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WE BUILD QUANTUM COMPUTERS

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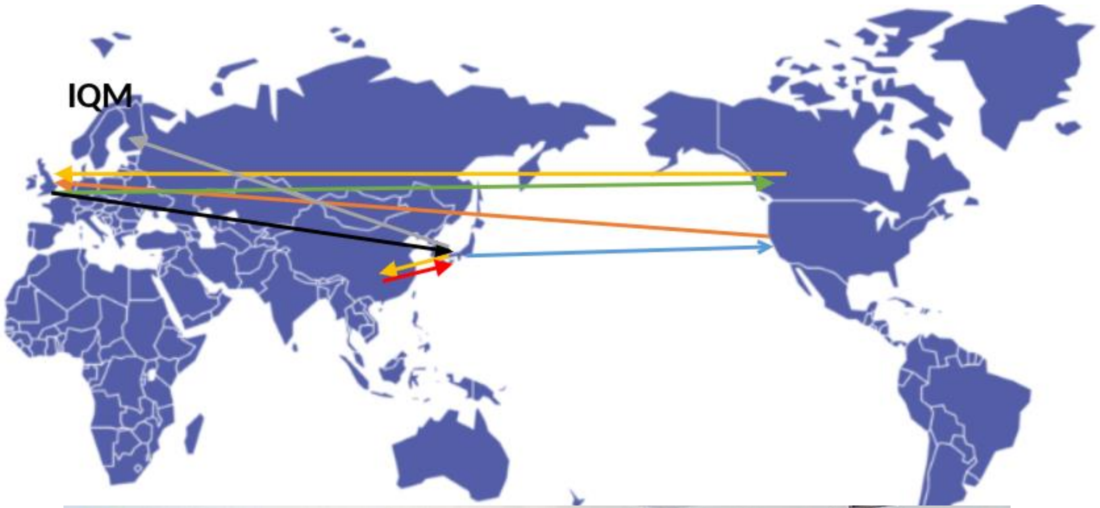
Lecture notes on PHYS-C0254 Quantum Circuits

www.meetiqm.com

About me: From science to industry

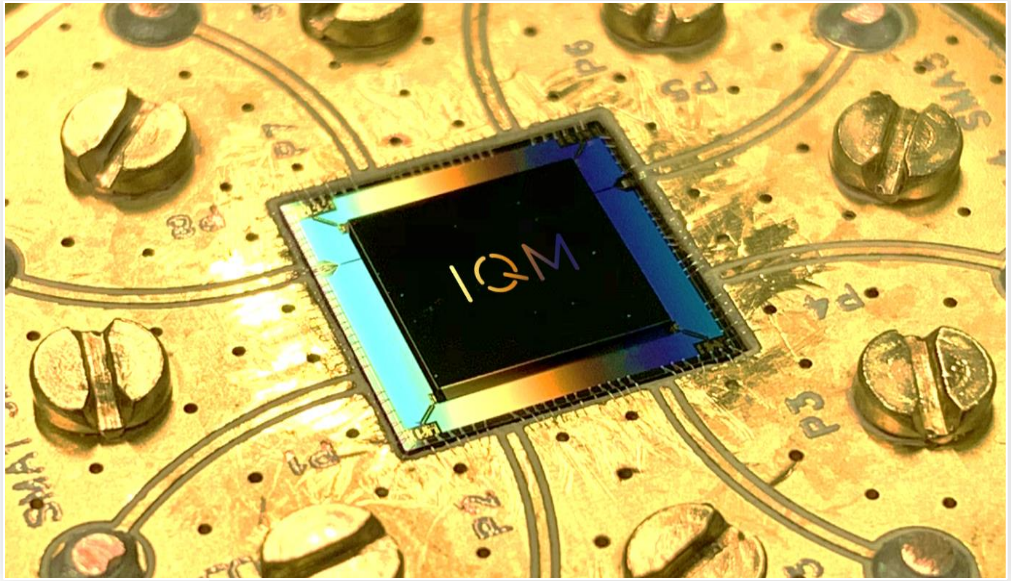
- Originally from Sasebo, Japan
- 1971 – 1975: Science student at Kyoto University, Japan
- 1975 – 1981: PhD student in Physics at Kyoto.
- 1980 – 1982: Research Fellow & Postdoc at University of Southern California, LA.
- 1982 – 1983: Math Student at King's College, London
- 1983 – 1985: Postdoc at University of Alberta, Canada
- 1985 – 1986: Postdoc at University of Sussex, UK
- 1986 – 1993: Associate Prof. at Shizuoka University, Japan
- 1993 – 2017: Associate Prof & Prof at Kindai University, Japan
- 2017 – 2020: Prof at Shanghai University, PR China
- 2023 – today: Quantum Education Manager at IQM
- 2001- ??: Lectured quantum computing at Helsinki University of Technology (TKK)



IQM in brief

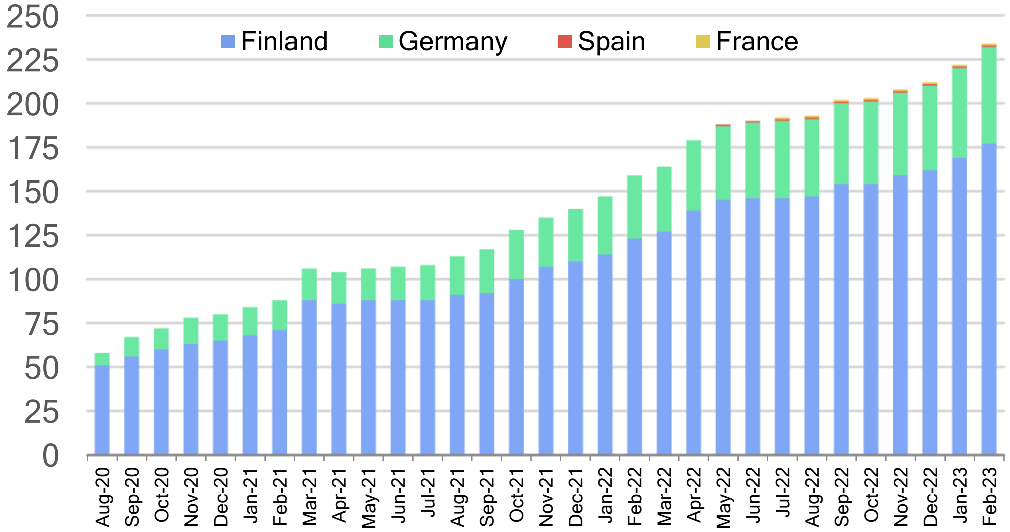
Quantum computing scale-up

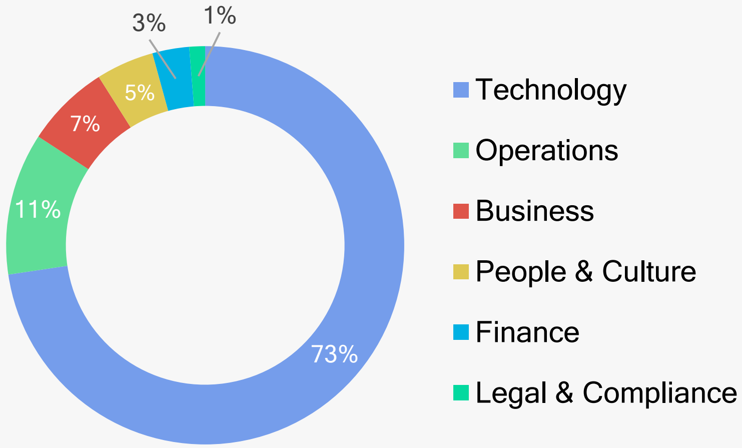
- Spinout of Aalto University and VTT in July 2019
- Develop and sell on-premises quantum computers based on superconducting technology
- Secured 2 rounds of private investment funding (Seed & A)
- Sold 2 quantum computers thus far (Finland, Germany)



Employees

- 220+ employees
- 105 PhDs
- 11 Professor-level tech leaders
- 45+ nationalities





IQM builds and delivers quantum computers

240+ experts

105+ PhDs

45+ nationalities

On-premises & full access

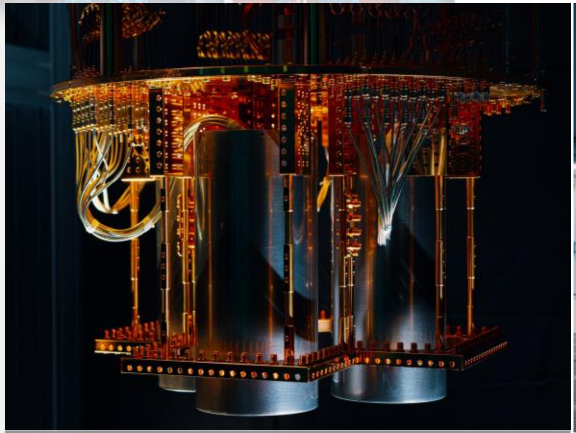
2 systems sold

1 delivered

Full-stack solutions with **co-design**

Own facilities → **fast turnaround**

IQM's Private Foundry: **600 m^2**



Industry partners

Atos

KEYSIGHT

VTT

Funding

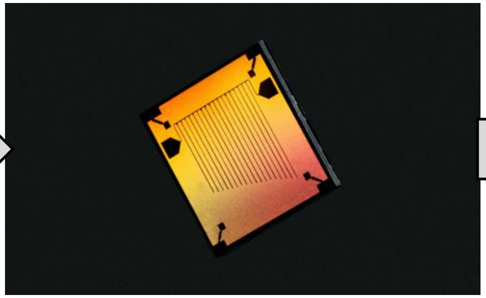
200+ M€

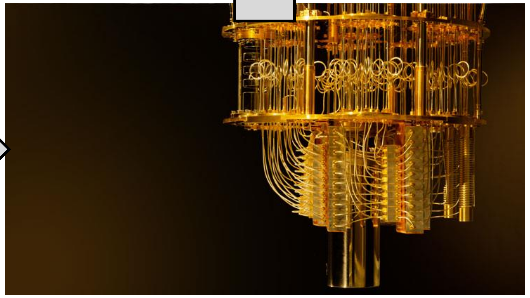
Our Mission: To build world-leading quantum computers for the well-being of humankind, now and for the future

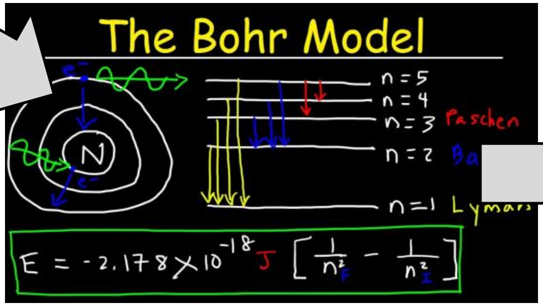
How does everything fit into the big picture?

