Lecture 7

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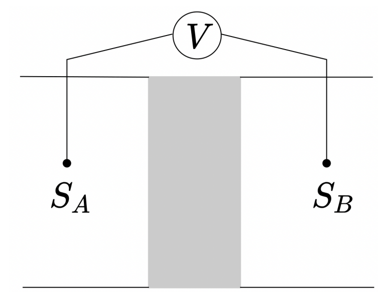
I. JOSEPHSON EFFECT

- What happens when we put a voltage across a weak link between two superconductors?



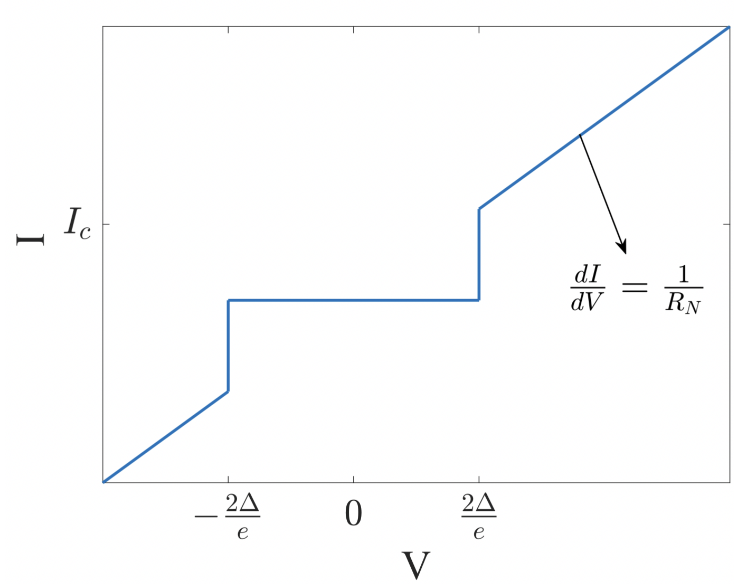
Weak link = can be a:

S-I-S (insulator between two superconductors)

S-N-S (a metal in-between)

S-s-S (a constriction)

- What we measure:



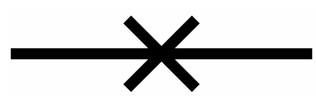
Δ = superconducting gap, $\Delta = 1.764 k_B T_c$, T_c = critical temperature (from BCS theory), R_N = normal-state resistance.

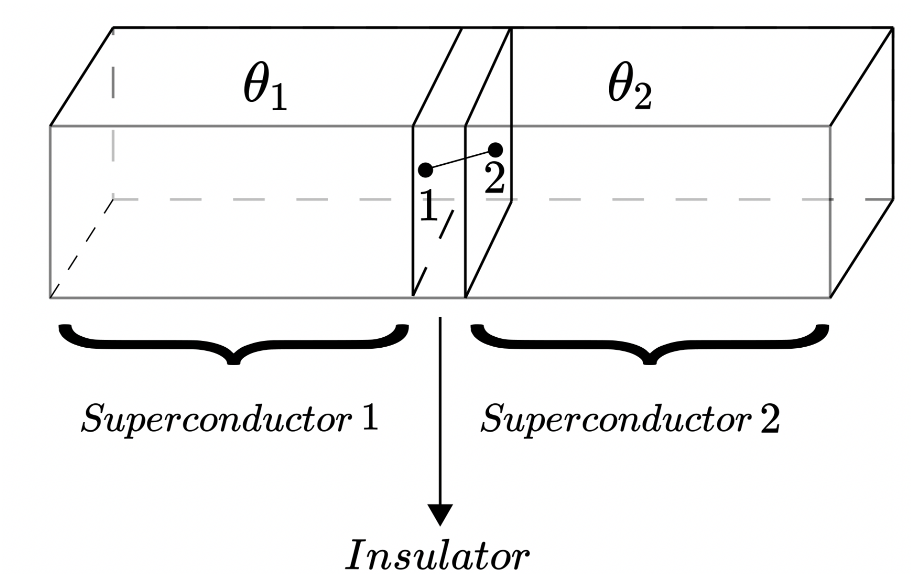
– Currents flowing for $|V| \geq \frac{2\Delta}{e}$ are no surprise – they are associated with breaking the

Cooper pairs by the voltage.

