Problem Set 1 - Soultions
Public Economics II: Public Expenditures
Spring 2023

1) (*50 pts*) **Lindahl Equilibrium**: Consider an economy consisting of three people - $i \in \{A, B, C\}$, who each have the utility function:

$$U_i = z^{1/3} c_i^{2/3}$$

where z is the quantity of Public good provided and c_i is person i's consumption of the private good. Person i's income y_i is divided between consumption of the private good and their contribution to the financing of the public good. Let t_i be the proportion of the public good person i contributes. This makes i's budget constraint:

$$c_i + t_i z = y_i$$

A's income is 90, B's income is 120 and C's income is 150. Find the Lindahl equilibrium, i.e. find the Lindahl prices $\{t_A, t_B, t_C\}$.

To get the Lindahl prices we first solve for the social planner problem for the optimal level of public goods z^*:

$$\mathcal{L} = \sum_i z^{1/3} c_i^{2/3} - \lambda \Big(\sum_i c_i + z - 360 \Big)$$

FOC's give:

$$\sum_i c_i = 2z$$

Plugging into the resource constraint to get z^* we get:

$$z^* = 120$$

Now solve individual maximization problem.

$$\mathcal{L} = z^{1/3} c_i^{2/3} - \mu \big(c_i + t_i - y_i \big)$$

FOCs give:

$$c_i = 2 t_i z$$

We now impose the solution to be $z = z^*$ plug the expression above into the individual budget constraint and find the set of t_is that ensure z^* is the solution.

$$t_i = \frac{y_i}{3 z_i^*}$$

Plugging in $z^* = 120$ and each i's income we the Lindahl prices:

$$t_A = 1/4, t_B = 1/3, t_C = 5/12$$

Which we can easily see sum to 1.

2) (*30 pts*) **Rothschild-Stiglitz model**: Consider the model we covered in class. Where there are two types of individuals $i \in \{H, L\}$ with wealth w who have different risks of incurring a loss l. These risks are given by $p^i \in \{p^H, p^L\}$ where $p^L < p^H$ and are unobservable to anyone but themselves.

Assume there is a risk neutral set of insurance companies who offer a menu of contracts: $\alpha^i = \{\alpha_1^I, \alpha_2^I\}, i \in \{H, L\}$, where α_1 is the transfer to the insurance premium, and α_2 is transfer to an individual by the insurer (net of α_1) in the case of a loss.

In what follows we will skip the proof that there is no pooling equilibrium and just characterize the separating equilibrium.

a) (2pts) Write down the insurer's *full* optimization problem (Hint: this has 6 constraints, ignoring the no profitable deviation condition).

$$\max_{\alpha_1^H, \alpha_2^H, \alpha_1^L, \alpha_2^L} (1 - p^H)u(w - \alpha_1^H) + p^H u(w - l + \alpha_2^H) + (1 - p^L)u(w - \alpha_1^L) + p^L u(w - l + \alpha_2^L)$$

subject to:

incentive compatibility:

$$(1 - p^H)u(w - \alpha_1^H) + p^H u(w - l + \alpha_2^H) \geq (1 - p^H)u(w - \alpha_1^L) + p^H u(w - l + \alpha_2^L) \quad (IC^H)$$

$$(1 - p^L)Hu(w - \alpha_1^L) + p^L u(w - l + \alpha_2^L) \geq (1 - p^L)u(w - \alpha_1^H) + p^L u(w - l + \alpha_2^H) \quad (IC^L)$$

individual rationality:

$$(1 - p^i)u(w - \alpha_1^i) + p^i u(w - l + \alpha_2^i) \geq (1 - p^i)u(w) + p^i u(w - l) \quad \textit{for} \quad i \in \{L, H\}$$

profitability:

$$(1 - p^i)(\alpha_1^i) + p^i(-\alpha_2^i) \geq 0 \quad i \in \{L, H\}$$

b) (4pts) Without solving the maximization problem, what is the contract that will be offered to type H? Why?

Given competition, in any separating equilibrium both types contract must fall on their fair odds line (profit=0). Type H will always prefer no insurance to a contract on the H-type fair odds line or below therefore there is no threat of deviation by L-types, and even if they did this would create positive profits for the insurance company. In this case, if the contract offered to type H provides less than full insurance there will be a profitable deviation where a firm could enter and make positive profits by offering a contract with slightly more insurance to the H types (draw it). Therefore the H-types must receive a contract offering full insurance in a separating equilibrium.