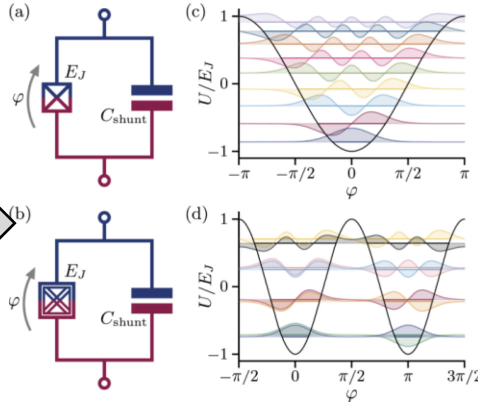
- Scientists, atoms, circuits, qubits, quantum computing

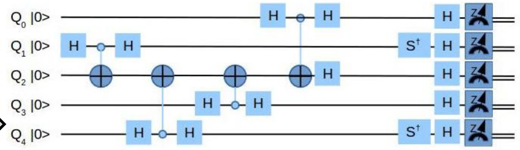
What is required?











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Di Vincenzo Criteria and where you can find them in this course

Statement of the criteria

1. A scalable physical system with well characterized qubit
2. The ability to initialize the state of the qubits to a simple fiducial state
3. Long relevant decoherence times
4. A "universal" set of quantum gates
5. A qubit-specific measurement capability

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Agenda for lectures 8-12

8. Quantization of electrical networks
    a. Harmonic oscillator: Lagrangian, eigenfrequency
    b. LC oscillator, Legendre transform to Hamiltonian
    c. Quantization of oscillators
9. Superconducting quantum circuits
    a. Qubits: Transmon qubit, Charge qubit, Flux qubit 1st DiVincenzo criteria
    b. Circuit-QED: Rabi model
    c. Rotating Wave approximation: Jaynes-Cummings model
10. Single-qubit operations:
    a. Initialization 2nd DiVincenzo criteria
    b. Readout 5th DiVincenzo criteria
    c. Control:T1, T2 measurements, Randomized benchmarking 3rd DiVincenzo criteria
11. Two-qubit operations: Architectures for 2-qubit gates 4th DiVincenzo criteria
    a. iSWAP
    b. cPhase
    c. cNot
12. Challenges in quantum computing
    a. Scaling
    b. SW-HW gap
    c. Error-correction

Agenda for today

7. Quantization of electrical networks
    a. Harmonic oscillator: Lagrangian, eigenfrequency
    b. LC oscillator, Legendre transform to Hamiltonian
    c. Quantization of oscillators

a.

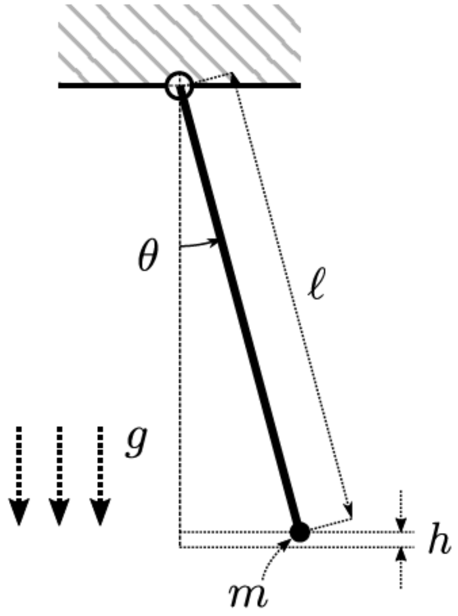


Figure 1: Classical pendulum.

b.

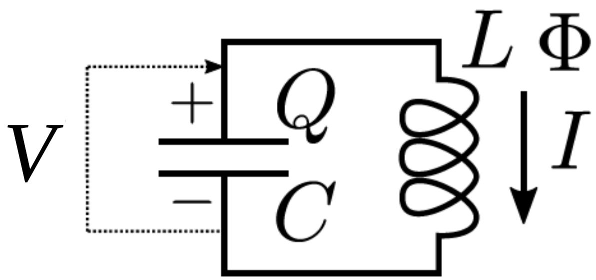
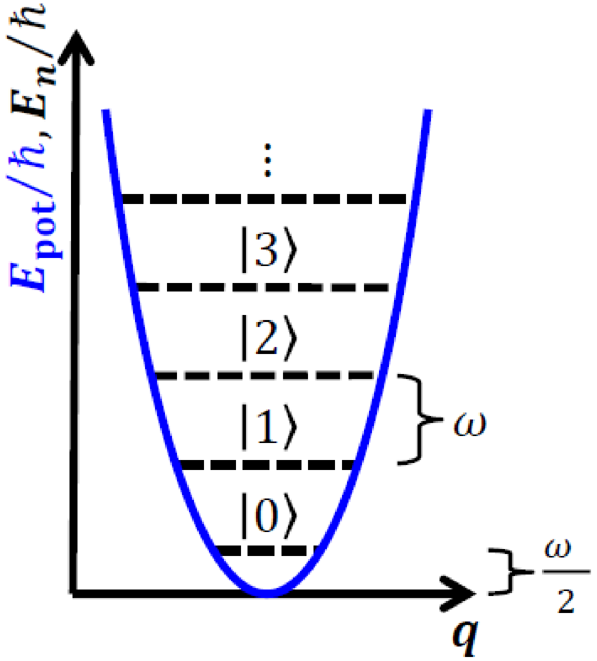


Figure 2: Superconducting LC oscillator.

c.



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General note: Harmonic oscillators

- General note: In physics, many phenomena can be explained by harmonic oscillators. They are the standard tool in our physics toolbox.

$$V(x) = V(0) + \cancel{V'(0)}x + \frac{1}{2}V''(0)x^2 + \cdots$$

- Usually, there are two important variables involved like position and momentum, *q* and *p*.

- One can often find analogies where two system variables are equivalent to *q* and *p*. For example, in an LC oscillator these are flux and charge.

Short review: Lagrangian & Hamiltonian

- During this course, Lagrangian and Hamiltonian mechanics are used for analyzing quantum computing circuits.

- Recall that the Lagrangian is defined as the kinetic energy T minus the potential energy V:

$$L(\dot{q}, q) \equiv T(\dot{q}, q) - V(q)$$

- Quite often the Hamiltonian represents the total energy of the system:

$$H(p, q) \equiv T(p, q) + V(q)$$

Legendre transformation

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Short review: Classical oscillator*

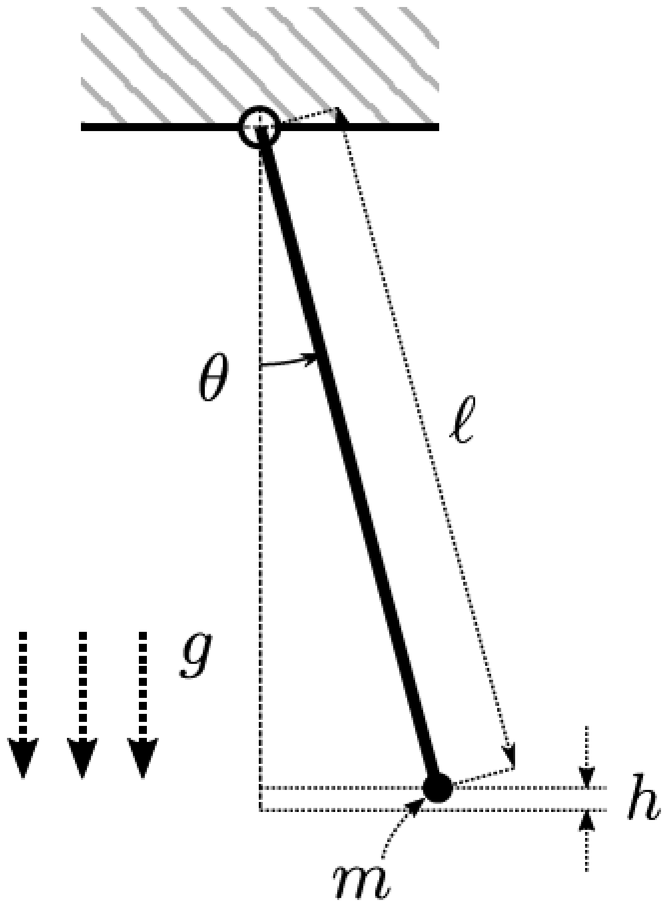

Figure 1: Classical pendulum.

The Euler-Lagrange equation states

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

If *T* does not depend on *q*,

$$\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q}$$

Since $p = \partial L / \partial \dot{q}$ we obtain Newton´s equation of motion

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial V}{\partial q}$$

* Details in lecturenotes-EQS_Helsinki_2019.pdf

Short review: Classical oscillator*

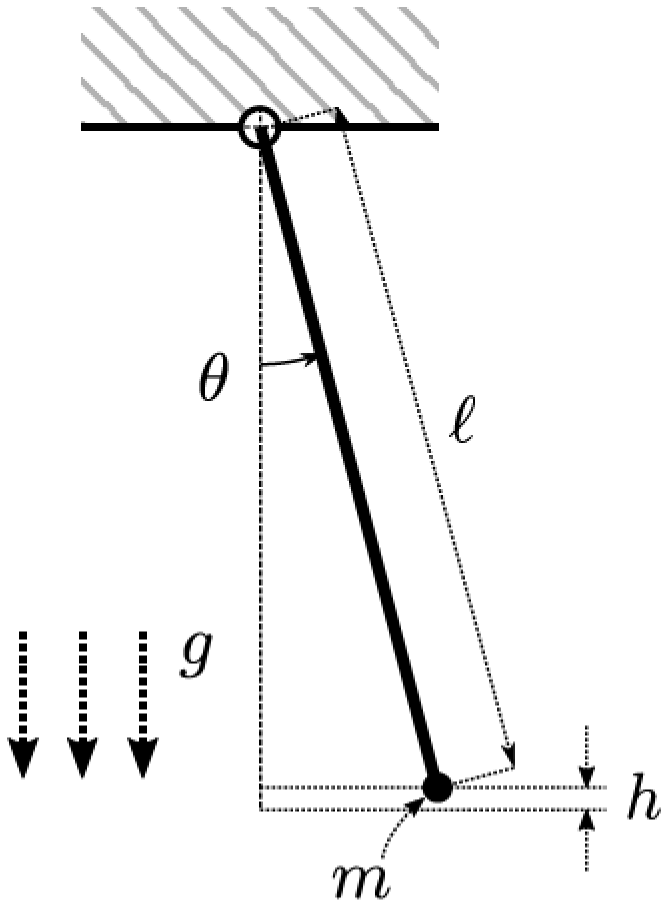

Figure 1: Classical pendulum.