But, at $V = 0$ there is a current flowing, with max. value $= I_c$ (critical current of the junction). This is the Josephson effect.

– Circuit symbol:





Current-phase :

Consider the gauge-invariant phase difference, obtained by integrating the gauge-invariant phase gradient,

$\varphi = \int_1^2 d\vec{r}\left(\vec{\nabla}\theta + \frac{2\pi}{\Phi_0}\vec{A}\right) = \theta_2 - \theta_1 + \frac{2\pi}{\Phi_0}\int_1^2 d\vec{r}\vec{A}(\vec{r}, t).$

This is the only quantity which is gauge-invariant and includes the difference in phases $\theta_2 - \theta_1$ as we cross the insulator. So perhaps $\mathcal{J}_s = \mathcal{J}_s(\varphi)$, that is, a function of φ. Which one? Well, we should have also:

1. periodicity $\mathcal{J}_s(\varphi) = \mathcal{J}_s(\varphi + 2\pi n)$
2. $\mathcal{J}_s(0) = 0$ (no current when there is no phase difference).

$$\Longrightarrow \quad \mathcal{J}_s(\varphi) = \mathcal{J}_c \sin\varphi + \underbrace{\sum_{m=2}^{\infty} \mathcal{J}_m \sin m\varphi}_{\text{This can be neglected.}} \quad , \tag{1}$$

where $\mathcal{J}_c$ is constant and is the critical Josephson current density.

Therefore for a given device we will have

$$I = I_c \sin\varphi \; , \tag{2}$$

$I_c =$ critical Josephson current.

Voltage-phase

Consider again the gauge-invariant phase difference
$\varphi = \theta_2 - \theta_1 + \frac{2\pi}{\Phi_0} \int_1^2 d\vec{r}\vec{A}(\vec{r}, t)$

Recall now the energy-phase relationship
$-\hbar \frac{\partial\theta}{\partial t} = \frac{1}{2} \frac{\mu_0 \lambda_L^2}{n_s} \mathcal{J}_s^2 + q^* V$.
This relation is valid for the phases θ_1 and θ_2 inside the superconductors 1 and 2.
$\Longrightarrow \quad \frac{\partial\varphi}{\partial t} = -\frac{1}{\hbar} \frac{\mu_0 \lambda_L^2}{2n_s} \Big(\mathcal{J}_s^2(2) - \mathcal{J}_s^2(1) \Big) - \frac{q^*}{\hbar} \Big(V(2) - V(1) \Big) + \frac{2\pi}{\Phi_0} \int_1^2 d\vec{r} \frac{\partial\vec{A}}{\partial t}$, but $\mathcal{J}_s(1) \equiv \mathcal{J}_s(2)$ (conservation of charge or Kirchoff's current law).

$\Longrightarrow \; \frac{\partial\varphi}{\partial t} = \frac{2\pi}{\Phi_0} \int_1^2 d\vec{r} \Big(\vec{\nabla} + \frac{\partial\vec{A}}{\partial t} \Big)$, but $\vec{E} = -\vec{\nabla} V - \frac{\partial\vec{A}}{\partial t}$

$\Longrightarrow \; \frac{\partial\varphi}{\partial t} = -\frac{2\pi}{\Phi_0} \int_1^2 d\vec{r} \cdot \vec{E}$.

Let us consider now the case of a thin junction (which is typically the case, the insulating layer is a few nm). We can then neglect the integral over the $vecA$ vector.

We get our final result

$$\frac{\partial\varphi}{\partial t} = \frac{2\pi}{\Phi_0} V \; . \tag{3}$$

where $V \equiv V_2 - V_1$.

Remember also that $\Phi_0 = \frac{h}{2e}$.

Or: $V = \frac{\partial}{\partial t}\left(\frac{\Phi_0}{2\pi} \cdot \varphi\right)$ very similar to Faraday's law. Indeed, notice that the quantity $\phi = \frac{\Phi_0}{2\pi}\varphi$ has dimensions of flux. Indeed, a phase difference φ between two nodes in the circuit means that there is a difference $\phi = \frac{\Phi_0}{2\pi}\varphi$ between the node fluxes at those points. So we have

$$V = \frac{\partial \phi}{\partial t} \tag{4}$$

which is Faraday's induction law.