So type H will receive full insurance with $\alpha_1 = p^h l$ *and* $\alpha_2 = (1 - p^h)l$

c) (3pts) Given the contract for type H what constraints can we ignore, and why?

We can ignore the L type's incentive compatibility constraint as we have already established they will not deviate to the H type contract. We can drop both the individual rationality constraints because we know it holds for H when they get full insurance. For the L types we know that any contract will fall on their fair odds line and they will prefer any contract on that line to no insurance

d) (3pts) Write down the optimization problem again, given we know the solution for type-H, with only the relevant constraints.

$$\max_{\alpha_1^L, \alpha_2^L} (1 - p^L)u(w - \alpha_1^L) + p^L u(w - l + \alpha_2^L)$$

subject to:

incentive compatibility:

$$(1 - p^H)u(w - \alpha_1^H) + p^H u(w - l + \alpha_2^H) \geq (1 - p^H)u(w - \alpha_1^L) + p^H u(w - l + \alpha_2^L) \quad (IC^H)$$

profitability:

$$(1 - p^L)(\alpha_1^L) + p^L(-\alpha_2^L) \geq 0$$

e) (4pts) Write down the Lagrangian for this maximization problem using λ_H as the Lagrange multiplier on the high types incentive compatibility constraint and μ_L as the Lagrange multiplier on the L-types profit constraint. Derive the full Kuhn-Tucker first order conditions (including the complimentary slackness conditions for each constraint).

$$\mathcal{L} = (1-p^L)u(w-\alpha_1^L) + p^L u(w-l+\alpha_2^L)$$

$$+\lambda_H\big[(1-p^H)u(w-\alpha_1^H) + p^H u(w-l+\alpha_2^H) - (1-p^H)u(w-\alpha_1^L) - p^H u(w-l+\alpha_2^L)\big]$$

$$+\mu_L\big[(1-p^L)\alpha_1^L - p^L\alpha_2^L\big]$$

FOCs:

$$\begin{aligned}
[\alpha_1^L]: \quad u'(w-\alpha_1^L) &= \frac{\mu_L}{1-\lambda_H\frac{1-p^H}{1-p^L}} \\
[\alpha_2^L]: \quad u'(w-l+\alpha_2^L) &= \frac{\mu_L}{1-\lambda_H\frac{p^H}{p^L}}
\end{aligned}$$

$$\begin{aligned}
[\lambda_H]: \quad & \lambda_H\big[(1-p^H)u(w-\alpha_1^H) + p^H u(w-l+\alpha_2^H) \\
& \quad -(1-p^H)u(w-\alpha_1^L) - p^H u(w-l+\alpha_2^L)\big] \\
[\mu_L]: \quad & \mu_L\big[(1-p^L)\alpha_1^L - p^L\alpha_2^L\big]
\end{aligned}$$

f) (3pts) Show that both constraints bind. What is the intuition here?

If IC_H does not bind then from the complimentary slackness condition we have $\lambda_H = 0$. This in turn would imply that $u'(w-\alpha_1^L) = \mu_L = u'(w-l+\alpha_2^L)$, which would mean type Ls would get full insurance. The IC_H constraint ensures that the high type does not deviate and try to buy the contract for type L. If this was not a concern it would be optimal to provide L types with full insurance. Because it is a concern we can only offer the L types insurance to the point where H types are indifferent between their own contract and the L type contract. When this happens, IC_H will bind.

If the profit constraint does not bind then by the complimentary slackness condition $\mu_L = 0$ which would mean $u'(w-\alpha_1^L) = u'(w-l+\alpha_2^L) = 0$.

g) (3pts) Show that the first order conditions for α_1^L and α_2^L imply that the optimal type-L contract will offer less than full insurance.

Because $p^H > p^L$ we have:

$$u'(w-\alpha_1^L) = \frac{\mu_L}{1-\lambda_H\frac{(1-p^H)}{(1-p^L)}} < \frac{\mu_L}{1-\lambda_H\frac{p^H}{p^L}} = u'(w-l+\alpha_2^L)$$