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$$

The potential energy for small oscillation is

$$V = mgh = mg\ell\,(1 - \cos\theta) \approx \frac{1}{2}mg\ell\theta^2$$

$$L \equiv T - V\,.$$

We introduce generalized coordinates q and generalized momentum p as

$$q \equiv \theta,$$

$$p \equiv \frac{\partial L}{\partial \dot{\theta}} \simeq \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}m\ell^2\dot{\theta}^2 - \frac{1}{2}mg\ell\theta^2\right) = m\ell^2\dot{\theta}$$

p is the angular momentum.

* Details in lecturenotes-EQS_Helsinki_2019.pdf

Short review: Classical oscillator*

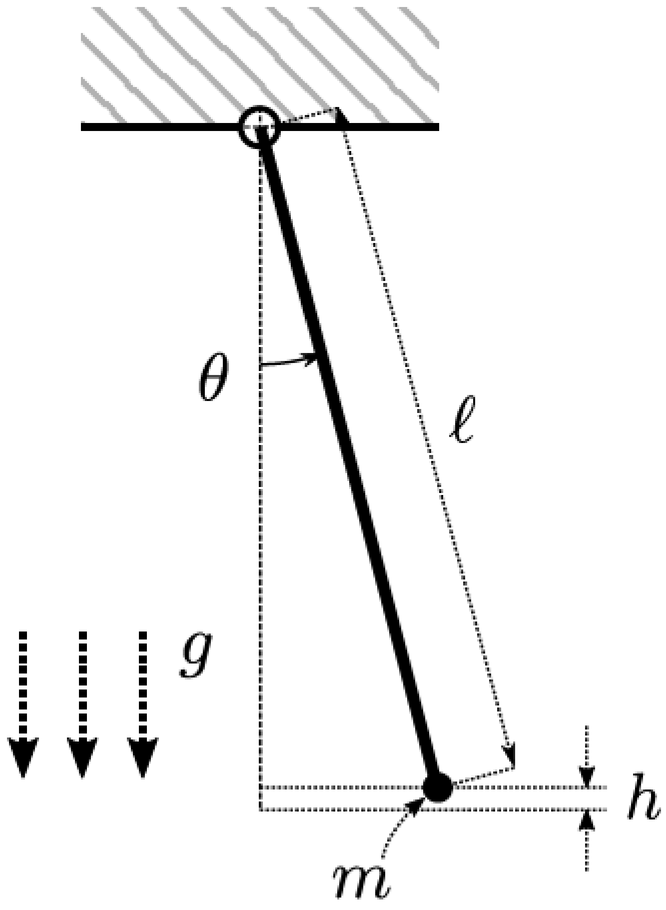

Figure 1: Classical pendulum.

Applying our example to the Euler-Lagrange equation gives

$$\dot{p} = -mg\ell\theta.$$

By differentiating $p = m\ell^2\dot{\theta}$ wrt time, we obtain

$$\dot{p} = m\ell^2\ddot{\theta}.$$

Equating these yields

$$m\ell^2\ddot{\theta} + mg\ell\theta = 0 \rightarrow \ddot{\theta} + \frac{g}{\ell}\theta = 0.$$

* Details in lecturenotes-EQS_Helsinki_2019.pdf

Short review: Classical oscillator*

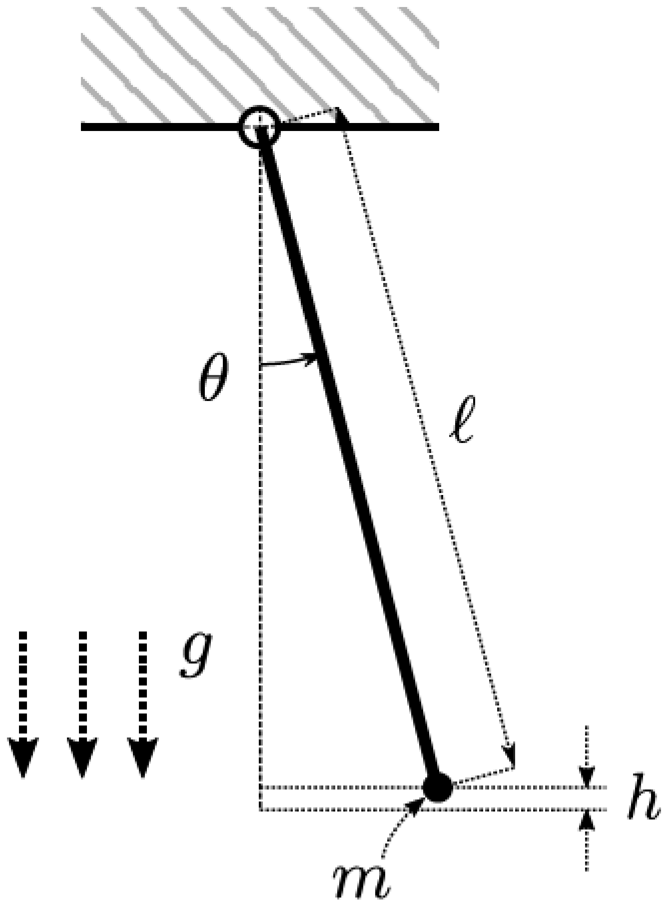

Figure 1: Classical pendulum.

Because we are smart, we chose a trial function

$$\theta = C\exp(i\omega t)$$

Inserting this function into the differential equation yields:

$$i^2\omega^2 C\exp(i\omega t) + \frac{g}{\ell}C\exp(i\omega t) = 0$$

This equation is satisfied for any t if we choose

$$\omega = \sqrt{g/\ell}$$

Key takeaway: Starting from the equation of motion, we derived the eigenfrequency of the system

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Agenda for today

7. Quantization of electrical networks
    a. Harmonic oscillator: Lagrangian, eigenfrequency
    b. LC oscillator, Legendre transform to Hamiltonian
    d. Quantization of oscillators

a.

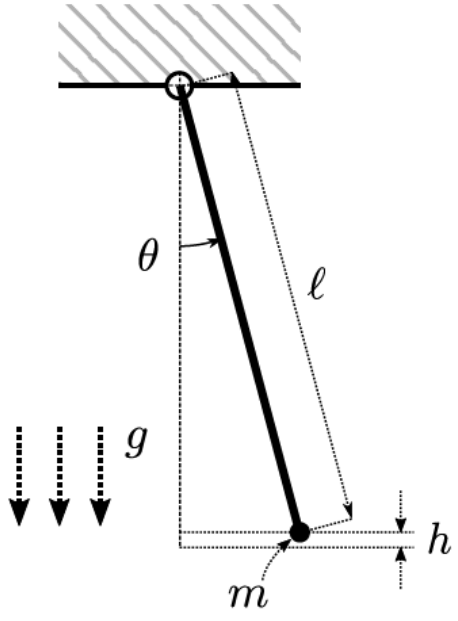


Figure 1: Classical pendulum.

b.

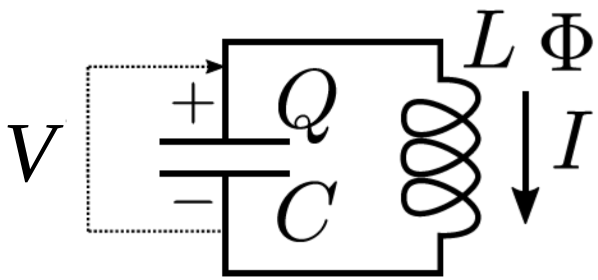
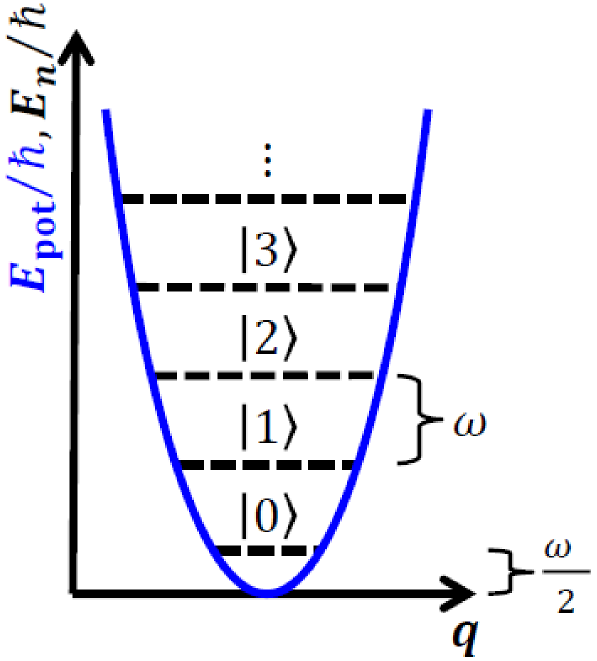


Figure 2: Superconducting LC oscillator.

c.



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General note: LC oscillators

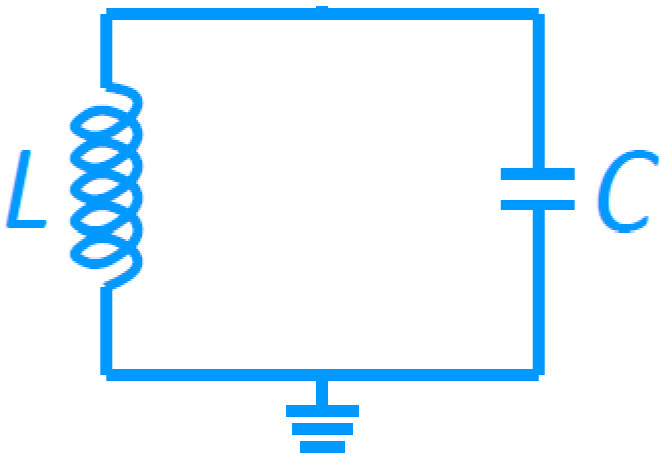
- General note: Once you understand the harmonic oscillator, you can easily apply the concept to any other oscillator.

$$\text{Position } \hat{q} \leftrightarrow \text{Flux } \hat{\Phi}$$

$$\text{Momentum } \hat{p} \leftrightarrow \text{Charge } \hat{Q}$$

$$\text{Mass } m \leftrightarrow \text{Capacitance } C$$

$$\text{Frequency } \omega \leftrightarrow \omega = 1/\sqrt{LC}$$



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Transfer step: LC oscillator*

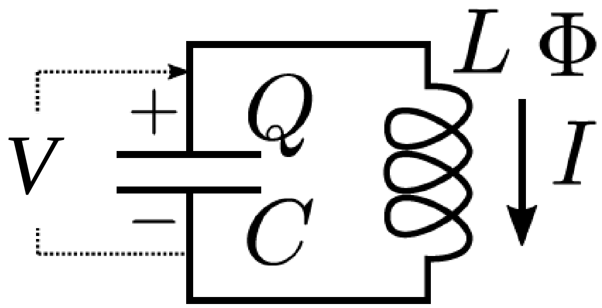

Figure 2: Superconducting LC oscillator.

