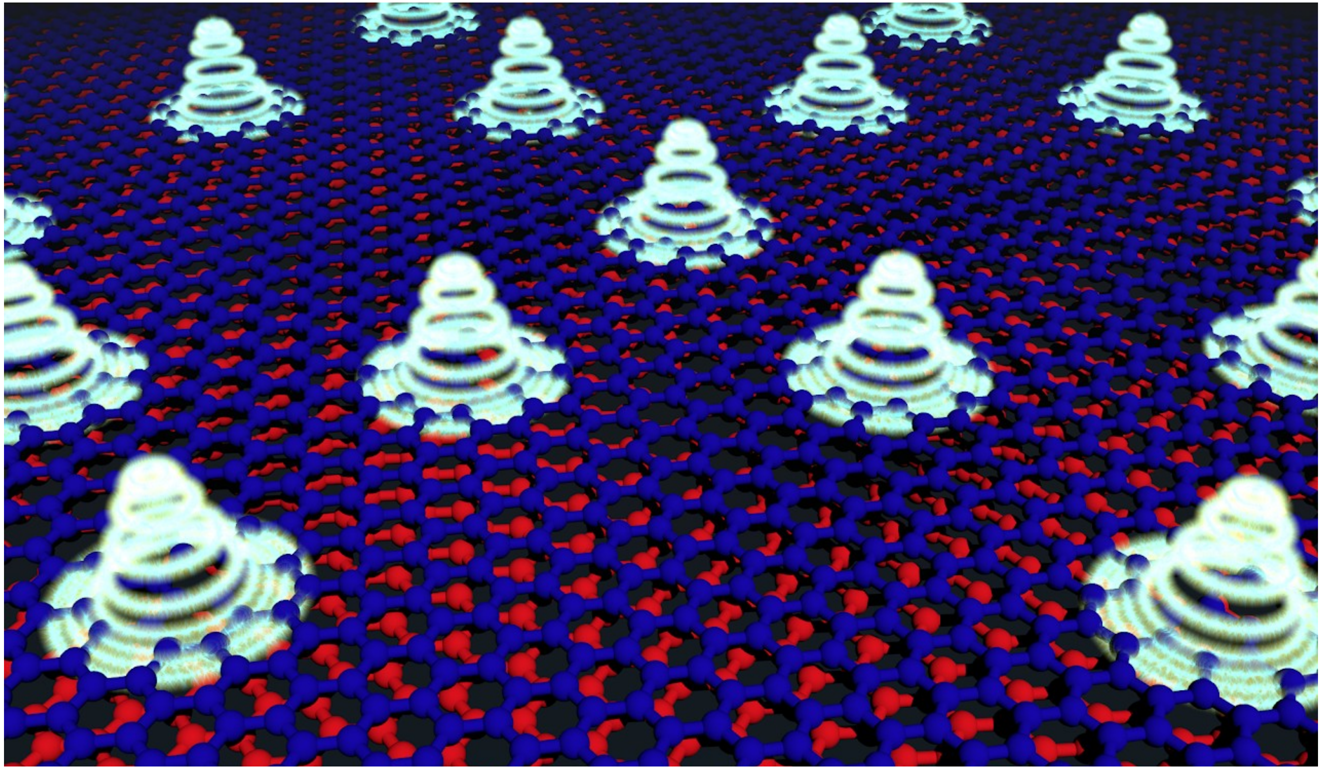
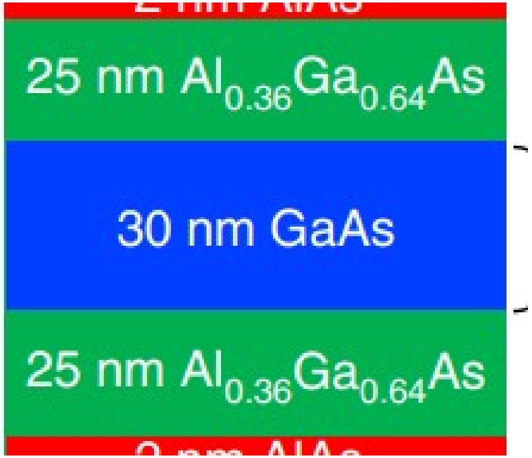
The quantum Hall effect



