Exercise Session 4

Problem 1:

Consider the following electric field in free space, which as function of coordinate z and time t reads as follows

$$\mathbf{E}(z,t) = \mathbf{a}_x E_0 \cos(\omega t - bz)$$

Here E_0 (unit V/m) and b (unit 1/m) are constants, and ω is the angular frequency of the wave.

The parameters of free space are ε_0, μ_0. There are no sources in the domain considered, in other words the current density and the charge density vanish: $\mathbf{J} = 0, \rho_v = 0$.

i. Using the (explicitely time-dependent) Fadaray's law (6-7), determine the magnetic field vector function $\mathbf{H}(z,t)$.

ii. Plug your electric and magnetic fields into the four Maxwell equations (6-45(a–d)) and check that all are valid.

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}$$

(4b) Faraday's law $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

Now $\nabla \times \bar{E} = \left(\bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \cancel{\frac{\partial}{\partial y}} + \bar{a}_z \cancel{\frac{\partial}{\partial z}}\right) \times \bar{a}_x \cos(\omega t - bz)\, E_0$

$$= -\bar{a}_y \sin(\omega t - bz) \cdot (-b)\, E_0$$

$$= -\frac{\partial \bar{B}}{\partial t} \quad \Rightarrow \quad \bar{B} = b\, \frac{\cos(\omega t - bz)}{\omega}\, \bar{a}_y\, E_0$$

$$\Rightarrow \bar{H}(z,t) = \frac{b}{\omega \mu_0}\, \bar{a}_y \cos(\omega t - bz)\, E_0$$

NOTE: a constant magnetic field $\bar{H}_0$ (with no dependence on space nor time) could be added, and it would have no effect as it vanishes in all differentiations $\nabla, \frac{\partial}{\partial t}$.

the other laws:

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} \quad ?$$

$$\nabla \times \bar{H} = \bar{a}_z \frac{\partial}{\partial z} \times \bar{a}_y \frac{b}{\omega \mu_0} \cos(\omega t - bz) E_0 = -\bar{a}_x \frac{b}{\omega \mu_0} \left(-\sin(\omega t - bz)\right)(-b)\, E_0$$

$$= -\bar{a}_x \frac{b^2}{\omega \mu_0} \sin(\omega t - bz)\, E_0$$

$$\frac{\partial \bar{D}}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t} \bar{a}_x \cos(\omega t - bz) E_0 = -\bar{a}_x\, \varepsilon_0 \sin(\omega t - bz) \cdot \omega\, E_0$$

are equal if $\varepsilon_0 \omega = \frac{b^2}{\omega \mu_0} \quad \Rightarrow \quad \underline{b^2 = \omega^2 \mu_0 \varepsilon_0}$

$$\nabla \cdot \underbrace{\bar{D}}_{\varepsilon_0 \bar{E}} = 0 \quad ? \qquad \nabla \cdot \bar{E} = \bar{a}_z \frac{\partial}{\partial z} \cdot \bar{a}_x E_0 \cos(\omega t - bz) \quad \checkmark$$

0

$$\nabla \cdot \underbrace{\bar{B}}_{\mu_0 \bar{H}} = 0 \quad ? \qquad \nabla \cdot \bar{H} = \bar{a}_z \frac{\partial}{\partial z} \cdot \bar{a}_y \frac{b}{\omega \mu_0} E_0 \cos(\omega t - bz) \quad \checkmark$$

$= 0$

Problem 2:

Given that the electric field intensity of an electromagnetic wave in a nonconducting dielectric medium with permittivity $\epsilon = 9\epsilon_0$ and permeability μ_0 is

$$\mathbf{E}(z, t) = \mathbf{a}_y 5 \cos(10^9 t - \beta z) \quad (\mathrm{V/m}),$$

find the magnetic field intensity $\mathbf{H}$ and the value of β.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$
$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E},$$
$$\nabla \cdot \mathbf{E} = \rho/\epsilon,$$
$$\nabla \cdot \mathbf{H} = 0.$$

Conversion of Instantaneous (time-dependent) electric field to Phasor form:

First, let's see how to convert the given instantaneous electric field into phasor form:

For cosine reference the instantaneous field can be written as the real part of the phasor form multiplied by $e^{j\omega t}$:

$$\mathbf{E}(x, y, z, t) = \mathcal{R}e[\mathbf{E}(x, y, z)e^{j\omega t}]$$

Where **E**(x,y,z) is the electric field in phasor form. Now given:

$$\mathbf{E}(z, t) = \mathbf{a}_y 5 \cos(10^9 t - \beta z) \quad (\mathrm{V/m}).$$

We can write this in the following form:

$$\boldsymbol{E}(z,t) = Re\left[\boldsymbol{a_y} 5 e^{j(\omega t - \beta z)}\right]$$

Note: You can use Euler's formula to expand the above term and then take the real part to get back the instantaneous field. Euler's formula:

$$e^{jx} = \cos(x) + j\sin(x)$$

Now continuing:

$$\boldsymbol{E}(z,t) = Re\left[\boldsymbol{a_y} 5 e^{-j\beta z} e^{j\omega t}\right]$$

We can also write:

$$\boldsymbol{E}(z,t) = Re\left[\boldsymbol{E}(z) e^{j\omega t}\right]$$

Comparing we get:

$$\boldsymbol{E}(z) = \boldsymbol{a_y} 5 e^{-j\beta z}$$

Using this phasor form we can then solve the given problem. Complete solution is on next page.

Given that the electric field intensity of an electromagnetic wave in a nonconducting dielectric medium with permittivity $\epsilon = 9\epsilon_0$ and permeability μ_0 is

$$\mathbf{E}(z, t) = \mathbf{a}_y 5 \cos (10^9 t - \beta z) \qquad (\text{V/m}), \tag{6-88}$$

find the magnetic field intensity $\mathbf{H}$ and the value of β.

SOLUTION

The given $\mathbf{E}(z, t)$ in Eq. (6-88) is a harmonic time function with angular frequency $\omega = 10^9$ (rad/s). Using phasors with a cosine reference, we have

$$\mathbf{E}(z) = \mathbf{a}_y 5e^{-j\beta z}. \tag{6-89}$$

The magnetic field intensity can be calculated from Eq. (6-80a).

$$\begin{aligned}
\mathbf{H}(z) &= -\frac{1}{j\omega\mu_0} \nabla \times \mathbf{E} \\
&= -\frac{1}{j\omega\mu_0} \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 5e^{-j\beta z} & 0 \end{vmatrix} = -\frac{1}{j\omega\mu_0}\left(-\mathbf{a}_x \frac{\partial}{\partial z} E_y\right) \\
&= -\frac{1}{j\omega\mu_0}(\mathbf{a}_x j\beta 5e^{-j\beta z}) = -\mathbf{a}_x \frac{\beta}{\omega\mu_0} 5e^{-j\beta z}.
\end{aligned} \tag{6-90}$$

In order to determine β, we use Eq. (6-80b). For a nonconducting medium, $\sigma = 0$, $\mathbf{J} = 0$. Thus,

$$\begin{aligned}
\mathbf{E}(z) &= \frac{1}{j\omega\epsilon} \nabla \times \mathbf{H} = \frac{1}{j\omega\epsilon}\left(\mathbf{a}_y \frac{\partial}{\partial z} H_x\right) \\
&= \mathbf{a}_y \frac{\beta^2}{\omega^2\mu_0\epsilon} 5e^{-j\beta z}.
\end{aligned} \tag{6-91}$$

Equating Eqs. (6-89) and (6-91), we require

$$\beta = \omega\sqrt{\mu_0\epsilon} = 3\omega\sqrt{\mu_0\epsilon_0} = \frac{3\omega}{c}$$

$$= \frac{3 \times 10^9}{3 \times 10^8} = 10 \qquad \text{(rad/m)}.$$

From Eq. (6-90) we obtain

$$\begin{aligned} \mathbf{H}(z) &= -\mathbf{a}_x \frac{5(10)}{(10^9)(4\pi 10^{-7})} e^{-j10z} \\ &= -\mathbf{a}_x 0.0398 e^{-j10z}. \end{aligned} \tag{6-92}$$

The phasor $\mathbf{H}(z)$ in Eq. (6-92) corresponds to the following instantaneous time function:

$$\mathbf{H}(z, t) = -\mathbf{a}_x 0.0398 \cos(10^9 t - 10z) \qquad \text{(A/m)}. \tag{6-93}$$