We consider an electrical circuit consisting of inductance *L* and capacitance *C*. For the magnetic flux Φ through a coil, it holds that

$$\Phi = LI$$

The Lenz law tells us that

$$\dot{\Phi} = V$$

Hence, the potential energy stored in the inductor is

$$U = \int_{t_0}^{t_1} P\, dt = \int_{t_0}^{t_1} VI\, dt = \int_{t_0}^{t_1} \frac{\Phi\dot{\Phi}}{L}\, dt = \frac{\Phi^2}{2L},$$

where we defined Φ as the generalized coordinate.

 * Details in lecturenotes-EQS_Helsinki_2019.pdf

Transfer step: LC oscillator*

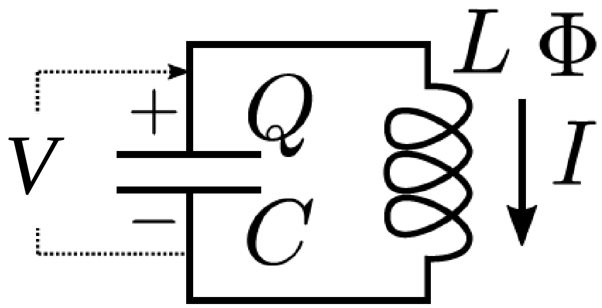

Figure 2: Superconducting LC oscillator.

The charge stored in the capacitor is

$$Q = CV.$$

The power fed into the circuit is $P = VI$ and consequently

$$P = V\dot{Q} = VC\dot{V}.$$

Hence, the kinetic energy stored in the capacitor is

$$T = \int_{t_0}^{t_1} P\,dt = \int_{t_0}^{t_1} VC\dot{V}\,dt = \frac{CV^2}{2} = \frac{C}{2}\dot{\Phi}^2$$

 * Details in lecturenotes-EQS_Helsinki_2019.pdf

Transfer step: LC oscillator*

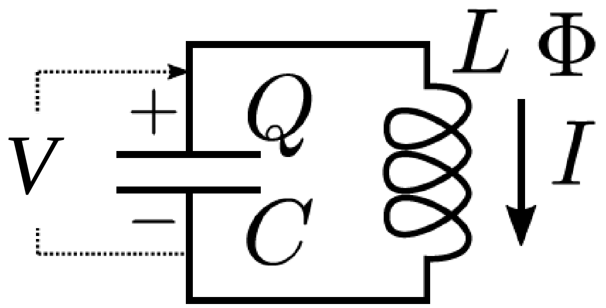

Figure 2: Superconducting LC oscillator.

To apply Lagrangian mechanics, we use the previous results

$$T = \frac{C}{2}\dot{\Phi}^2, \qquad U = \frac{\Phi^2}{2L}$$

allowing us to write the Lagrangian as

$$L = \frac{C}{2}\dot{\Phi}^2 - \frac{\Phi^2}{2L}$$

To derive the equation of motion, we again introduce generalized coordinate and momentum

$$q = \Phi, \qquad p = \frac{\partial L}{\partial \dot{\Phi}} = C\dot{\Phi} = CV = Q.$$

 * Details in lecturenotes-EQS_Helsinki_2019.pdf

Transfer step: LC oscillator*

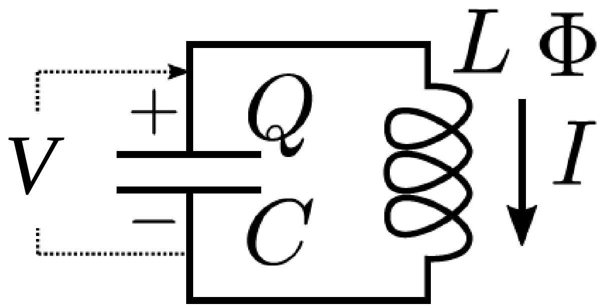

Figure 2: Superconducting LC oscillator.

Remind yourself again of Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Using the above results gives the equation of motion for flux:

$$C\ddot{\Phi} + \frac{\Phi}{L} = 0 \rightarrow \ddot{\Phi} + \frac{\Phi}{LC} = 0.$$

Using a similar ansatz for the trial function yields the resonance frequency

$$\omega = \frac{1}{\sqrt{LC}} \quad \Leftrightarrow \quad \text{Pendulum: } \omega = \sqrt{g/\ell}$$

Key takeaway: Starting from the equation of motion, we derived the eigenfrequency of the system

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Legendre transformation to Hamiltonian*

Hamiltonian gives two 1st order differential equations, while Euler Lagrange gives one 2nd order

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\delta H = p\delta\dot{q} + \dot{q}\delta p - \frac{\partial L}{\partial \dot{q}}\delta\dot{q} - \frac{\partial L}{\partial q}\delta q$$

$$= \dot{q}\delta p - \dot{p}\delta q = \frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial q}\delta q$$

$$\rightarrow \frac{dq}{dt} = \frac{\partial H}{\partial p}, \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

We need the Hamiltonian to write down the Schrodinger Equation. The general definition of a Hamiltonian is

$$H(p,q) = \dot{q}p - L(\dot{q},q).$$

We take the total time derivative to show H is conserved;

$$\frac{dH}{dt} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial t}.$$

Since $p = \partial L/\partial\dot{q}$ and $\frac{\partial L}{\partial t} = 0$, we have

$$\frac{dH}{dt} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - p\ddot{q} = \dot{q}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] = 0$$

The Hamiltonian is a constant of motion, i.e. the energy is conserved.

* Details in lecturenotes-EQS_Helsinki_2019.pdf

Legendre transformation to Hamiltonian*

Hamiltonian gives two 1st order differential equations, while Euler Lagrange gives one 2nd order

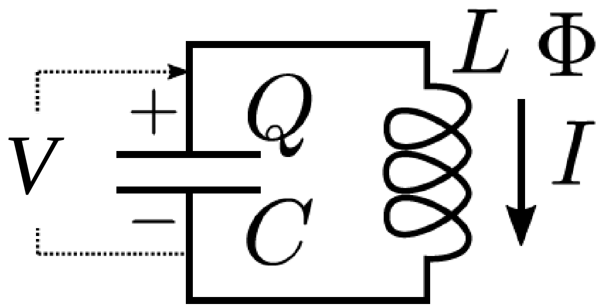

Figure 2: Superconducting LC oscillator.

We can use the general definition for the Hamiltonian to find

$$H = Q\dot{\Phi} - \left(\frac{C}{2}\dot{\Phi}^2 - \frac{\Phi^2}{2L}\right) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$T = \frac{Q^2}{2C}$ is the energy of a capacitor while $V = \frac{\Phi^2}{2L}$ is the energy of the inductor.

Hence, the Hamiltonian represents the total energy of the system.

$$H(\Phi, Q) = T(Q) + V(\Phi)$$

Key takeaway: Starting from Lagrangian, we derived the Hamiltonian of the system. This is necessary to derive energy quantization.

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Agenda for today

7. Quantization of electrical networks
   a. Harmonic oscillator: Lagrangian, eigenfrequency
   b. Transfer step: LC oscillator, Legendre transform to Hamiltonian
   d. Quantization of oscillators

a.

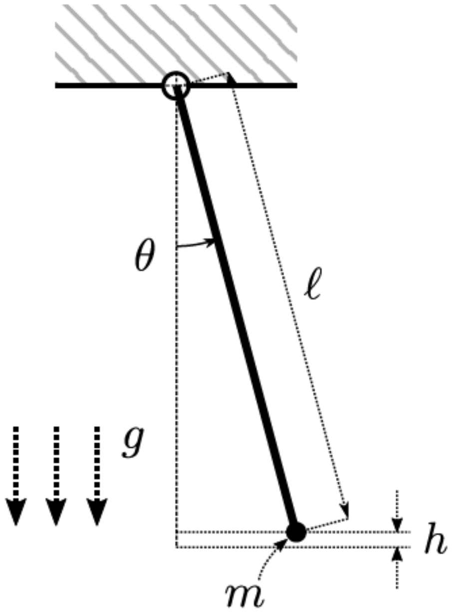


Figure 1: Classical pendulum.

b.

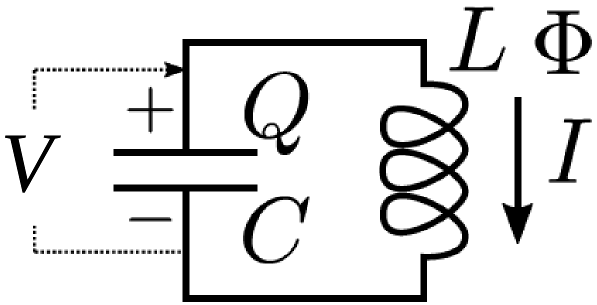
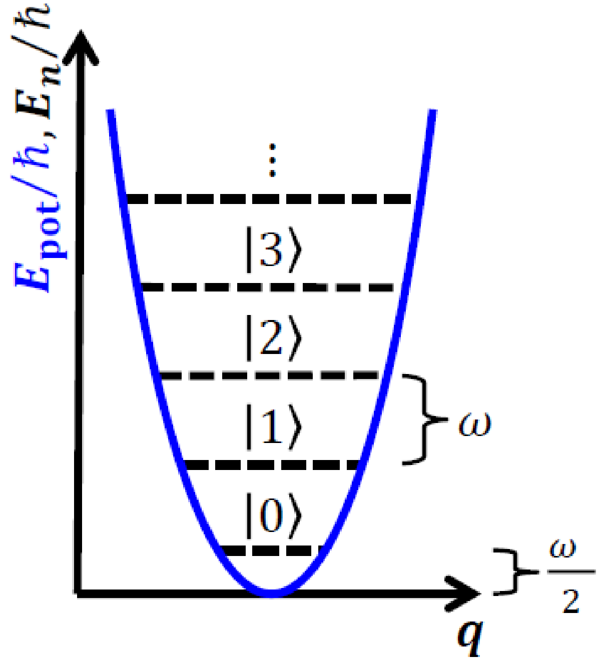


Figure 2: Superconducting LC oscillator.

c.