March 27th 2023

Materials showing quantum Hall effect

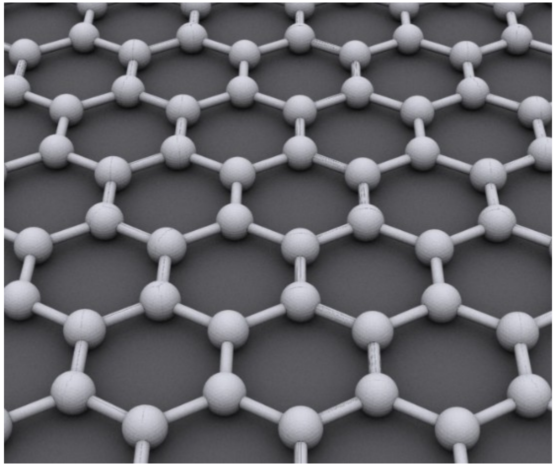
GaAs quantum wells



$$E \sim B$$

$$T \sim 1K$$

Graphene

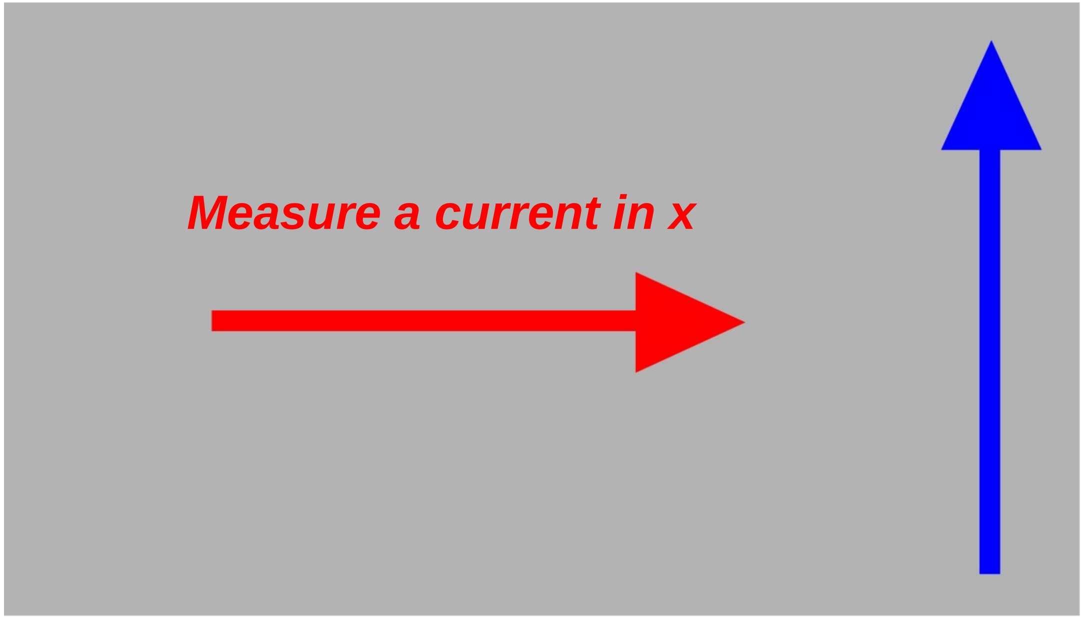


$$E \sim \sqrt{B}$$

$$T \sim 100K$$

The quantum Hall state

Take a two-dimensional material

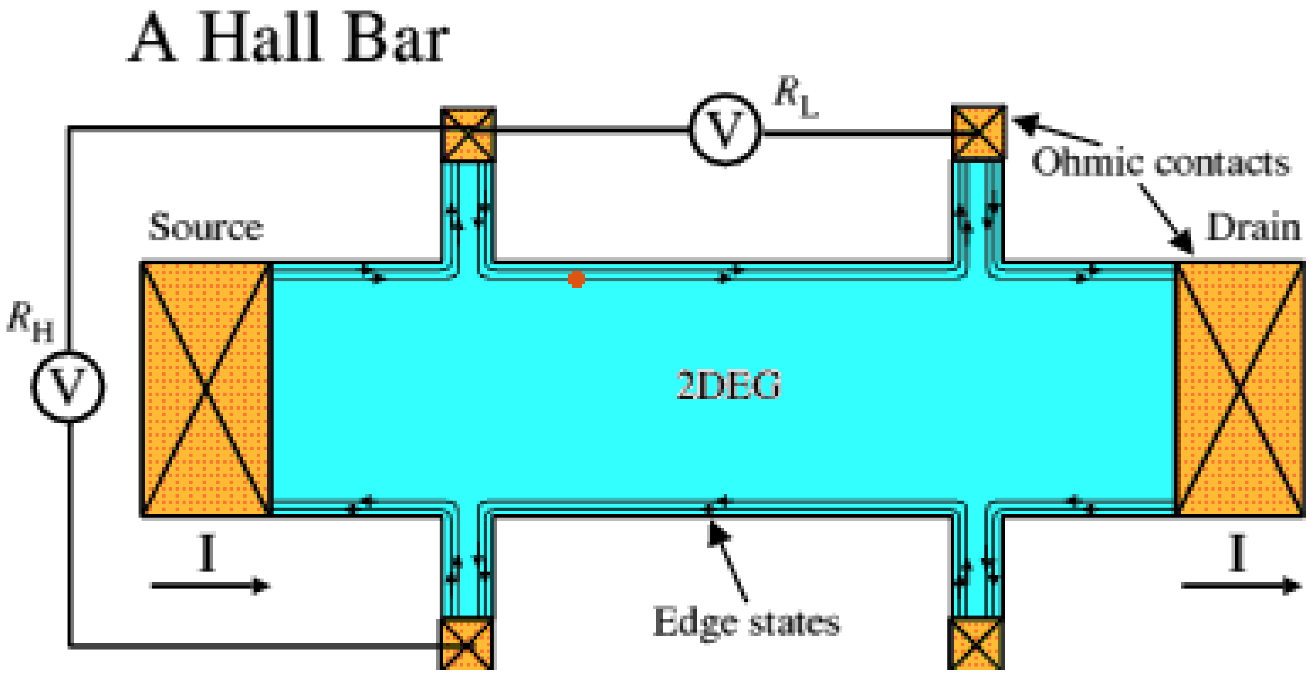


Apply a voltage in y

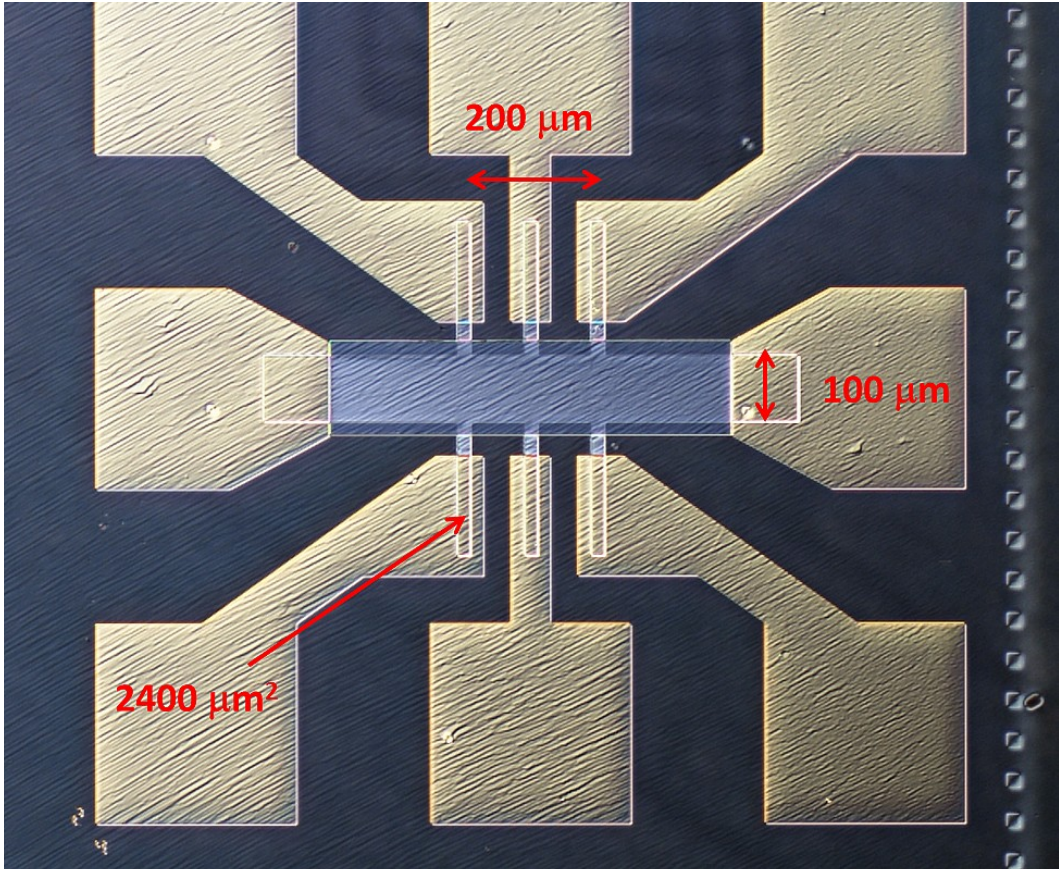
Hall conductance

$$J_x = \sigma_{xy} V_y$$

Quantum Hall devices



Quantum Hall devices



Reminder: Linear response for transverse current

$$J_x = \sigma_{xy} V_y$$

The Hall conductivity is obtained as $\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k}$

with $\Omega_\alpha(\mathbf{k}) = i \sum_{\beta \neq \alpha} \frac{\langle \Psi_\alpha | \partial H / \partial k_x | \Psi_\beta \rangle \langle \Psi_\beta | \partial H / \partial k_y | \Psi_\alpha \rangle}{(\epsilon_\alpha - \epsilon_\beta)^2} - \alpha \leftrightarrow \beta$

Berry curvature of a band

Expression coming from perturbation theory

The Hall conductivity

The Hall conductivity is obtained as $$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k}$$

Using $$\langle \Psi_\alpha | \partial H / \partial k_\mu | \Psi_\beta \rangle = \langle \partial_{k_\mu} \Psi_\alpha | \Psi_\beta \rangle (\epsilon_\alpha - \epsilon_\beta)$$