To recap, we have derived the two Josephson relations

$$\boxed{I = I_c \sin\varphi} \qquad \text{current-phase relation ,} \tag{5}$$

$$\boxed{\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V} \qquad \text{phase-voltage relation ,} \tag{6}$$

II. CONSEQUENCES

The Josephson relations look very simple but they have very profound consequences that lead to many applications in sensing, metrology, and quantum computing.

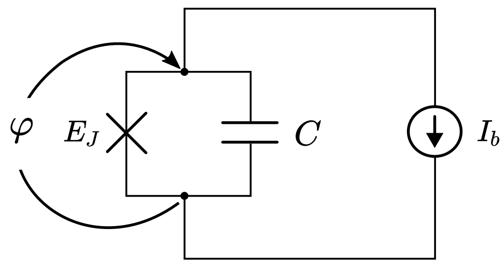
DC Josephson Effect: $V = 0 \implies \frac{\partial \varphi}{\partial t} = 0 \implies \varphi = \text{const.}$
$I = I_c \sin\varphi$ — the current can reach a max. value of I_c.

AC Josephson Effect: $V = \text{const} \neq 0 \implies \varphi = \frac{2e}{\hbar} V \cdot t$.
$\implies \quad I = I_c \sin\left(\frac{2e}{\hbar} V \cdot t\right) = I_c \sin\left(2\pi \frac{V}{\Phi_0} t\right)$.
$f_J = \frac{V}{\Phi_0}$ = Josephson frequency $= 483 \times 10^{12}\ V_0$ (Hz).

Josephson Inductance: $\frac{\partial I}{\partial t} = \frac{\partial I}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial t} = I_c \cos\varphi \cdot \frac{2\pi}{\Phi_0} V$
or: $V = L_J(\varphi) \frac{\partial I}{\partial t}$, where $L_J(\varphi)$ = Josephson inductance.
$L_J(\varphi) = \frac{\Phi_0}{2\pi I_c \cos\varphi}$ — depends on phase! Can be ∞ if $\varphi = \frac{\pi}{2} + n\pi$.

Josephson Energy: $E_J(\varphi) = \int dt I \cdot V = \int d\varphi \cdot I_c \sin\varphi \cdot \frac{\Phi_0}{2\pi}$
$E_J(\varphi) = -\frac{I_c \Phi_0}{2\pi} \cos\varphi = -E_J \cos\varphi$.
$E_J = \frac{\Phi_0 I_c}{2\pi}$ = Josephson energy.

The Josephson junction as a circuit element



A current-biased Josephson junction.

Circuit equations will include the superconducting phase φ. All the usual other circuit laws (e.g. Kirchhoff's laws) remain valid. In the next lectures we will learn the very important Lagrange method to deal with complicated circuits, which shows how to identify the correct variables and their canonically conjugates, and writing the equations of motion. This method is based on analyzing the energies in the circuit. Let's do that here, with a simple example, a current-biased Josephson junction. We have:

$$E_J = -E_J \cos\varphi \tag{7}$$

the Josephson energy as above. Note the minus sign. What does it mean? Then, the Josephson junction, by the way in which it is fabricated, has an intrinsic capacitance C. The energy stored in this capacitor is

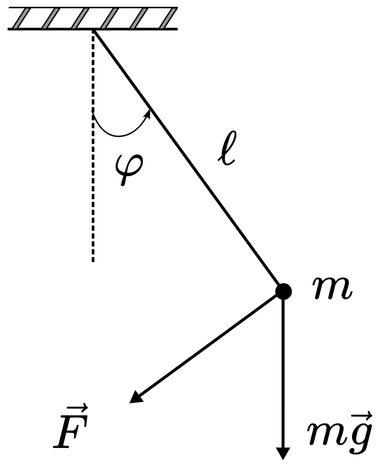
$$E_C = \int dt I_C V = \int dt \frac{d(CV)}{dt} V = C \int dV V = \frac{1}{2} C V^2 = \frac{1}{2} C \left(\frac{d\phi}{dt}\right)^2 = \frac{1}{2} C \left(\frac{\Phi_0}{2\pi}\right)^2 \left(\frac{d\varphi}{dt}\right)^2 \tag{8}$$

The energy associated with the phase difference across the current bias source

$$E_b = \int dt I_b V = \int dt I_b \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt} = \frac{\Phi_0}{2\pi} I_b \varphi, \tag{9}$$

where we have used $I_b =$ constant (in time).