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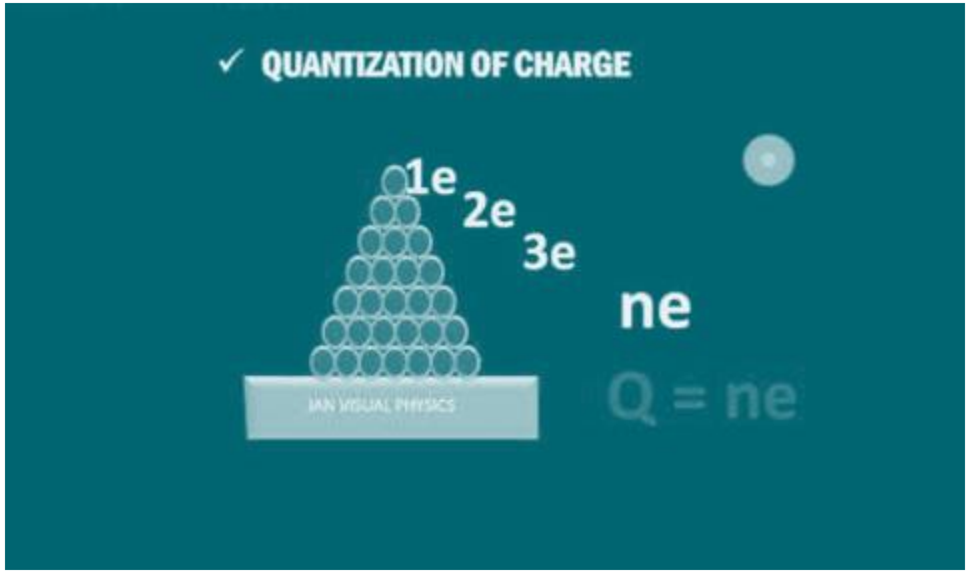
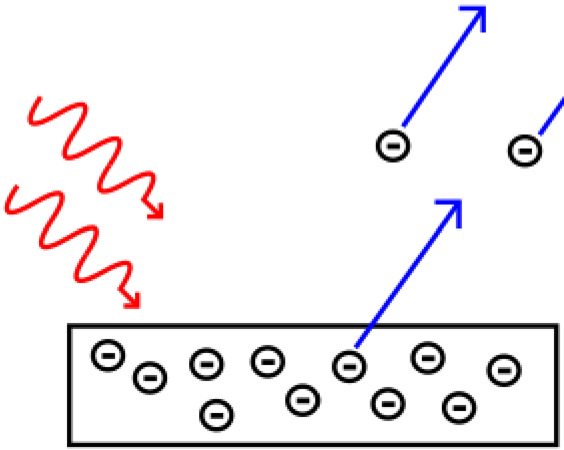
Classical → Quantum



Photoelectric effect → Electromagnetic field is quantized → $E = \hbar\omega(n + \frac{1}{2})$

Energy is quantized:



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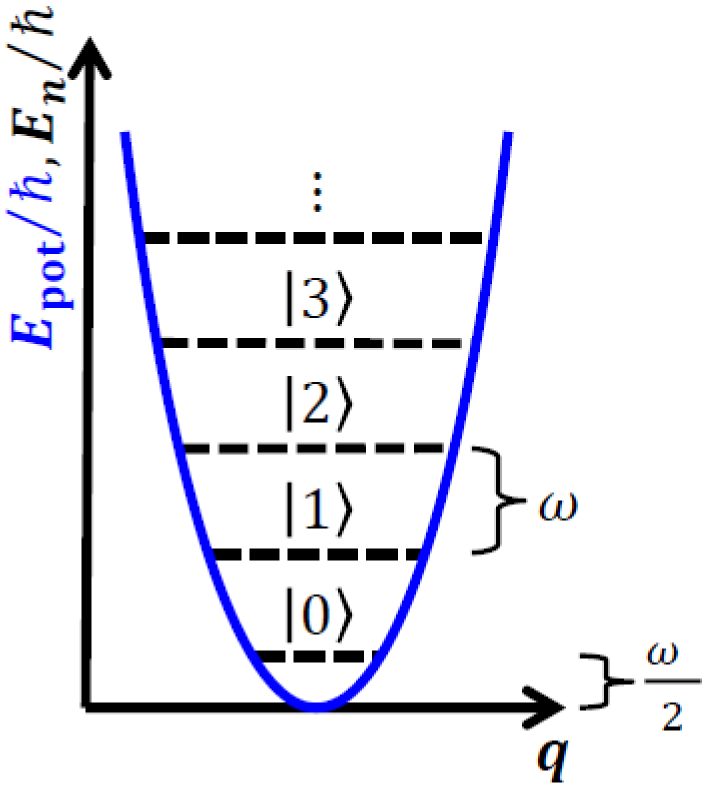
General note: Quantization of oscillators

- General note: In quantum mechanics, the energy of a system is given by an eigenvalue of the Hamiltonian.
- In a harmonic oscillator, the energy is quantized equidistantly.
- Energy quantization can be seen as counting the number of photons stored in the oscillator.

Quantization of an oscillator



In quantum mechanics, variables are replaced by operators:

$$q \rightarrow \hat{q},\ p \rightarrow \hat{p} = -i\hbar \frac{\partial}{\partial q}$$

For practical reasons, we often use matrix representations

$$O_{k\ell} = \langle e_k | \hat{O} | e_\ell \rangle$$

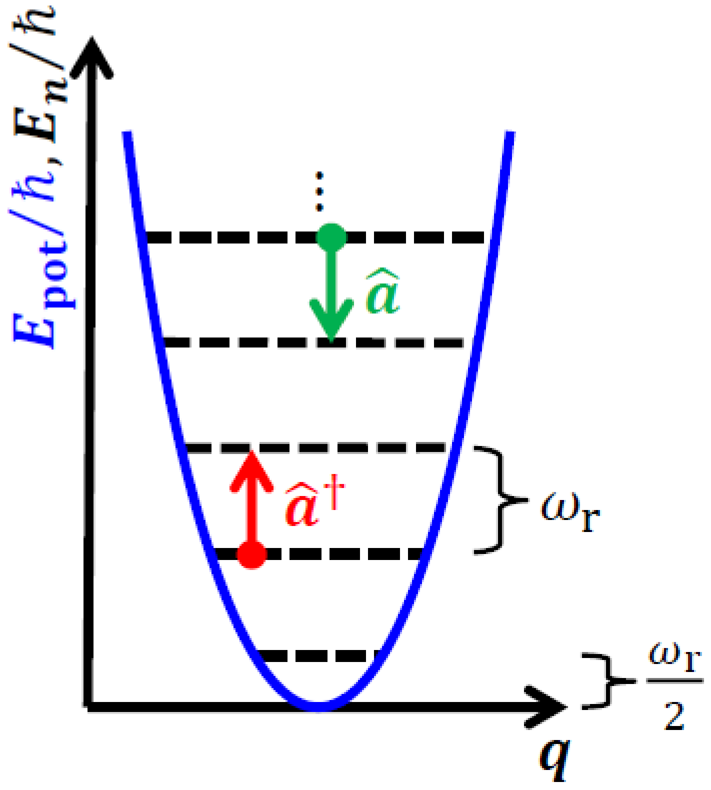
$$\begin{pmatrix} (O\psi)_1 \\ (O\psi)_2 \\ \ldots \\ (O\psi)_i \\ \ldots \end{pmatrix} = \begin{pmatrix} O_{11} & O_{12} & \ldots & O_{1j} & \ldots \\ O_{21} & O_{22} & \ldots & O_{2j} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ O_{i1} & O_{i2} & \ldots & O_{ij} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \ldots \\ \psi_j \\ \ldots \end{pmatrix}$$

Two conjugate variables follow the commutation relation

$$[\widehat{p}, \widehat{q}] \equiv \widehat{p}\widehat{q} - \widehat{q}\widehat{p} = -i\hbar$$

 * Details in lecturenotes-EQS_Helsinki_2019.pdf

Quantization of an oscillator



It is convenient to transform $\hat{q}$ and $\hat{p}$ to $\hat{a}$ and $\hat{a}^\dagger$ to find the eigenvalues algebraically.

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p}, \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} - i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p}$$

$[\hat{p}, \hat{q}] = -i\hbar \rightarrow [\hat{a}, \hat{a}^\dagger] = 1.$ (Exercise)

$$\rightarrow H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$$

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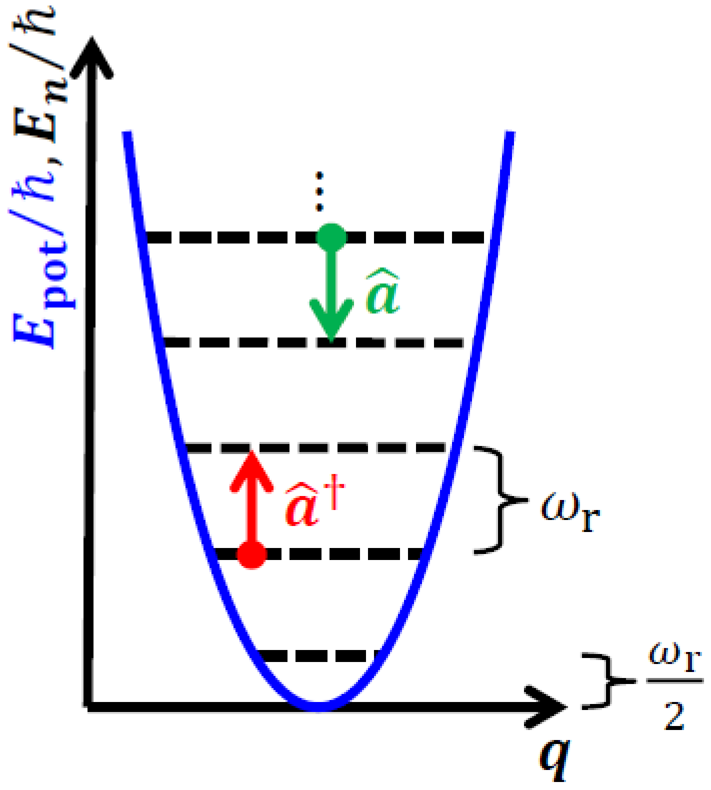
Quantization of an oscillator



Let us work out the eigenvalue problem of a harmonic oscillator.

Let $\hat{n} = \hat{a}^\dagger \hat{a}$ and $|n\rangle$ be an eigenvector of $\hat{n}$, $\hat{n}|n\rangle = n|n\rangle$.