the Hall conductivity can be expressed in terms of

Berry curvature

$$\Omega_\alpha = \partial_{k_x} A^\alpha_y - \partial_{k_y} A^\alpha_x$$

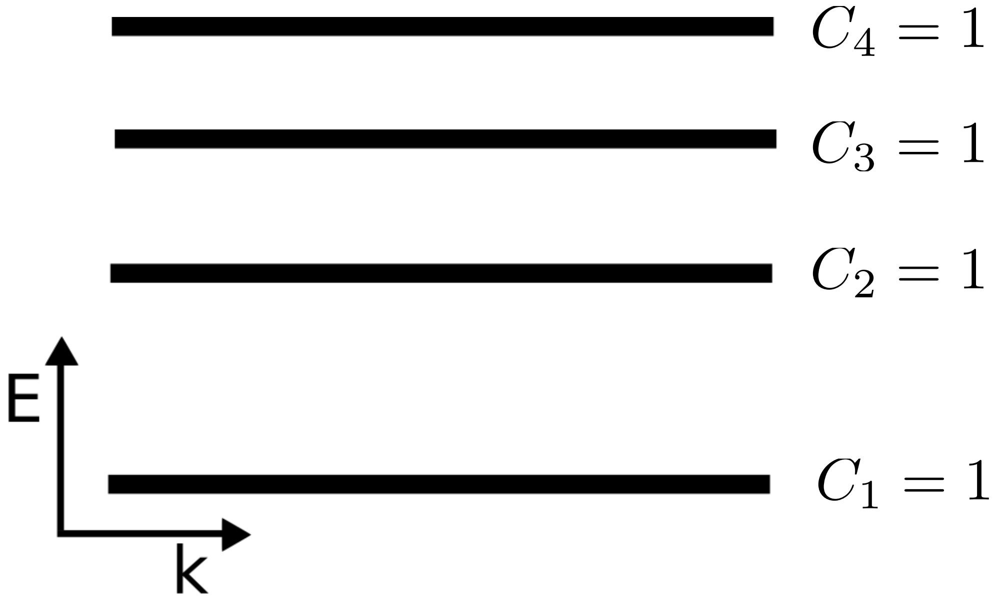
Berry connection

$$A^\alpha_\mu = i \langle \partial_{k_\mu} \Psi_\alpha | \Psi_\alpha \rangle$$

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$ ← Chern number

Chern numbers in the quantum Hall state

Band-structure in the quantum Hall state



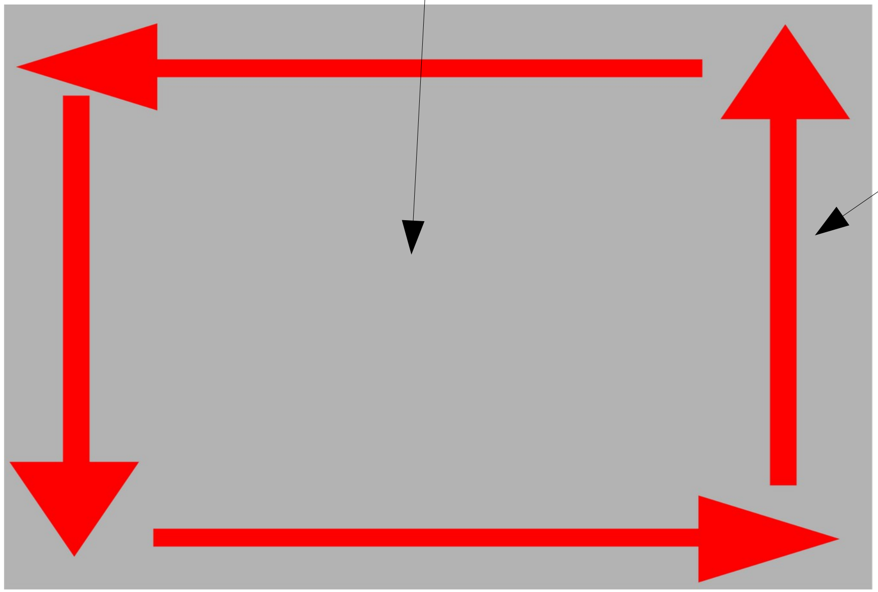
Hall conductivity

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

Each band (a.k.a Landau level), contributes with Chern number +1

The puzzling quantum Hall effect

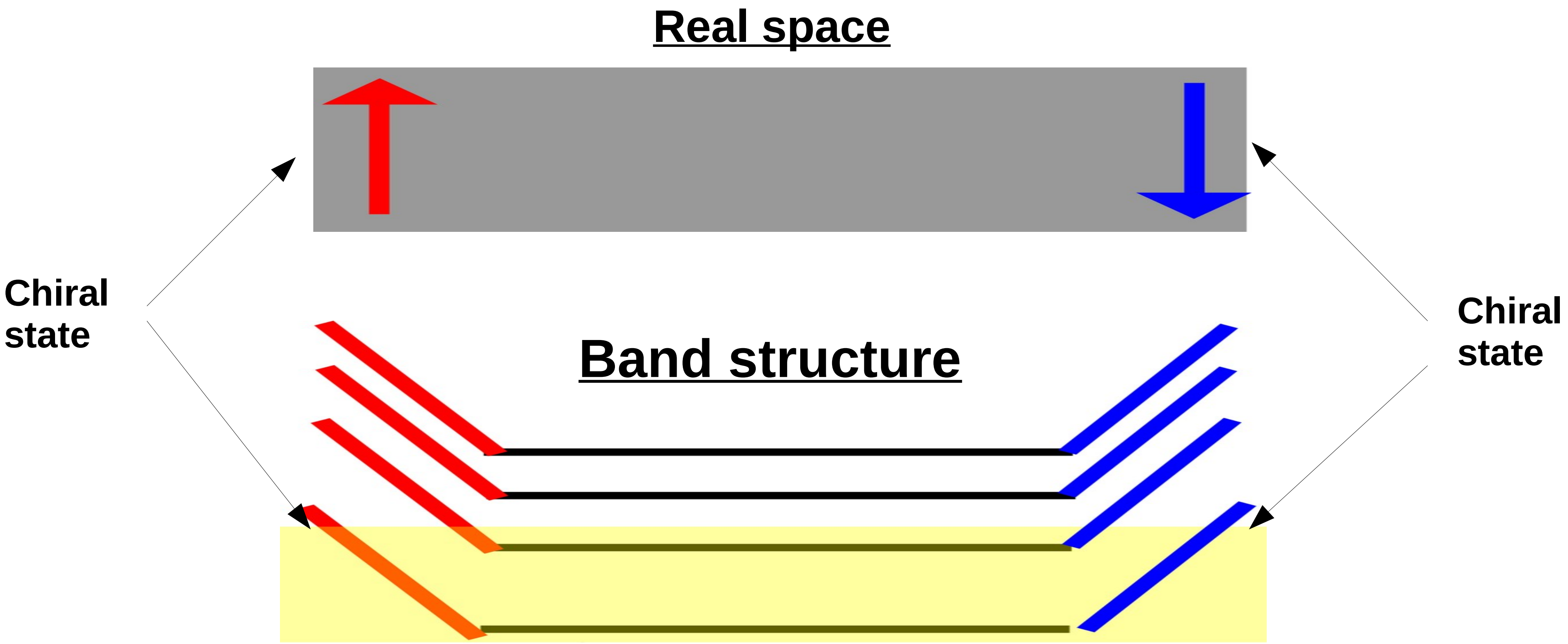
The bulk of a quantum Hall state is insulating



The edge has chiral states

Chiral: propagating
only in one direction

The quantum Hall effect



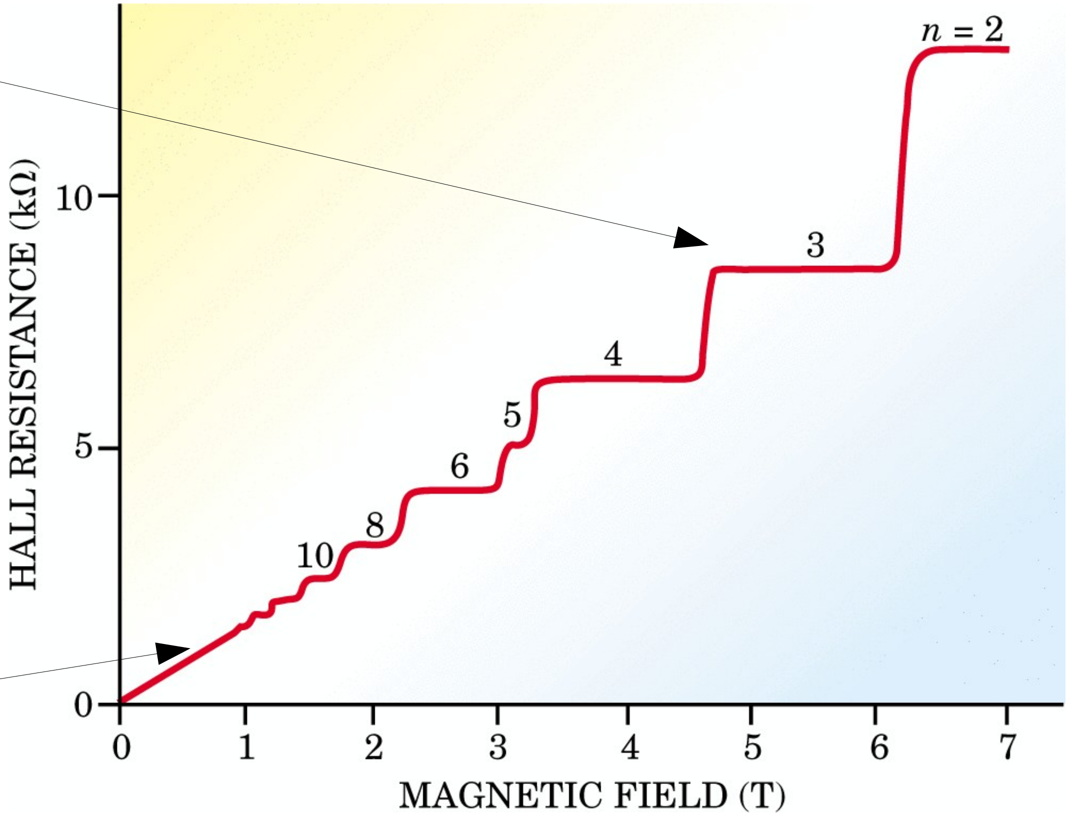
The quantum Hall state

Quantum regime

In the quantum Hall regime

$$\rho_{xy} \sim \sigma_{xy}^{-1}$$

Classical regime



Electrons in a magnetic field

Coupling (classical) electrons to a gauge field

The continuum limit

$$H = \frac{p^2}{2}$$

Given the kinetic energy without magnetic field

By replacing momentum by the canonical momentum
we recover the equations of motion in a magnetic field

$$H = \frac{\Pi^2}{2}$$

$$\Pi = \mathbf{p} + \mathbf{A}$$

Canonical momentum

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Magnetic field

Magnetic vector potential

Coupling (quantum) electrons to a gauge field

$$H = \frac{\Pi^2}{2}$$

$$\Pi = \mathbf{p} + \mathbf{A}$$

Canonical momentum

$$p_\alpha = -i\partial_\alpha$$

Quantized momentum

Hamiltonian for electrons in a magnetic field

$$H = \frac{\left(\sum_\alpha -i\partial_\alpha + A_\alpha\right)^2}{2}$$

$$\Psi(\mathbf{r})$$

Gauge #1

$$H = \frac{\left(\sum_\alpha -i\partial_\alpha\right)^2}{2}$$

$$e^{i\int_0^{\mathbf{r}} \mathbf{A}(\mathbf{r}')d\mathbf{r}'}\Psi(\mathbf{r})$$

Gauge #2

Coupling electrons to a magnetic field in a tight binding model

Given a tight-binding model without magnetic field

$$H = \sum t_{ij} c_i^\dagger c_j$$

How do we rewrite it in the presence of a magnetic field?