Which, assuming $u(w)$ is concave means:

$$\iff w-\alpha_1 > w-l+\alpha_2^L$$

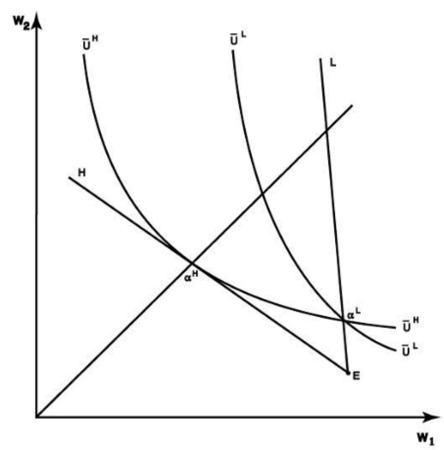
Consumption in the good state > in the bad state → less than full insurance

h) (2pts) How does α_2^L change with p^H, what is the intuition?

$$\frac{\partial \alpha_2^L}{\partial p^H} = -\frac{\lambda_H}{p^L} \frac{\mu_L}{(1-\lambda_H \frac{p^H}{p^L})^2} < 0$$

α_2^L is decreasing in p^H. Intuitively: when type H is riskier the more likely they are to want to deviate to a L type contract and therefore the L type contract that makes type H indifferent between it and their own contract will need to offer less consumption in the bad state. In terms of the classic Rothschild-Stiglitz graph: higher p^H means that the fair odds line for type H will be flatter, placing them on a lower indifference curve, and that indifference curve will cut the L type fair-odds line at a lower point.

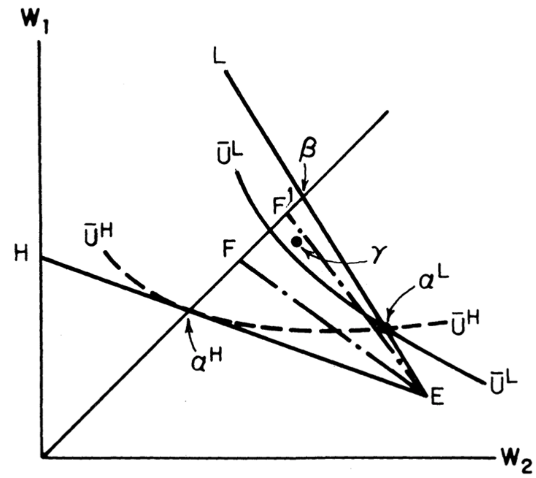
i) (2pts) Draw a graph showing this equilibrium. Include the fair-odds line and indifference curves for each type.



j) (2pts) Are the constraints used in part e) sufficient to ensure this separating equilibrium exists?

No. We also need to ensure that there is no profitable deviation for firms. i.e. a pooled contract $\{\bar{\alpha}_1, \bar{\alpha}_2\}$ s.t. $(1-\bar{p})\bar{\alpha}_1 - \bar{p}\bar{\alpha}_2 > 0$

k) (2pts) Show on the graph drawn above that the no profitable deviation condition is not satisfied if the pooled fair-odds is sufficiently steep. When may this occur? What is the the intuition for this?



If the pooled fair-odds line is F there is no profitable deviation if you offer any contract on or below it and above the L type indifference curve only L types will buy the contract and profits will be negative. If the pooled fair-odds line is the steeper and lies above the L type indifference curve (F') then a contract such as γ that falls between F' and $\bar{u}^L$ will earn positive profits.

Let f be the fraction of H types in the economy then the slope of pool fair-odds line is given by

$$\bar{F} = \frac{1 - (fp^H + (1-f)p^L)}{fp^H + (1-f)p^L}$$

and it is easy to show that $\frac{\partial \bar{F}}{\partial f} < 0$, meaning that the fair-odds line gets steeper when the fraction of H types is lower. Intuitively: when their are fewer H types the cost of subsidizing them for the L type grows smaller.

Note: the Fair odds line could also end up falling above the L type indifference curve when the difference $p^H - p^L$ is smaller, either by a reduction in the risk of the H type or an increase in the risk of the L type.

3) (*20pts*) **Pre-Existing Conditions**: Now we will consider an alternative type of contract in the Rothschild-Stiglitz world. Assume that when a costumer applies for insurance they sign a contract where they voluntarily disclose their type, but if a claim is made the insurance company can investigate and reveal their true type. If the consumer is found to be untruthful then the contract states that the insurance company can refuse to pay the agreed upon coverage. This set up mirrors practices in some real-world insurance markets, most notably when coverage is denied due to the presence of pre-existing conditions in private health insurance markets.