From $[\hat{n}, \hat{a}] = -\hat{a}$ and $\left[\hat{n}, \hat{a}^\dagger\right] = \hat{a}^\dagger$ (exercise) we find $\hat{n}(\hat{a}|n\rangle) = (n-1)(\hat{a}|n\rangle)$, $\hat{n}(\hat{a}^\dagger|n\rangle) = (n+1)(\hat{a}^\dagger|n\rangle)$, showing $\hat{a}|n\rangle \propto |n-1\rangle$ and $\hat{a}^\dagger|n\rangle \propto |n+1\rangle$.

$\hat{a}^k|n\rangle$ has eigenvalue $n-k$, which must be nonnegative;

$$\langle n-k|\hat{a}^\dagger \hat{a}|n-k\rangle = (n-k) = \big|||a|n-k\rangle\big||^2 \geq 0$$

$$\rightarrow k \leq n$$

This is possible if there is $|0\rangle$ such that $\hat{n}|0\rangle = 0$. There is no "$|-1\rangle = \hat{a}|0\rangle$". $|0\rangle$ is called the vacuum state.

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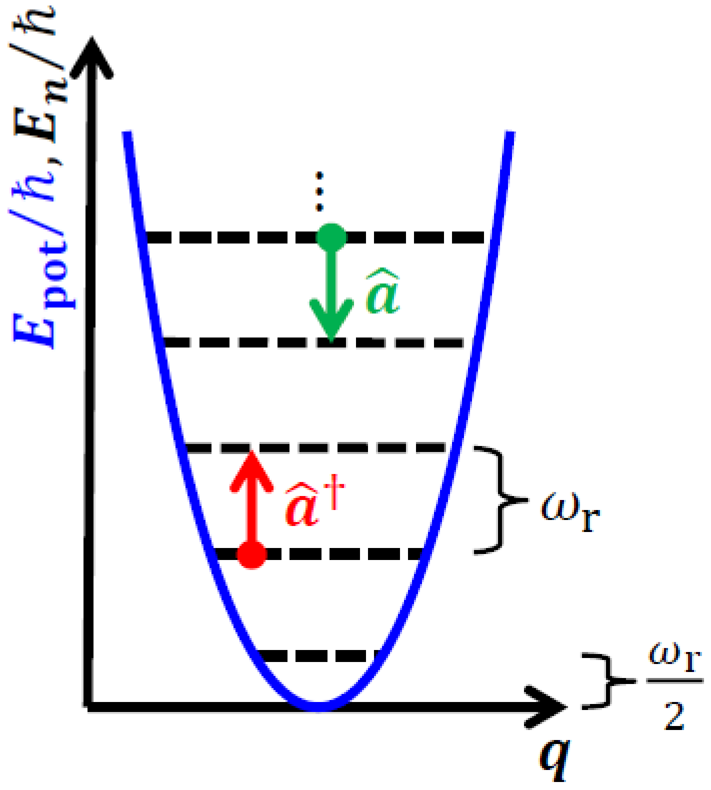
Quantization of an oscillator



It follows from $\langle n|\hat{a}\hat{a}^\dagger|n\rangle = n + 1$ that (exercise)

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|0\rangle, \qquad n = 0,1,2,\ldots$$

$|n\rangle$ is an eigenvector of $\hat{n}$ such that $\hat{n}|n\rangle = n|n\rangle$.

Matrix elements:

$$\langle m|\hat{a}|n\rangle = \sqrt{n}\langle m|n-1\rangle = \sqrt{n}\delta_{m,n-1}$$

$$\langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\langle m|n+1\rangle = \sqrt{n+1}\delta_{m,n+1}.$$

$$\langle m|\hat{n}|n\rangle = n\,\delta_{mn}$$

Explicitly

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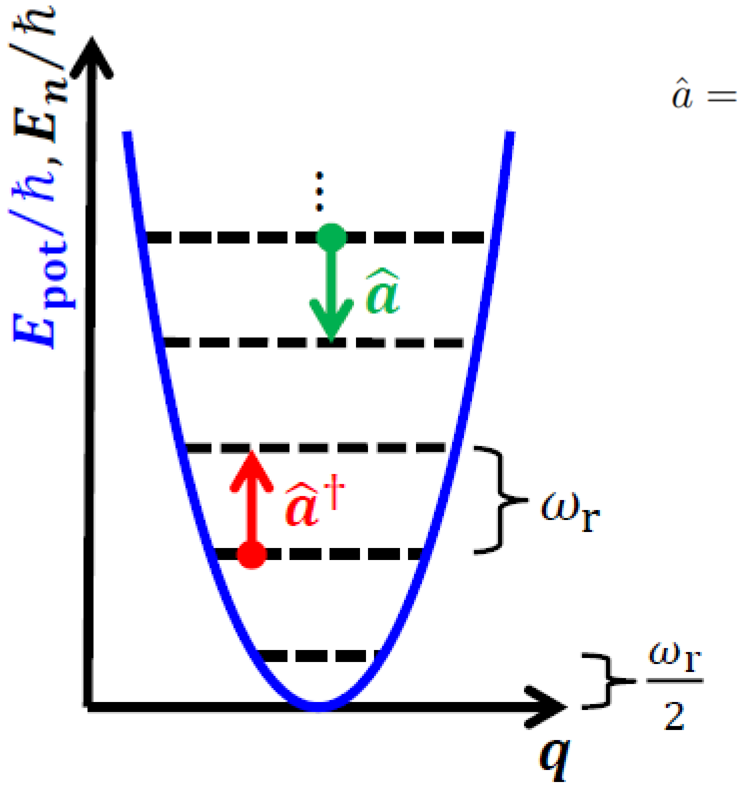
Quantization of an oscillator



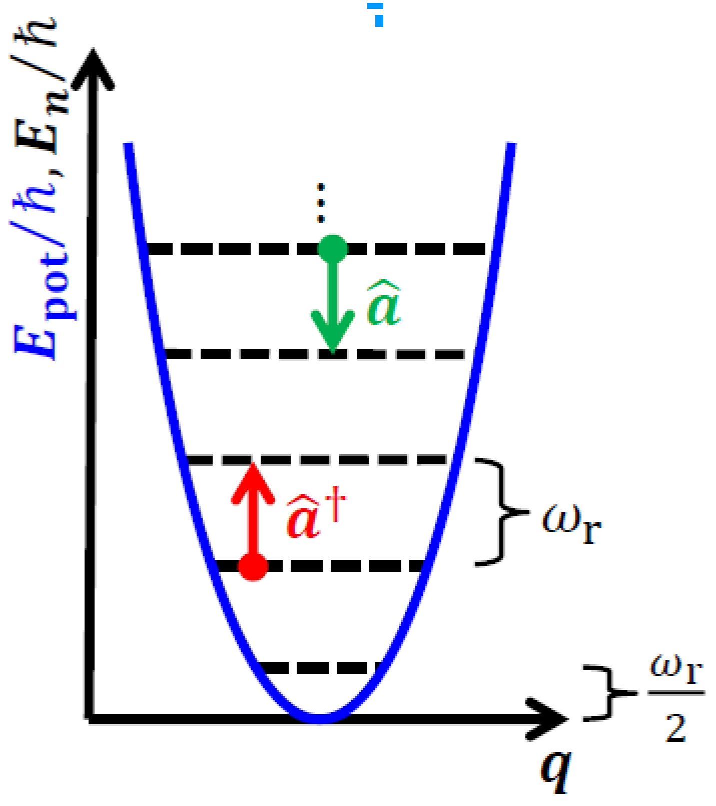
$$\hat{a} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \sqrt{n} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\hat{a}^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \sqrt{n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\hat{n} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 3 & \ddots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & n & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Exercise: Show that $\hat{a}^\dagger \hat{a} = \hat{n}$. Show also that $[\hat{a}, \hat{a}^\dagger] = 1$ by using the above matrix representations.
Is $\mathrm{Tr}\, \hat{a}\hat{a}^\dagger = \mathrm{Tr}\, \hat{a}^\dagger \hat{a}$?

Quantization of an oscillator



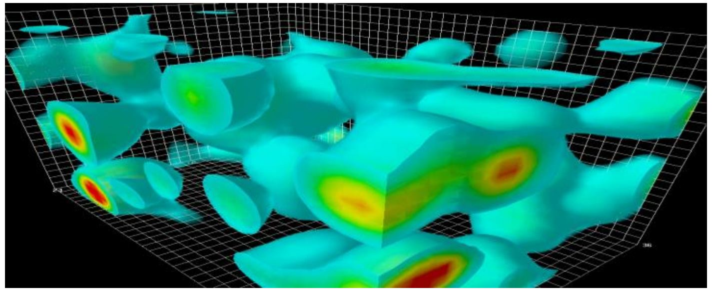
In summary, the eigenvalues of a harmonic oscillator are quantized as

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right),\ n = 0,1,2, \dots$$

The difference between the adjacent eigenvalues is $\hbar\omega$ independent of n.

This may be interpreted as “there are n photons (or quanta) in $|n\rangle$ state”. $\hat{a}$ annihilates one photon while $\hat{a}^\dagger$ creates one photon. $\hat{a}$ ($\hat{a}^\dagger$) is called the annihilation (creation) operator.

Vacuum $|0\rangle$ is not empty. There is zero-point fluctuation. $\mathrm{E}_0 = \frac{\hbar\omega}{2}$ is called the vacuum energy or the zero-point energy.



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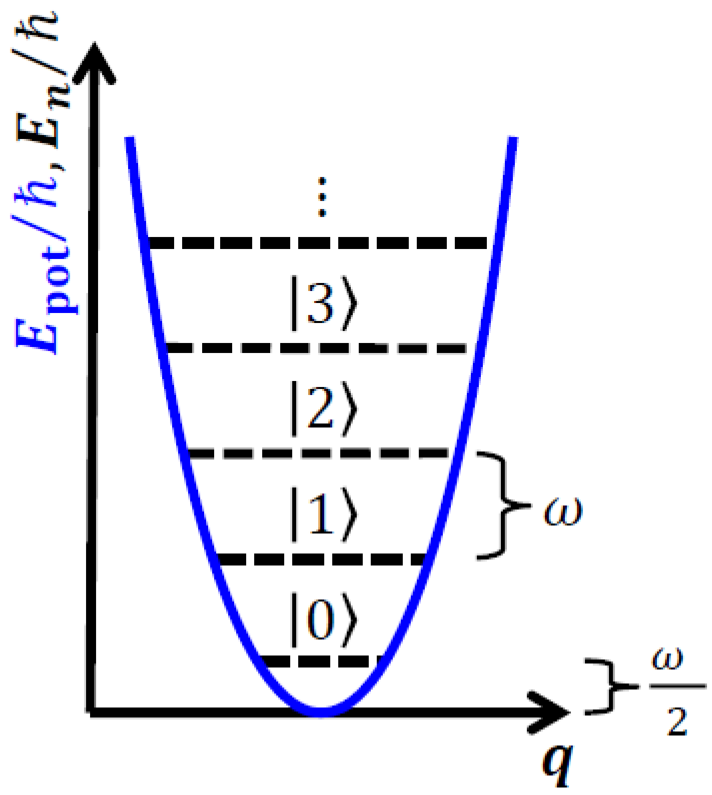
Quantization of the LC oscillator



For the superconducting LC resonator, we have

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = \frac{\hat{Q}^2}{2C} + \frac{1}{2}C\omega^2\hat{\Phi}^2$$

We aim to diagonalize $\hat{H}$ by rewriting it in terms of

$$\hat{\Phi} = \sqrt{\frac{\hbar}{2C\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \quad \hat{Q} = -i\sqrt{\frac{\hbar C\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right)$$

Here $\omega = 1/\sqrt{LC}$. The square root factors have been inserted so that $\left[\hat{a}, \hat{a}^\dagger\right] = 1$. Then $\hat{H}$ is written as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

as before.