$$\Psi(\mathbf{r}) \rightarrow e^{i \int_0^{\mathbf{r}} \mathbf{A}(\mathbf{r}') d\mathbf{r}'} \Psi(\mathbf{r})$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Coupling electrons to a magnetic field in a tight binding model

$$H = \sum t_{ij} c_i^\dagger c_j$$

$$\Psi(\mathbf{r}) \to e^{i\int_0^{\mathbf{r}} \mathbf{A}(\mathbf{r}')d\mathbf{r}'} \Psi(\mathbf{r})$$

Recall that, by definition

$$|\mathbf{r}_i\rangle = c_i^\dagger |\Omega\rangle \qquad t_{ij} = \langle \mathbf{r}_i | H | \mathbf{r}_j \rangle \qquad c_i|\Omega\rangle = 0$$

Coupling a gauge field becomes

$$|\mathbf{r}_i\rangle \to e^{i\int_0^{\mathbf{r}_i} \mathbf{A}(\mathbf{r}')d\mathbf{r}'} |\mathbf{r}_i\rangle$$

Which transforms the hoppings as

$$t_{ij} \to e^{i\int_{\mathbf{r}_i}^{\mathbf{r}_j} \mathbf{A}(\mathbf{r}')d\mathbf{r}'} t_{ij}$$

Two ways of coupling electrons to a magnetic field

Continuum limit

Schrodinger electrons $$H = \frac{\Pi^2}{2}$$

Dirac electrons $$H = \sum_\alpha \Pi_\alpha \sigma_\alpha$$

$$\Pi = \mathbf{p} + \mathbf{A}$$

Canonical momentum

Tight binding model

$$H = \sum t_{ij} c_i^\dagger c_j$$

$$t_{ij} \rightarrow e^{i\phi_{ij}} t_{ij}$$

$$\phi_{ij} = \int_{\mathbf{r}_i}^{\mathbf{r}_j} \mathbf{A}(\mathbf{r}') d\mathbf{r}'$$

Peierls substitution

For typical materials

$$\phi_{ij} \sim 10^{-4} \sim 10T$$

Landau levels

Landau levels in a nutshell

Band-structure in the quantum Hall state

The energy levels are

$$E \sim \left(n + \frac{1}{2}\right) B$$

$$E \sim 1 meV$$

For a Dirac equation they would be

$$E \sim \sqrt{nB}$$

$$E \sim 100 meV$$

The gauge of the magnetic potential

Let us consider the simplest magnetic field

$$\mathbf{B} = (0, 0, B_z)$$

We can write down two different gauges for the magnetic potential $\mathbf{B} = \nabla \times \mathbf{A}$

Landau gauge

$$\mathbf{A} = (-B_z y, 0, 0)$$

Respects one translational symmetry

$$[p_x, A_\alpha] = 0$$

Symmetric gauge

$$\mathbf{A} = \frac{1}{2}(-B_z y, B_z x, 0)$$

Convenient for the fractional quantum Hall wavefunction

Electrons coupled to a magnetic field

Let us take a conventional electron gas coupled to a gauge field

$$H = \frac{\left(\sum_\alpha -i\partial_\alpha + A_\alpha\right)^2}{2}$$

Momentum

Gauge potential

Minimal gauge coupling

Landau levels of Schrodinger electrons in a nutshell

Lets take a quadratic Hamiltonian $H \sim p_x^2 + p_y^2$

And add a magnetic field (minimal coupling) $\mathbf{p} \rightarrow \mathbf{p} + \mathbf{A}$

Take the Landau gauge

$$\mathbf{A} = (0, -Bx, 0)$$
$$\nabla \times \mathbf{A} = (0, 0, B)$$

Plugging the magnetic potential in we get $H \sim p_x^2 + B^2 x^2$

(this looks like an harmonic oscillator)

Quantized levels in a magnetic field $E_n \sim nB$

Landau levels of Dirac electrons in a nutshell

Lets take a Dirac Hamiltonian

$$H \sim \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix}$$

And add a magnetic field (minimal coupling)

$$\mathbf{p} \rightarrow \mathbf{p} + \mathbf{A}$$

Take the Landau gauge

$$\mathbf{A} = (0, -Bx, 0)$$

$$\nabla \times \mathbf{A} = (0, 0, B)$$

Plugging the magnetic potential in we get

$$H^2 \sim p_x^2 + B^2 x^2$$

(this looks like an harmonic oscillator)

