi) = -\bar{E}_r$ $\quad\bar{E}(\omega t = \frac{3\pi}{2}) = \bar{E}_i$

ELLIPSE!

$$\begin{cases} \bar{E}_r = (2+j)\,\bar{a}_x \\ \bar{E}_i = \bar{a}_y \end{cases}$$





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x = Graphics[Arrow[{{0, 0}, {1.2, 0}}]];
y = Graphics[Arrow[{{0, 0}, {0, 1.2}}]];
c = Graphics[{Dashed, Circle[]}];
er[erx_, ery_] =
 Graphics[{Arrowheads → Large, Blue, Thickness[.01], Arrow[{{0, 0}, {erx, ery}}]}];
ei[eix_, eiy_] =
 Graphics[{Arrowheads → Large, Red, Thickness[.01], Arrow[{{0, 0}, {eix, eiy}}]}];
e[t_, erx_, ery_, eix_, eiy_] :=
  Graphics[{Arrowheads → Large, Black, Thickness[.01],
    Arrow[{{0, 0}, {erx Cos[t] - eix Sin[t], ery Cos[t] - eiy Sin[t]}}]}]];

Manipulate[Show[{c, x, y, er[erx, ery], ei[eix, eiy], e[t, erx, ery, eix, eiy]},
  PlotRange → {{-2, 2}, {-2, 2}}], {t, 0, 2 π}, {erx, 1, 0}, {ery, 0, 1},
 {eix, 0, 1}, {eiy, 1, -1}]
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