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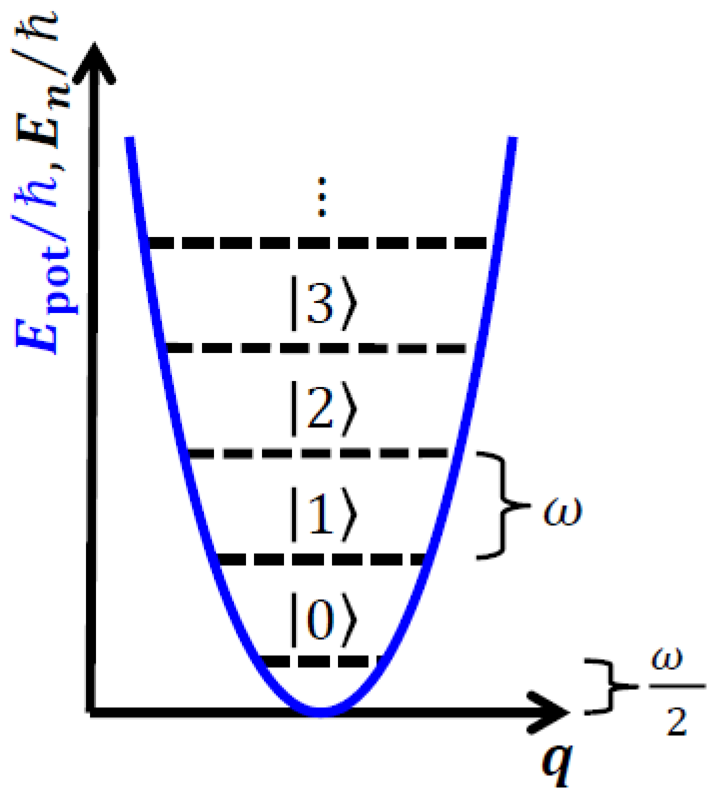
Quantization of the LC oscillator



The Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

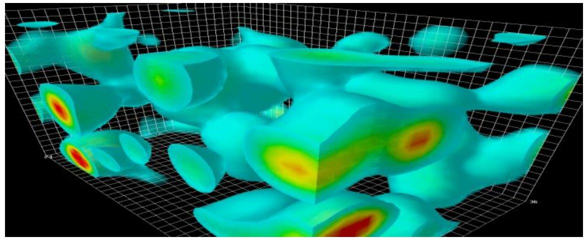
with $\omega = 1/\sqrt{LC}$. The eigenvalues and eigenvectors

$$n = 0,1,2, \ldots, |0\rangle, |1\rangle, |2\rangle, \ldots$$

The energy eigenvalues are

$$E_0 = \frac{1}{2}\hbar\omega, E_1 = \frac{3}{2}\hbar\omega, E_2 = \frac{5}{2}\hbar\omega, \ldots$$

n denotes the number of photons in the circuit. Zero-point energy E_0 even when $n = 0$. Φ and Q cannot vanish simultaneously, $[\hat{Q}, \hat{\Phi}] = -i\hbar$.



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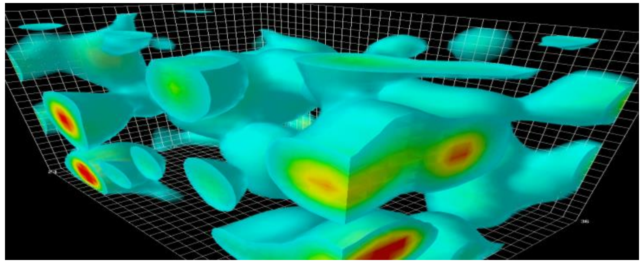
Zero-Point Energy from Uncertainty Relation

$\Delta Q \Delta \Phi \geq \frac{1}{2} |\langle [Q, \Phi] \rangle| = \frac{\hbar}{2}$ where $\Delta Q^2 \equiv \langle Q^2 \rangle - \langle Q \rangle^2$ etc.

$$\text{Let } \Delta Q \, \Delta \Phi \simeq \frac{\hbar}{2}$$

$$\Delta Q^2 \Delta \Phi^2 \simeq \frac{\hbar^2}{4} \rightarrow \Delta \Phi^2 \simeq \frac{\hbar^2}{4 \Delta Q^2} \rightarrow H \simeq \frac{\Delta Q^2}{2C} + \frac{\Delta \Phi^2}{2L} \simeq \frac{\Delta Q^2}{2C} + \frac{\hbar^2}{8L \Delta Q^2}$$

$$\frac{\partial H}{\partial \Delta Q^2} \simeq \frac{1}{2C} - \frac{\hbar^2}{8L(\Delta Q^2)^2} = 0 \rightarrow \Delta Q^2 = \frac{\hbar}{2}\sqrt{\frac{C}{L}} \rightarrow H \simeq \frac{\hbar}{4}\sqrt{\frac{1}{LC}} + \frac{\hbar}{4}\sqrt{\frac{1}{LC}} = \frac{\hbar \omega}{2}.$$



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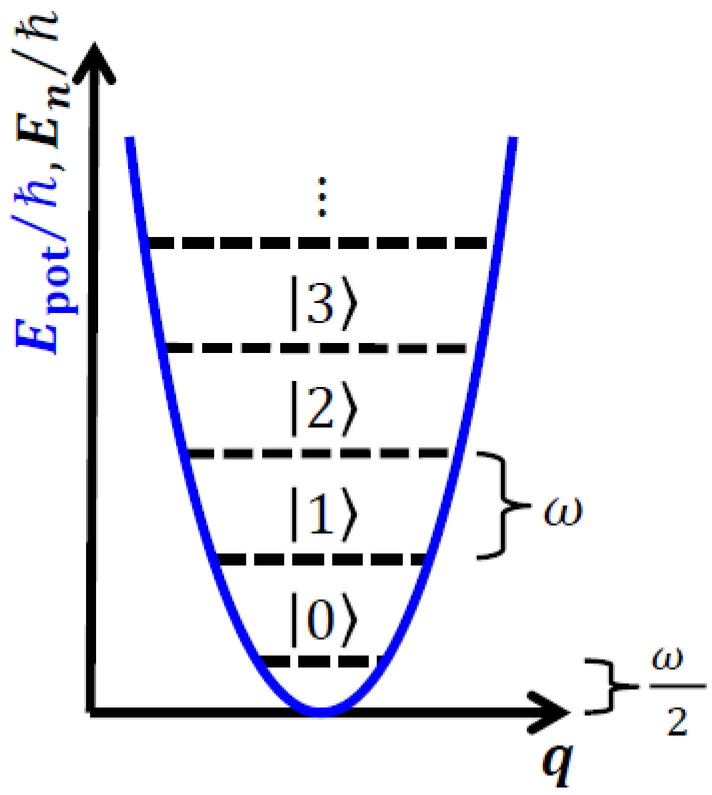
Problem of an LC oscillator as a qubit



Does an LC oscillator work as a qubit? $\{|0\rangle, |1\rangle\}$?

Transition between $|0\rangle \leftrightarrow |1\rangle$ is realized by absorption & emission of a photon of energy $\hbar\omega$. But the energy eigenvalues are equidistant, and the same photon induces transition between arbitrary $|n\rangle$ and $|n \pm 1\rangle$. It is impossible to confine the qubit state within Span$\{|0\rangle, |1\rangle\}$.

This problem is circumvented by introducing nonlinearity in V (Φ). We see in the following lectures that this will be realized by replacing the inductor in the circuit with a Josephson junction.

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Agenda for today (done)

7. Quantization of electrical networks
   a. Harmonic oscillator: Lagrangian, eigenfrequency
   b. Transfer step: LC oscillator, Legendre transform to Hamiltonian
   c. Quantization of oscillators

a.

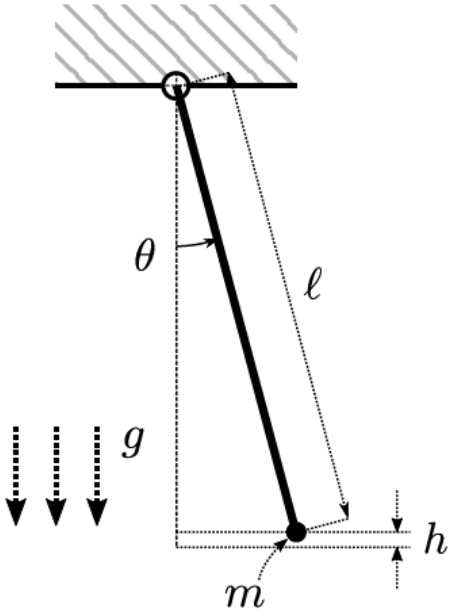


Figure 1: Classical pendulum.

b.