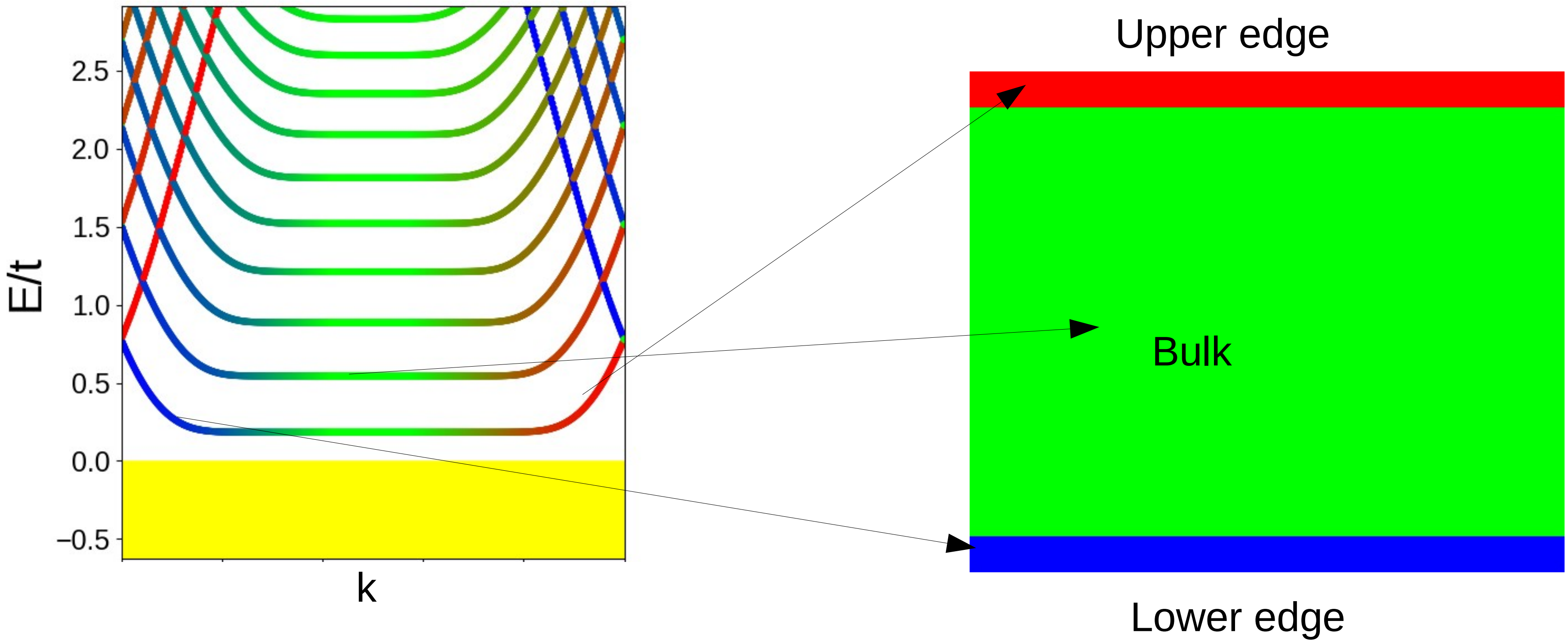
$$E_n^2 \sim nB$$

Quantized levels in a magnetic field

$$E_n \sim \sqrt{nB}$$

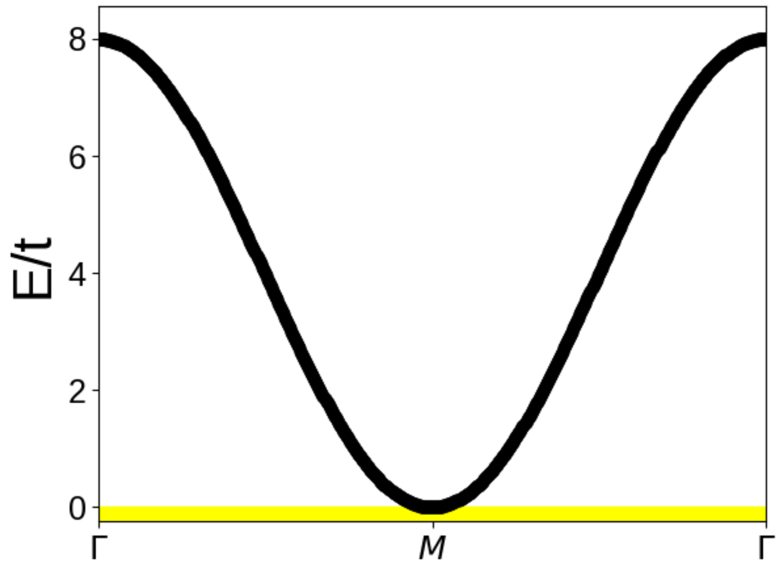
Landau levels in a nutshell

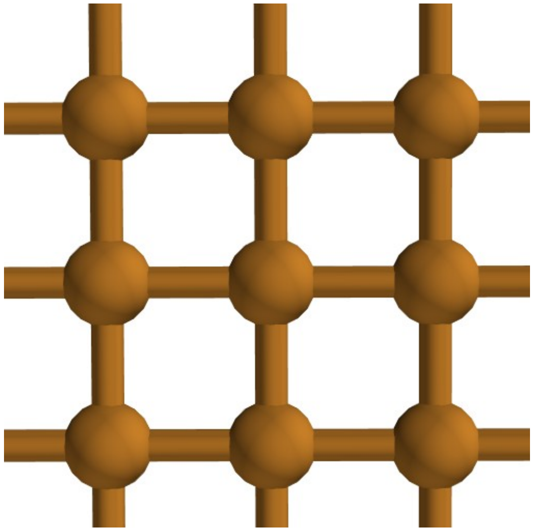
Electronic structure | **Real space**



The Landau levels of quadratic electronic gas

Band structure of a square lattice

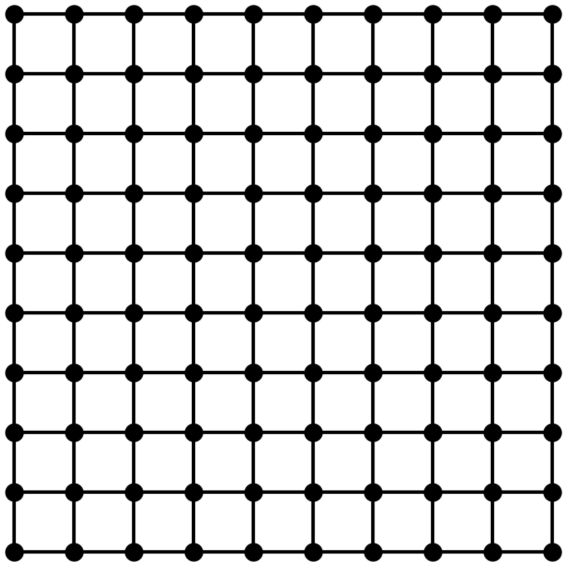




$$E \sim k_x^2 + k_y^2$$

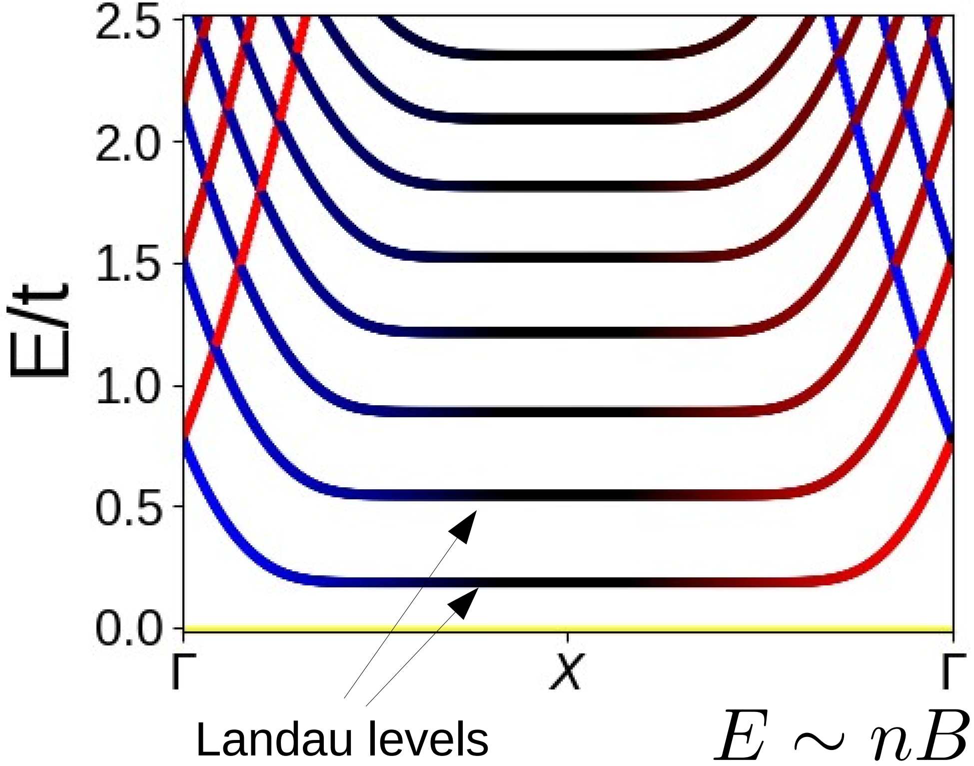
The Landau levels of quadratic electronic gas

Structure of a ribbon



```
from pyqula import geometry,ribbon
g = geometry.square_lattice() # create the lattice
g = ribbon.bulk2ribbon(g,n=30) # create a ribbon
h = g.get_hamiltonian() # get the tight binding Hamiltonian
h.add_orbital_magnetic_field(0.03) # add magnetic field
h.add_onsite(4.) # shift Fermi energy
(k,e,c) = h.get_bands(operator="yposition") # get the band structure
```

