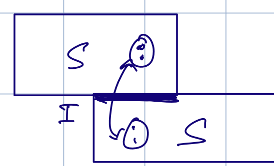
Problem I: Charge - phase relation

At $T=0$ and $V=0$, charge can only flow through units of cooper pairs. We want to describe the behaviour of junction in terms of number of cooper pairs.



Define $\hat{N}$ which represents number of cooper pair on side of the junction.

$$\hat{N}|n\rangle = n|n\rangle$$

$\to$ $\hat{N}$ tells you number of cooper pair on one side of junction.

Phase is another quantity that defines a superconductor. We want to relate these

$$|\varphi\rangle = \sum_{n=-\infty}^{\infty} e^{in\varphi}|n\rangle$$

Switching between the basis,

$$|n\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{-in\varphi}|\varphi\rangle$$

Fourier like relation just like $\hat{x}-\hat{p}$.

(a) $|n\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{-in\varphi} \sum_{m=-\infty}^{\infty} e^{im\varphi} |m\rangle$

When evaluating this, use Kronecker delta relation

$$\frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-i(x-x')k} = \delta_{xx'}$$

(b) $\langle\varphi|\varphi'\rangle = \sum_{n=-\infty}^{\infty} e^{-in\varphi}\langle n| \sum_{m=-\infty}^{\infty} e^{im\varphi'}|m\rangle$

For charge states, orthogonality imposes $\langle n|m\rangle = \delta_{nm}$

Recognize discrete Dirac delta function

$$\delta(\varphi-\varphi') = \frac{1}{2\pi}\sum_n e^{-in(\varphi-\varphi')}$$

Problem 2: Exponent of a phase operator

$$e^{i\hat{\varphi}} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi'\, e^{i\varphi'} |\varphi'\rangle\langle\varphi'|$$

(a) Evaluate $e^{i\hat{\varphi}}|\varphi\rangle$

$$\Rightarrow e^{i\hat{\varphi}}|\varphi\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi'\, e^{i\varphi'}|\varphi'\rangle\langle\varphi'|\varphi\rangle$$

Recognize $\langle\varphi|\varphi'\rangle$ from Problem 1 (Dirac delta)

(b) $e^{i\hat{\varphi}}|n\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi' \, e^{i\varphi'} |\varphi'\rangle\langle\varphi'|n\rangle$

$\langle\varphi'|n\rangle \to \sum_{m=-\infty}^{\infty} e^{-im\varphi'}|m\rangle$

Use orthogonality δ_{mn} and configure the expression as in eq^n (3) of problem 1

(c) $e^{i\hat{\varphi}} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi' \, e^{i\varphi'} \sum_{m=-\infty}^{\infty} e^{im\varphi'}|m\rangle \sum_{n=-\infty}^{\infty} e^{-in\varphi'}\langle n| \quad = \sum_n |n-1\rangle\langle n|$

1° Use the definition of $e^{i\hat{\varphi}}$

2° Use definition of $|\varphi'\rangle$ and $\langle\varphi'|$

3° Combine the exponential

4° Form Kronecker delta form

5° Apply Kronecker delta to get rid of one sum.

$$\boxed{e^{-i\hat{\varphi}} = \sum_n |n\rangle\langle n-1|}$$

Final result !

Problem 3 : Josephson Junction

$H_J = -\frac{E_J}{2}\sum_n |n\rangle\langle n+1| + |n+1\rangle\langle n|$

(a) $H_J|n\rangle = -\frac{E_J}{2}\sum_m |m\rangle\langle m+1|n\rangle + |m+1\rangle\langle m|n\rangle$

Use orthogonality with dirac delta.

(b) $H_J = -\frac{E_J}{2}\sum_n |n\rangle\langle n+1| + |n+1\rangle\langle n|$

Express in terms of $\exp(\pm i\hat{\phi})$. Then, use $\frac{e^{i\varphi}+e^{-i\varphi}}{2} = \cos\varphi$

Problem 4:

$$L = \frac{1}{2}m\dot{x}^2 + e\dot{x}\cdot A(x,t) - e\varphi(x)$$

generalized co-ordinate : $(\bar{x}, \dot{\bar{x}}) = (x, y, z, \dot{x}, \dot{y}, \dot{z})$

(a) Euler - Lagrange equation [equation of motion]