Where have you seen these before? Let's have a look at a driven pendulum (a pendulum of length l and mass m in a gravitational field with a constant force F applied as in the figure, producing a constant torque).



A current-biased Josephson junction.

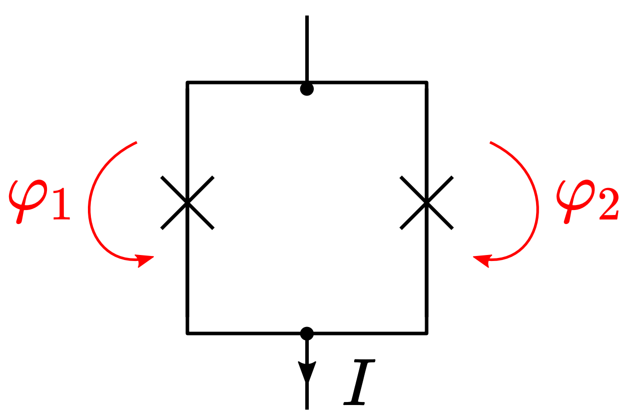
gravitational potential energy $= -mgl\cos(\varphi)$. This depends where you set the zero-energy level (if I write it like this the zero-energy level is where the pendulum is suspended), but you can add any constant and it will not matter for the equations of motion. For example you can have $mgl(1-\cos\varphi)$. Similarly you can write the Josephson energy as $E_J(1-\cos\varphi)$ - nothing will change!

kinetic energy $= \frac{1}{2}ml^2\left(\frac{d\varphi}{dt}\right)^2$, because the velocity is $v = l\frac{d\varphi}{dt}$.

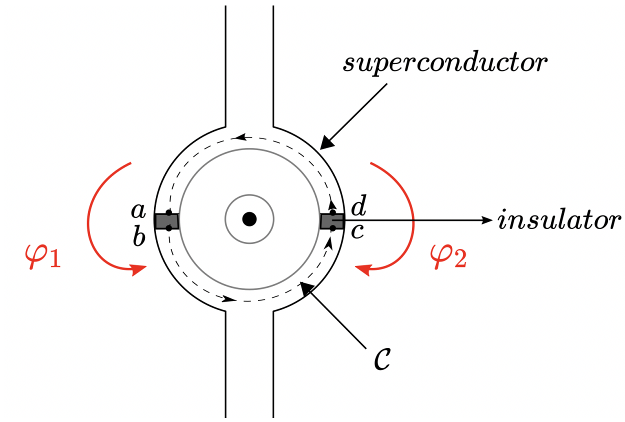
torque potential energy $= Fl\varphi$ is the energy due to the force F.

III. APPLICATION: THE DC–SQUID

– Superconducting quantum interference device:



How it is fabricated:



$\oint_{\mathcal{C}} \vec{\nabla}\theta \cdot d\vec{r} = 2\pi n = (\theta_b - \theta_a) + (\theta_c - \theta_b) + (\theta_d - \theta_c) + (\theta_a - \theta_d)$, where

$$\begin{cases} \theta_b - \theta_a = \varphi_1 - \frac{2\pi}{\Phi_0}\int_a^b \vec{A} d\vec{r} \\ \theta_c - \theta_b = \int_b^c d\vec{r}\cdot\vec{\nabla}\theta = \underbrace{-\frac{2\pi}{\Phi_0}\mu_0\lambda_L^2 \int_b^c d\vec{r}\vec{\mathcal{J}}_s}_{0, \quad \mathcal{J}_s=0 \text{ inside the superconductor}} - \frac{2\pi}{\Phi_0}\int_b^c d\vec{r}\cdot\vec{A} \\ \theta_d - \theta_c = -\varphi_2 - \frac{2\pi}{\Phi_0}\int_c^d \vec{A} d\vec{\ell} \\ \theta_a - \theta_d = \int_d^a d\vec{r}\cdot\vec{\nabla}\theta = \underbrace{-\frac{2\pi}{\Phi_0}\mu_0\lambda_L^2 \int_d^a d\vec{r}\vec{\mathcal{J}}_s}_{0} - \frac{2\pi}{\Phi_0}\int_d^a d\vec{r}\cdot\vec{A} \end{cases}$$