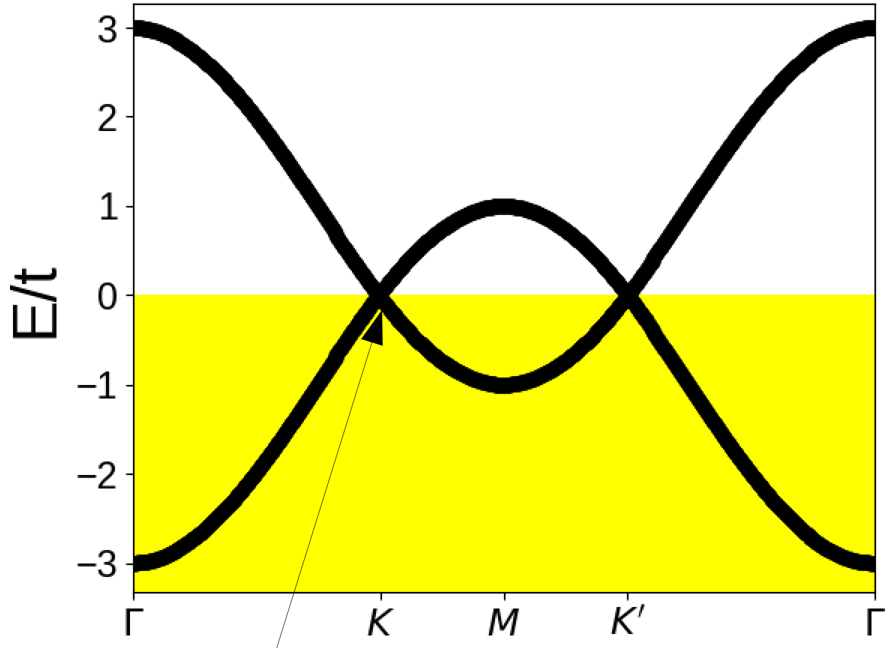
Band structure



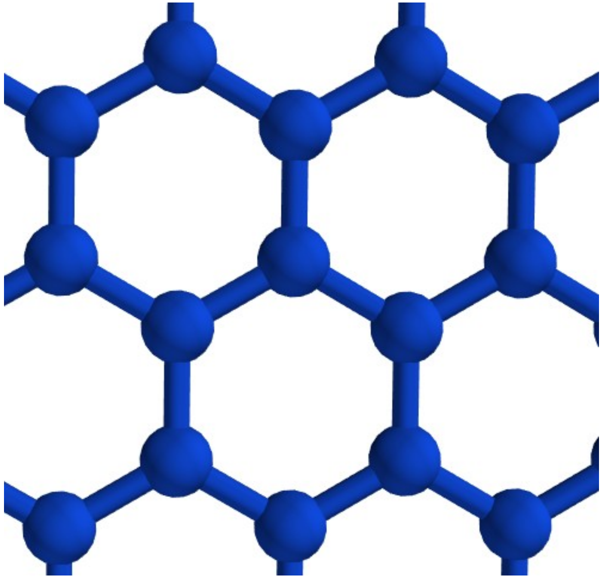
$$E \sim nB$$

The Landau levels of a Dirac electronic gas

Band structure of a honeycomb lattice

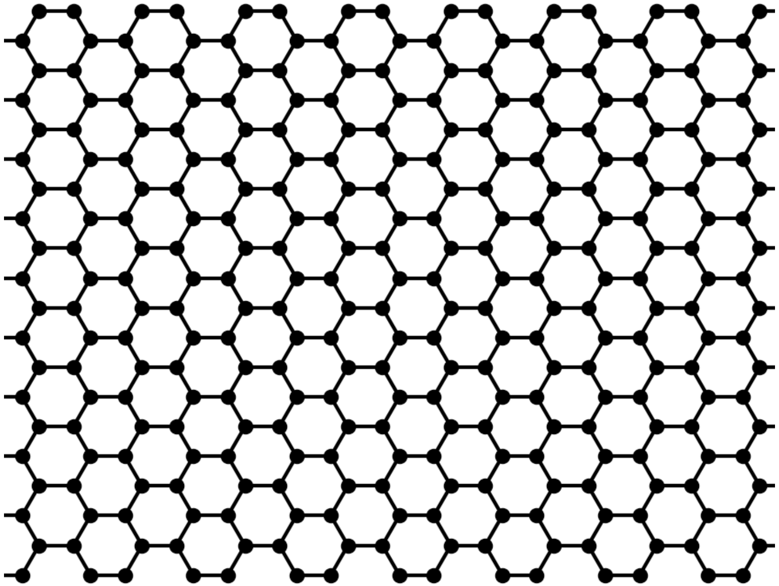


Effective Dirac equations



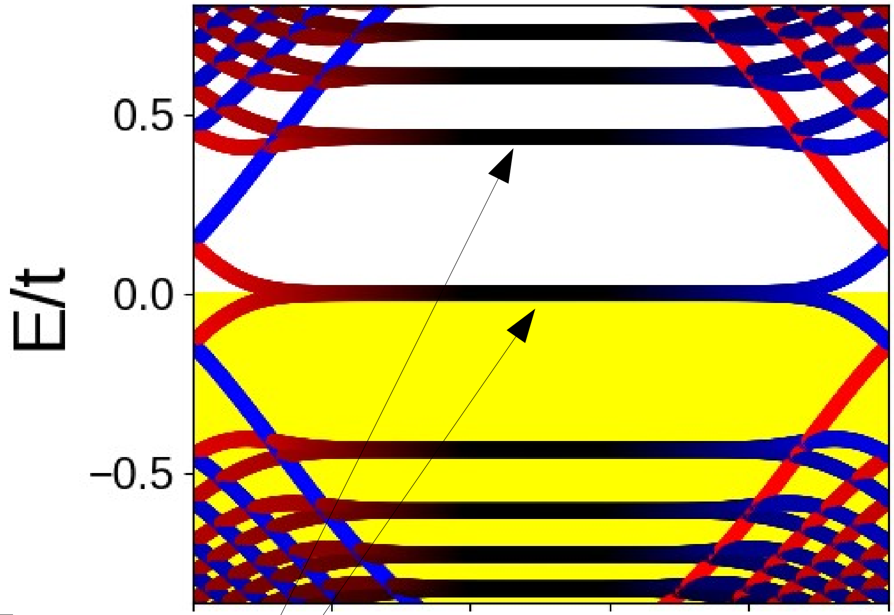
The Landau levels of a Dirac electronic gas

Structure of a ribbon



```
from pyqula import geometry
g = geometry.honeycomb_armchair_ribbon(30) # create the lattice
h = g.get_hamiltonian() # get the tight binding Hamiltonian
h.add_orbital_magnetic_field(0.007) # add magnetic field
(k,e,c) = h.get_bands(operator="yposition",kpath=["M","G","M"]) # get bands
```

Band structure



$$E \sim \sqrt{nB}$$

The many-body Landau level ground state

Ladder operators for Landau levels

Let us define new operators for the full Hamiltonian

$$a^\dagger = \frac{1}{\sqrt{2B}}\left((p_x - A_x) + i(p_y - A_y)\right)$$ "creation"

$$a = \frac{1}{\sqrt{2B}}\left((p_x - A_x) - i(p_y - A_y)\right)$$ "annihilation"

Canonical commutation relation

$$\nabla \times \mathbf{A} = (0, 0, B)$$

$$aa^\dagger - a^\dagger a = 1$$

Ladder operators for Landau levels

The Hamiltonian for the bulk quantum Hall state can be written as

$$H = \frac{(\sum_\alpha -i\partial_\alpha + A_\alpha)^2}{2}$$

$$H = B\left(a^\dagger a + \frac{1}{2}\right)$$

$$\nabla \times \mathbf{A} = (0, 0, B)$$

$$aa^\dagger - a^\dagger a = 1$$

With eigenenergies

$$E_n = B\left(n + \frac{1}{2}\right)$$

The Landau levels in the symmetric gauge

Take the symmetric gauge

$$\mathbf{A} = \frac{B}{2}(-y, x, 0) \qquad \nabla \times \mathbf{A} = (0, 0, B)$$

0th Landau level wavefunctions $$\Psi_m(z) \sim z^m e^{-|z|^2/(4\ell^2)}$$

m is the angular momentum $$z = x + iy \qquad \ell \sim 1/\sqrt{B}$$

This wavefunction will be our starting point for the fractional quantum Hall state

Single-particle wavefunction

Landau level wavefunction

$$\Psi_m(z) \sim z^m e^{-|z|^2/(4\ell^2)}$$

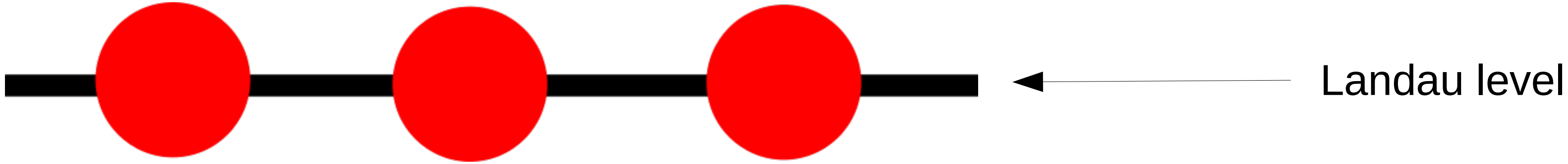
(another) Landau level wavefunction

$$\Psi(z) \sim p(z) e^{-|z|^2/(4\ell^2)}$$

Polynomial

This is for a single electron, how do we extend it to many-electrons?

The filled lowest Landau level



How to build the many-body wavefunction:
- Take all the single particle states
- Make them antisymmetric

The filled lowest Landau level

Many-body wavefunction of the filled lowest Landau level

$$\Psi_{\mathrm{LLL}}(z_1, \cdots, z_{N_\mathrm{p}}) \propto \prod_{i<j}^{N_\mathrm{p}} (z_i - z_j) \exp\left[-\sum_{l}^{N_\mathrm{p}} |z_l|^2/(4\ell^2)\right]$$

Fully antisymmetrical
(Pauli's principle)

Landau level like

$\ell \sim 1/\sqrt{B}$ $\quad$ $z_n = x_n + iy_n$

The quantum Hall effect without Landau levels

Quantum Hall effect without Landau levels

- Net magnetic field is not necessary to have QH physics
- To have a non-zero Chern number, we just need to break time-reversal symmetry

Berry curvature

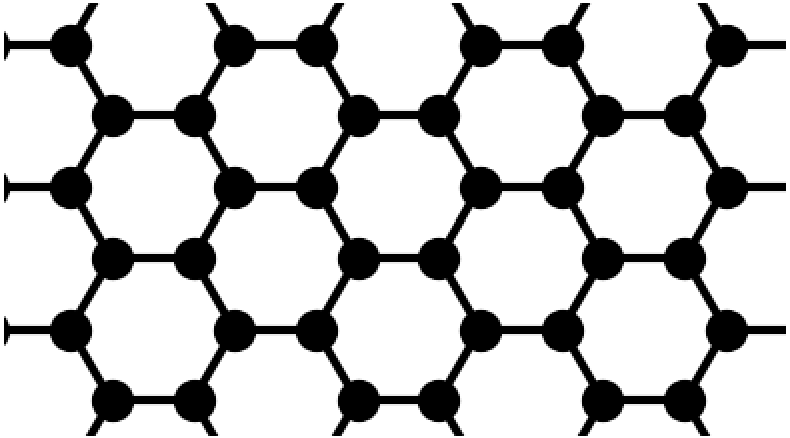
$$\Omega_\alpha = \partial_{k_x} A_y^\alpha - \partial_{k_y} A_x^\alpha$$

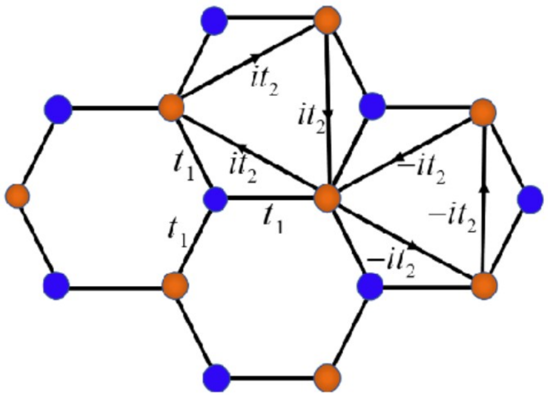
Chern number

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k} = \sum_\alpha C_\alpha = C$$