Use $\frac{d}{dx}(\bar{a}\cdot\bar{b}) = \frac{d\bar{a}}{dx}\cdot\bar{b} + \bar{a}\cdot\frac{db}{dx}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$\bar{A}(\bar{x},t) = (A_x(\bar{x},t), A_y(\bar{x},t), A_z(\bar{x},t))$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \left[L = \frac{1}{2}m(v_x^2+v_y^2+v_z^2) + e(v_x A_x + v_y A_y + v_z A_z) - e\varphi(x)\right]$$

1° Consider x - coordinate first.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} = 0$$

$\dot{\bar{x}} = (v_x, v_y, v_z)$

$\bar{x}$ is a 3d vector (x, y, z)

$\dot{\bar{x}}$ is a derivative of the position x, y, z

First evaluate.

$\frac{\partial L}{\partial v_x} =$ # only v_x term survives and $\varphi(\bar{x})$ does not depend on v_x

You should get something like this.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) = \frac{d}{dt}(m\dot{x}) + e\cdot\frac{d}{dt}A(\bar{r},t)$$

Use $\frac{d}{dt}\bar{A}(\bar{r},t) = \frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z} + \frac{\partial A}{\partial t}$

Then evaluate.

Product rule & $\frac{\partial v_i}{\partial i} = 0. \quad i \in \{x, y, z\}$

$$\frac{\partial L}{\partial x} = -e\frac{\partial \varphi}{\partial x} + e \cdot \frac{\partial}{\partial x}\left(v_x A_x(\bar{r}, t) + v_y A_y(\bar{r}, t) + v_z A_z(\bar{r}, t)\right)$$

Use $\bar{E} = -\left(\frac{\partial \bar{A}}{\partial t} + \nabla\varphi\right)$ and $\bar{B} = \bar{\nabla} \times \bar{A}$ to get $\bar{v} \times \bar{\nabla} \times \bar{A} = \bar{v} \times \bar{B}$.

to recognise the resulting term for x. Similar method can be used to find y, and z component.

(b) $\bar{p} = \frac{\partial L}{\partial \dot{x}}$ # Evaluate the derivative

(c) $H = \dot{x} p - L$ # Use result 4(b) in here

express $\dot{x}$ in terms of $\bar{p}$ from 4(b).

Problem 5

(a) Use the definition: $|n\rangle = \frac{(\hat{a}^{+})^{n}}{\sqrt{n!}}|0\rangle$ and use the fact

$$|n+1\rangle = \frac{(\hat{a}^{+})^{n+1}}{\sqrt{(n+1)!}}|0\rangle$$

(b) Use the complete basis: $\hat{I} = \sum_n |n\rangle\langle n|$ and the fact that

$\hat{a}^{+} = \hat{a}^{+}\hat{I}$ and $\hat{a} = \hat{a}\hat{I}$. Use the definition of $|n\rangle$.

(c) Write down the matrix (see lecture notes) and compute the terms.

Problem 6

(a) Use $|z\rangle = \hat{I}_n|z\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$ with completeness $\hat{I}_n = \sum_{n=0} |n\rangle\langle n|$.

- Use the definition of $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$. Remember to take the conjugate for $\langle n|$.

- $\hat{a}|z\rangle = |z\rangle$ (given)

(b) Use $\langle z|z\rangle = 1$

- Plug $\langle z|$, and $|z\rangle$ from 6 (a). Remember to conjugate $\langle z|$.

- Recognize taylor expansion

$$e^x = \sum_n \frac{x^n}{n!}$$

(c) Find $P(n) = |\langle n|z\rangle|^2$

- Use $|z\rangle$ from 6 (a) with proper normalization constant found in 6 (b).

(d) 1° $\langle z|\hat{n}|z\rangle = \langle z|\hat{a}^+\hat{a}|z\rangle$ Use $\hat{n} = \hat{a}^+\hat{a}$.

- $\hat{a}|z\rangle = z|z\rangle$ given

- Remember $(\hat{a}|z\rangle)^+ = \langle z|\hat{a}^+$.

2° $\langle z|\hat{n}^2|z\rangle = \langle z|\hat{n}\hat{n}|z\rangle = \langle z|\hat{a}^+\hat{a}\,\hat{a}^+\hat{a}|z\rangle$

- Use $\hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}$. Evaluate $(\Delta n)^2$.