$\therefore \quad \oint_{\mathcal{C}} \vec{\nabla}\theta \cdot d\vec{r} = \varphi_1 - \varphi_2 - \frac{2\pi}{\Phi_0} \underbrace{\oint_{\mathcal{C}} d\vec{r}\cdot\vec{A}}_{=\phi}$, where ϕ = the magnetic flux piercing the SQUID.

$$\phi_1 - \phi_2 = 2\pi n + \frac{2\pi\phi}{\Phi_0} \; . \tag{10}$$

So $\quad I = I_1 + I_2 = I_c \sin\varphi_1 + I_c \sin\varphi_2 = 2I_c \sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}$.

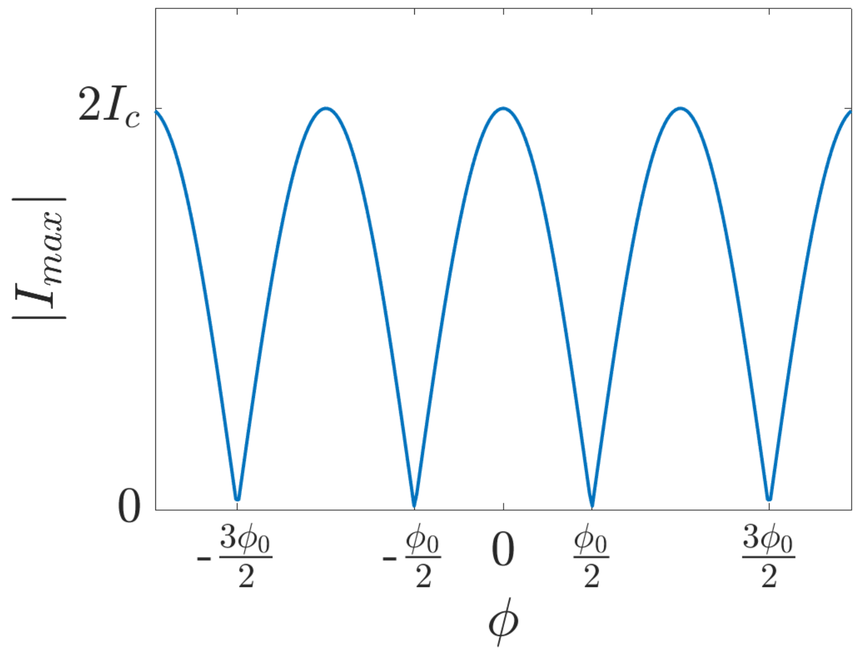
Let $\varphi \equiv \frac{\varphi_1-\varphi_2}{2} \implies I = 2I_c \sin\varphi \cos\left(\frac{\pi\phi}{\Phi_0} + \pi n\right)$.

$$I \equiv I_{\max}(\phi)\cdot\sin\varphi \; , \tag{11}$$

where $I_{\max}(\phi) = 2I_c \cos\left(\frac{\pi\phi}{\Phi_0} + \pi n\right)$.

The SQUID behaves as a single Josephson junction with critical current controlled by the magnetic flux.

The maximum current will be



- $|I_{\max}|$ never exceeds $2I_c$.
- $I_{\max}$ can be zero! This is understood as destructive interference of the currents in the two branches of the SQUID.

References

- Terry P. Orlando and Kevin A. Delin — Foundations of Applied Superconductivity.
- D.R. Tilley and J. Tilley — Superfluidity and Superconductivity.
- Antonio Barone and Giafranco Paternò — Physics and Applications of the Josephson Effect.