The Haldane model

Take a model in the honeycomb lattice and include second-neighbor hopping breaking time-reversal symmetry

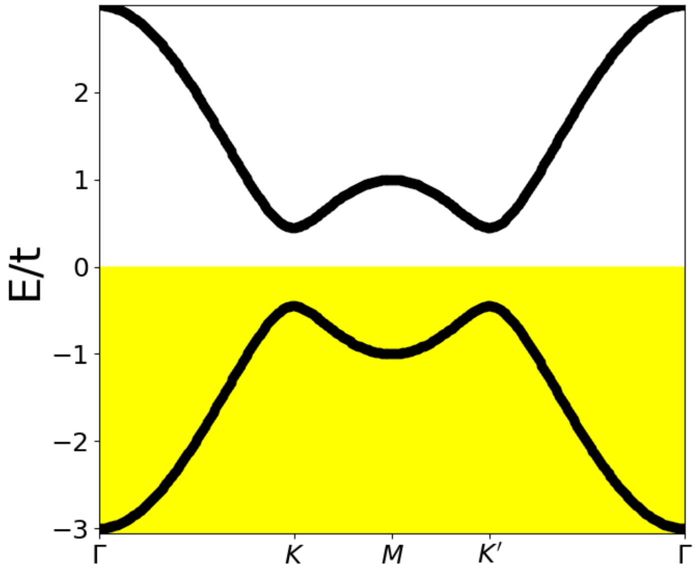




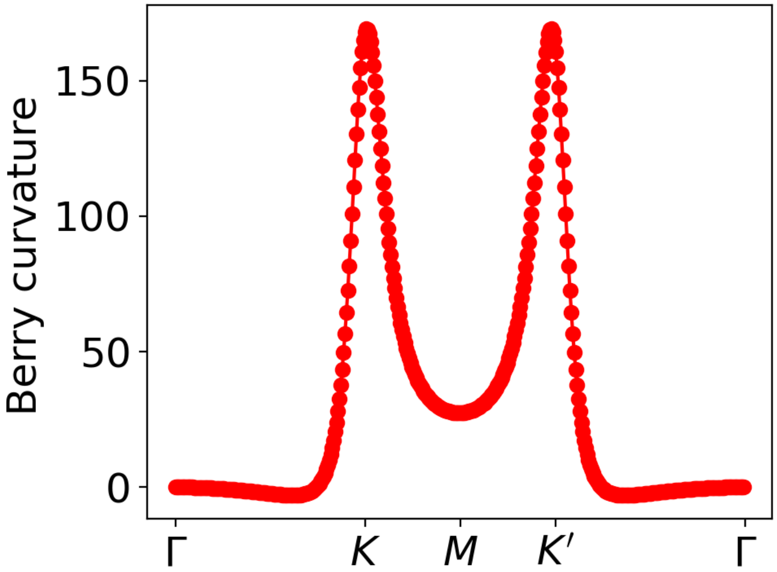
Equivalent to having a spatially dependent field that averages to zero in space