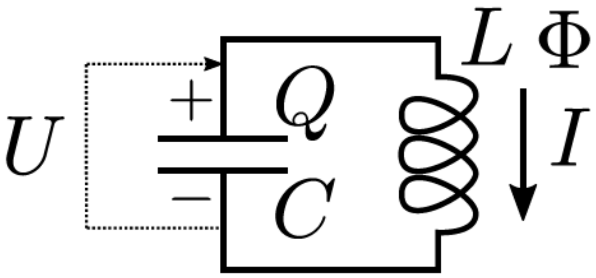
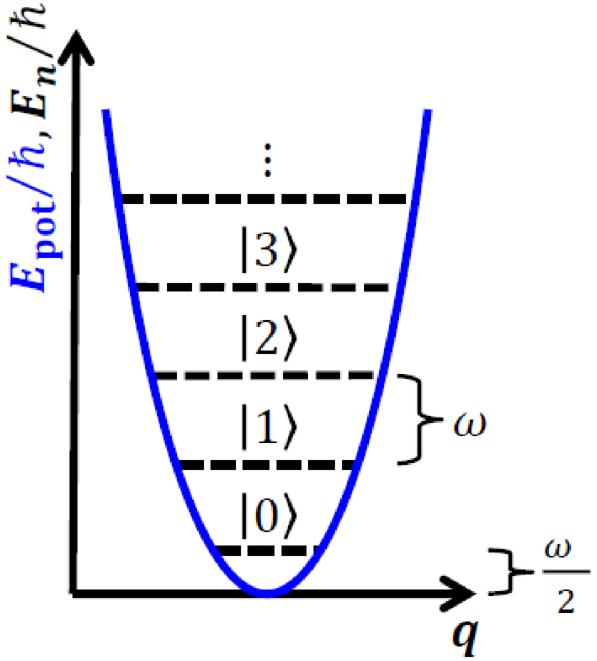


Figure 2: Superconducting LC oscillator.

c.



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Add-on: Vacuum fluctuations & thermal photons (if time allows)

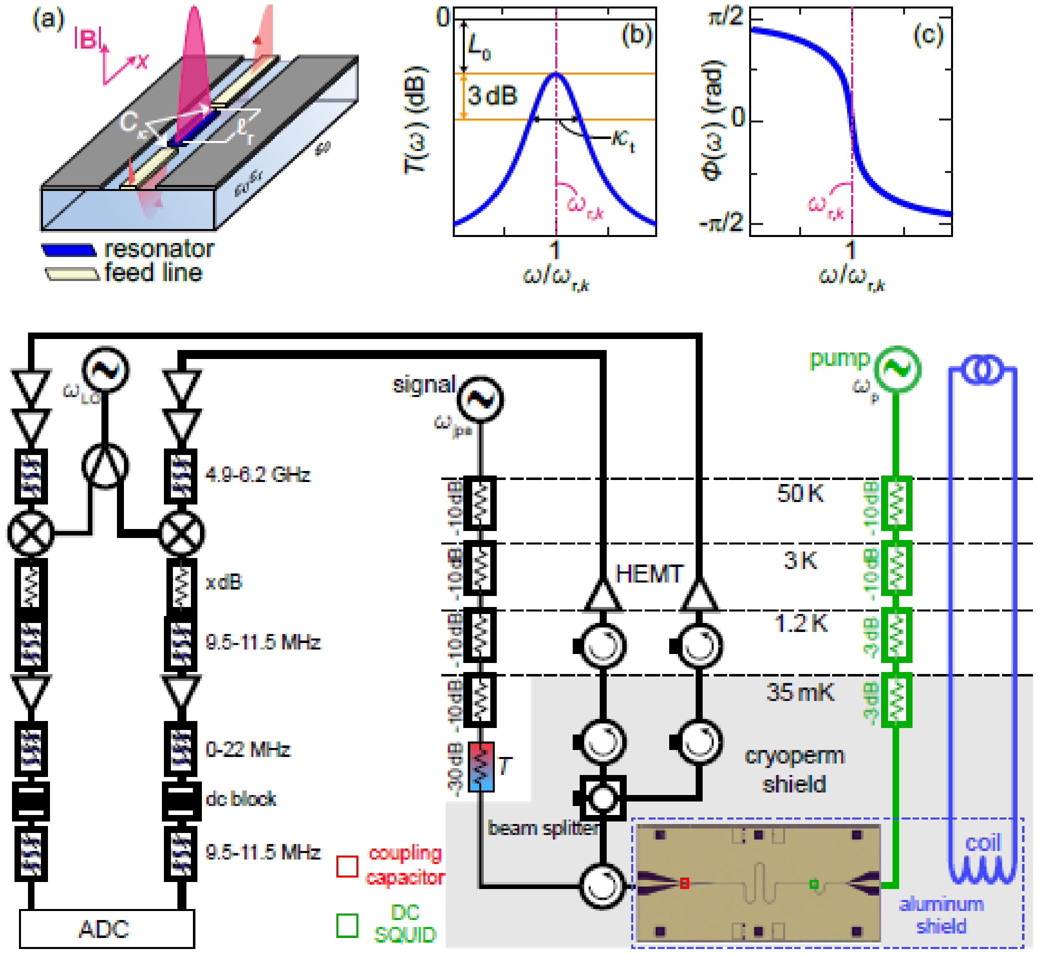
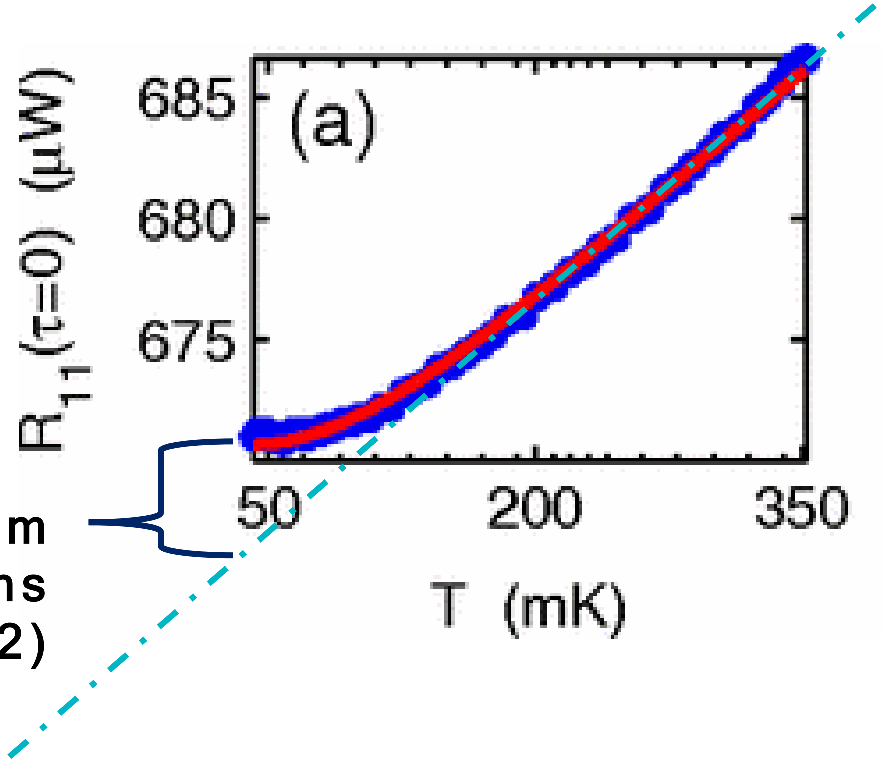


Figure 4.38: Schematics of the dual-path setup.



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Legendre transformation to Hamiltonian*

Hamiltonian gives two 1st order differential equations, while Euler Lagrange gives one 2nd order

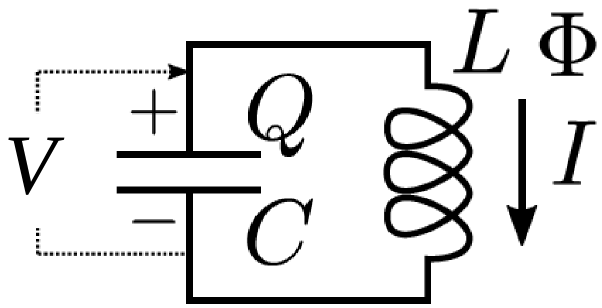

Figure 2: Superconducting LC oscillator.

Using the above terms in the total time derivative yields

$$\frac{dH}{dt} = \ddot{q}p + \dot{q}\dot{p} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial t}$$

Simplifying this formula further results in

since $\frac{\partial L}{\partial t} = 0$. The RHS vanishes due to Euler-Lagrange equation, and hence

since $\frac{\partial L}{\partial t} = 0$. The RHS vanishes due to Euler-Lagrange equation, and hence

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0$$

The Hamiltonian is a constant of motion, i.e. energy is conserved.

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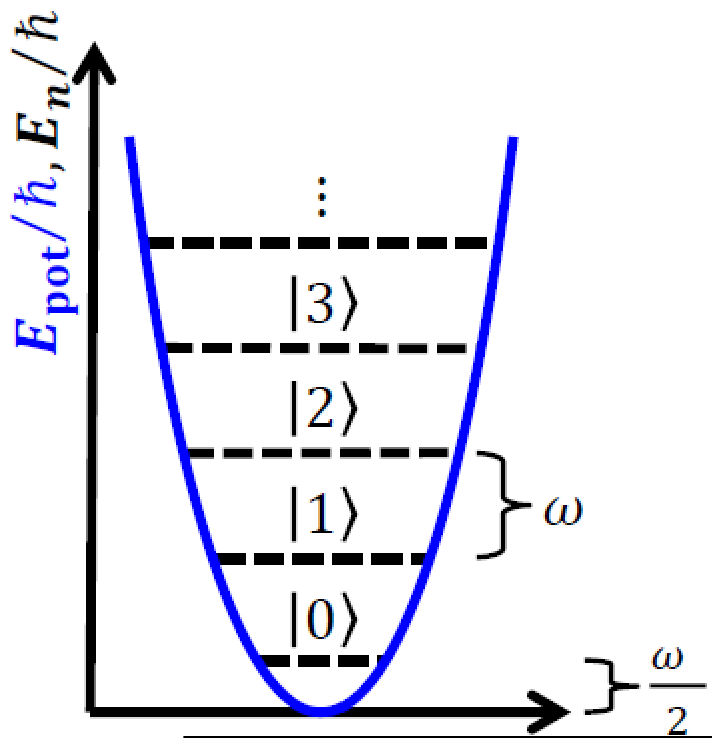
Quantization of the LC oscillator



The previous Hamiltonian becomes

$$\hat{H} = \frac{\left[\sqrt{\frac{\hbar\omega L}{2}}\left(\hat{a}+\hat{a}^\dagger\right)\right]^2}{2L} + \frac{\left[\sqrt{\frac{\hbar\omega C}{2}} i\left(\hat{a}-\hat{a}^\dagger\right)\right]^2}{2C}$$

$$= \frac{\hbar\omega}{4}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)$$

$$= \frac{\hbar\omega}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) .$$

Using $[\hat{a}, \hat{a}^\dagger] = 1$ it follows that $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

Key takeaway: The total energy of the system is given by vacuum fluctuations (+1/2) and the number of photons stored at frequency ω

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$$

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