The Haldane model: bulk electronic structure

Electronic structure

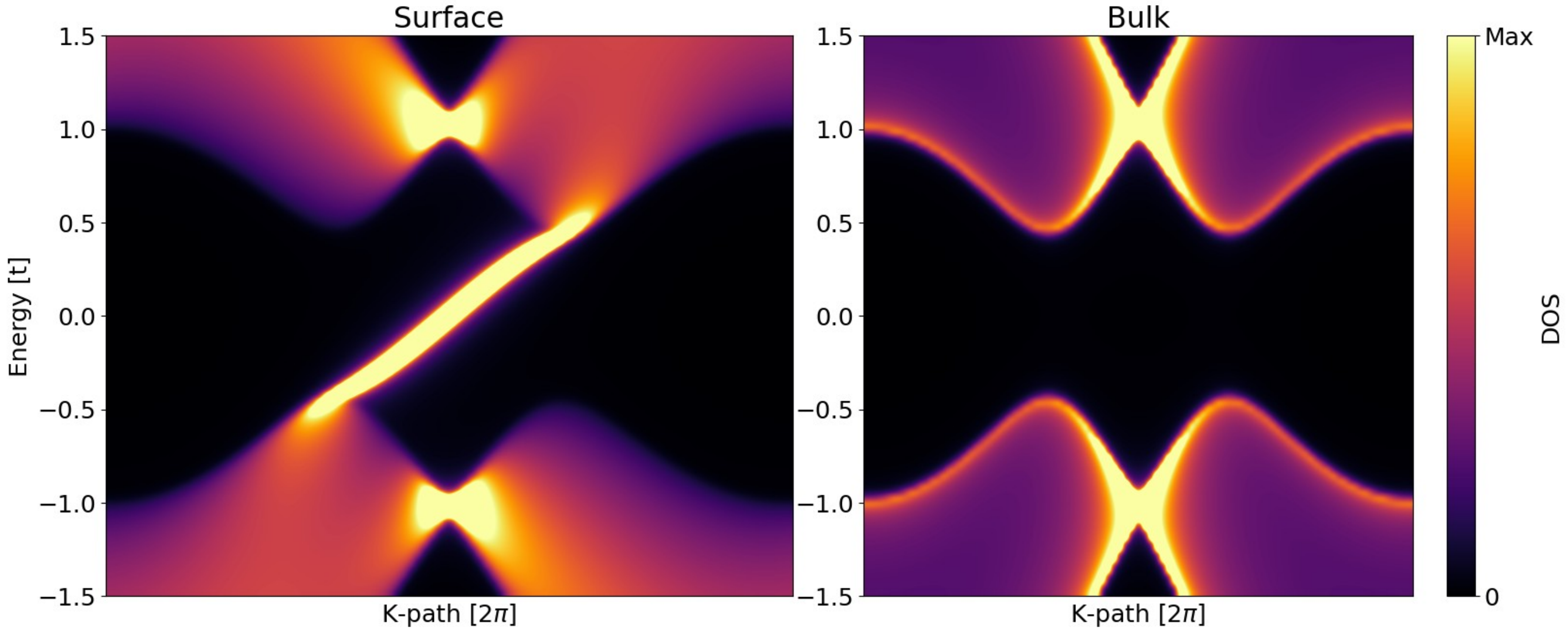


Berry curvature

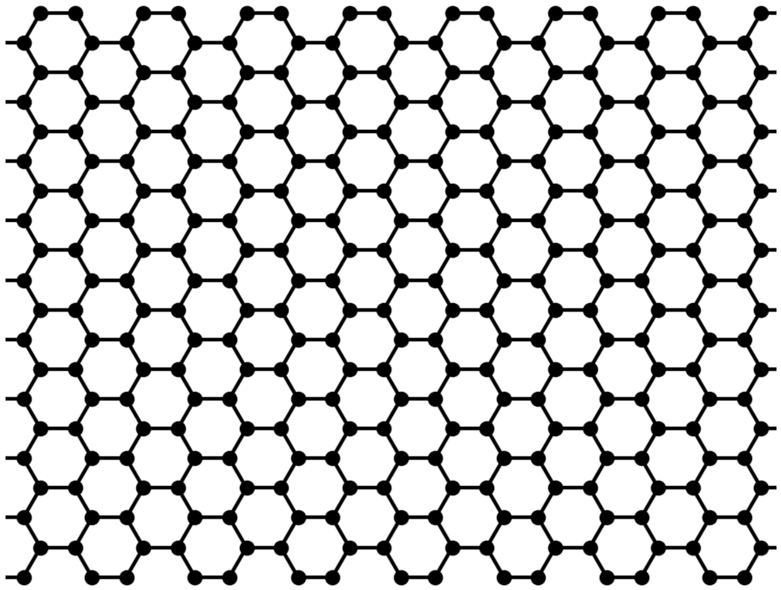


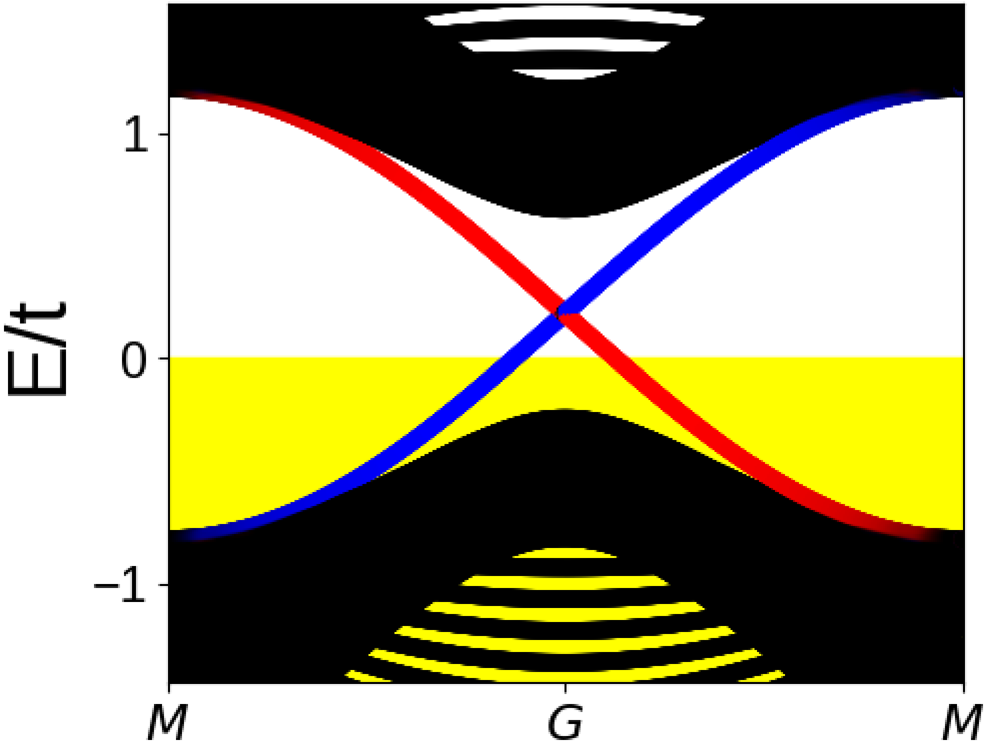
The Haldane model: edge electronic structure

Surface and bulk spectral functions

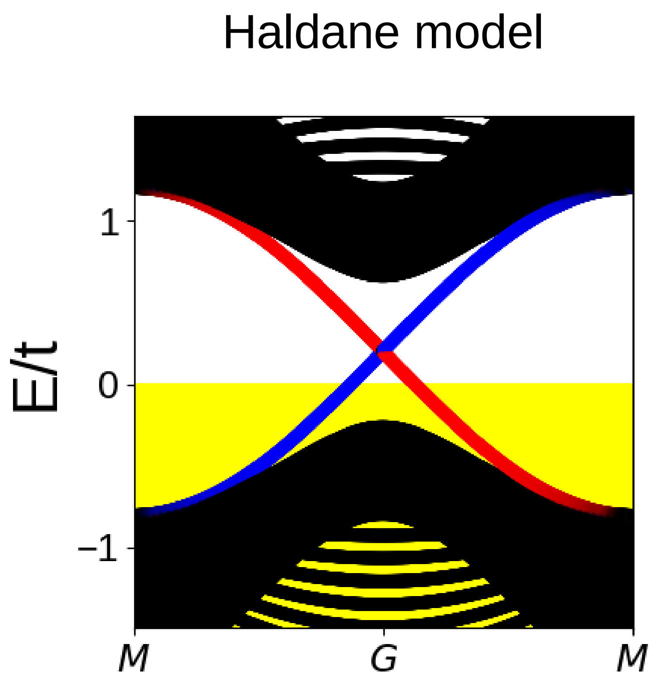


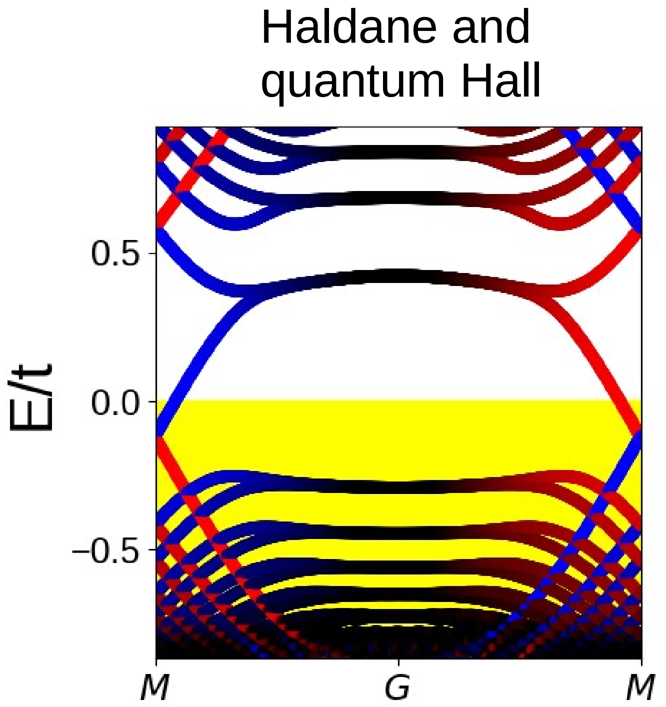
The Haldane model: ribbon electronic structure

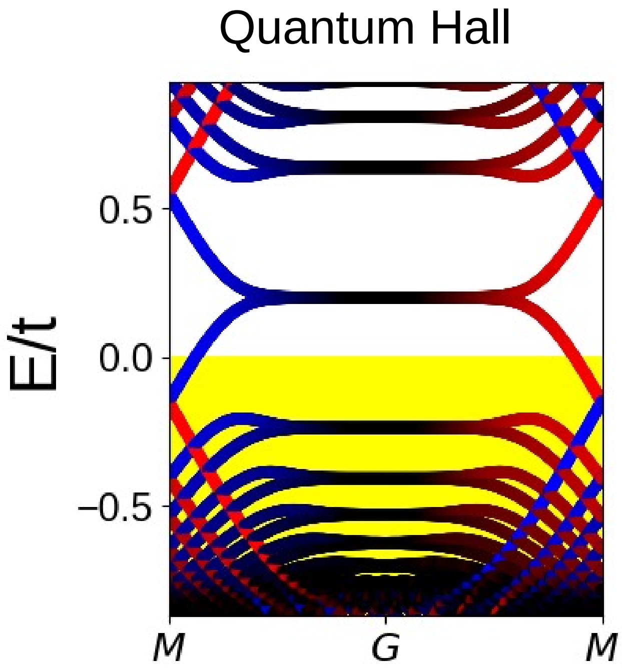




Topological equivalence between Haldane model and quantum Hall





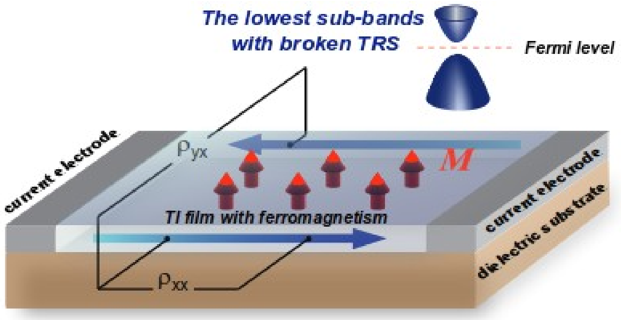


The bulk gap remains open as Haldane model is transformed in the quantum Hall state

Both Haldane model and quantum Hall have the same topological invariant

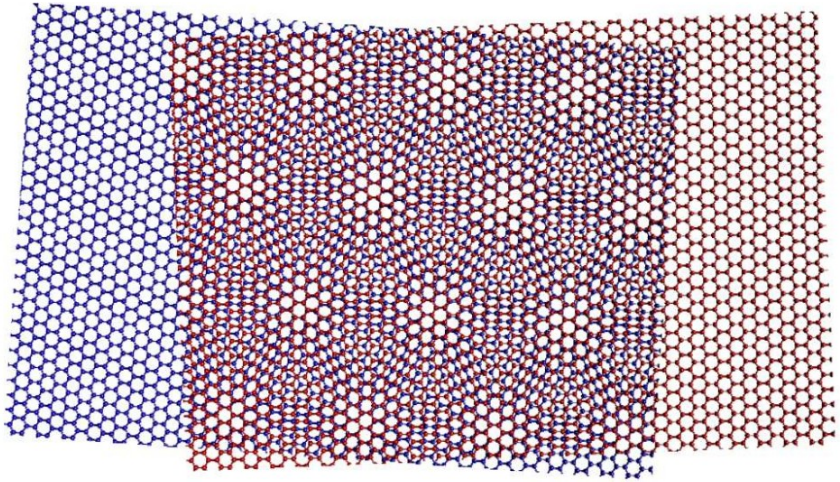
Real Chern insulating materials

Magnetically doped topological insulators



Cr-doped
$(Bi,Sb)_2Te_3$

Twisted graphene bilayers



(aligned with BN substrate)

Take home

- Quantum Hall effect can exist with and without external magnetic field
- Landau levels are topological flat bands of the quantum Hall state

In the next session: the fractional quantum Hall effect

Lets put the Fermi energy in the 0 Landau level

and assume that we have a repulsive interactions

Single particle wavefunction for the lowest Landau level

$$\Psi(z) \sim p(z) e^{-|z|^2/(4\ell^2)}$$

Polynomial

What is the wavefunction when we have repulsive interactions?

$$\Psi(z_1, z_2, \ldots, z_n)$$