In this scenario when a consumer (with initial wealth w) applies for insurance they report their type $i \in \{H, L\}$, where the probability of a loss for each type is $p^L < p^H$. When i files a claim the contract states that with probability π_i the insurer will perform an investigation with cost c to reveal the consumer's type. If there is an investigation and it is found the consumer was truthful then the insurer pays δ_i. If it is found that the consumer was untruthful the insurer will pay γ_i. If there is no investigation the consumer is paid α_i, where like above these payments are net of the insurance premium β_i.

Given this the insurer offers a menu of contracts of the form $\{\beta_i, \pi_i, \delta_i, \gamma_i, \alpha_i\}$ for $i \in \{H, L\}$ making a truthful person's utility when buying a contract:

$$(1 - p^i)u(w - \beta_i) + p^i\Big[(1 - \pi_i)u(w - l + \alpha_i) + \pi_i u(w - l + \delta_i)\Big]$$

Like above, we will skip the proof that there is no pooling equilibrium in this market.

a) (2pts) What will be the value of π_H? i.e. what will the probability of investigating a claim in the H-type contract? Explain.

$\pi_H = 0$. L types would not deviate to the H type contract as they are strictly better off with no insurance. Further, if they did deviate this would cause positive profits for the insurer so there would be no reason to investigate.

b) (2pts) What will the contract for type-H be?

Since $\pi_H = 0$ then $\delta_H = 0$ and $\gamma_H = 0$ as they are irrelevant. Then the problem for the H type contract is identical to the regular RS model with $\beta_H = \alpha_1^H$ and $\alpha_H = \alpha_2^H$. Therefore type-H will receive full insurance with $\pi_H = \delta_H = \gamma_H = 0$ and $\beta_H = p^H l$, $\alpha_H = (1 - p^H)l$.

c) (2pts) Write down the maximization problem for determining the type-L contract, including the incentive compatibility constraint for type H and the profitability constraint for the insurer.

$$\max \ (1 - p^L)u(w - \beta_L) + p^L[\pi_L u(w - l + \delta_L) + (1 - \pi_L)u(w - l + \alpha_L)]$$

subject to:

$$u(w - p^H l) \geq (1 - p^H)u(w - \beta_L) + p^H[\pi_L u(w - l + \gamma_L) + (1 - \pi_L)u(w - l + \alpha_L)] \quad (IC_H)$$

and

$$(1 - p^L)\beta_L - p^L(\pi_L(c + \delta_L) + (1 - \pi_L)\alpha_L) \geq 0$$

d) (3pts) Construct the Lagrangian this maximization problem, using μ_L as the Lagrange multiplier on the profit constraint and λ_H for the the incentive-compatibility constraint of type H. Write down the first order conditions for the $\beta_L, \pi_L, \delta_L, \gamma_L$ and α_L.

$$\begin{aligned}\mathcal{L} =& (1 - p^L)u(w - \beta_L) + p^L[\pi_L u(w - l + \delta_L) + (1 - \pi_L)u(w - l + \alpha_L)] \\ &+ \mu_L[(1 - p^L)\beta_L - p^L(\pi_L(c + \delta_L) + (1 - \pi_L)\alpha_L)] \\ &+ \lambda_H\Big[u(w - p^H l) - [(1 - p^H)u(w - \beta_L) + p^H(\pi_L u(w - l + \gamma_L) + (1 - \pi_L)u(w - l + \alpha_L))]\Big]\end{aligned}$$

FOCs:

$$\begin{aligned}
[\beta_L]: &\quad u'(w-\beta_L) &=& \quad \frac{\mu_L}{1-\lambda_H\frac{1-p^H}{1-p^L}} \\
[\alpha_L]: &\quad u'(w-l+\alpha_L) &=& \quad \frac{\mu_L}{1-\lambda_H\frac{p^H}{p^L}} \\
[\delta_L]: &\quad u'(w-l+\delta_L) &=& \quad \mu_L \\
[\gamma_L]: &\quad -\lambda_H p^H \pi_L u'(w-l+\gamma_L) &<& \quad 0
\end{aligned}$$

e) (7pts, 2pts each part except 2) 1 point) Show that:

1) It is optimal to set $\gamma_L = 0$

From above we can see that $\frac{\partial \mathcal{L}}{\partial \gamma_L} < 0$ always. Therefore the insurance company is going to want to set γ as low as possible to deter the H-type from taking the contract (Here I am assuming that the insurance company can't impose a fine on the H type or lying)

2) The incentive-compatibility constraint for type-H must be binding.

Similar to the previous question, if the constraint doesn't bind then $\lambda_H = 0$ which given the FOCs would mean that $u'(w-\beta_L) = u'(w-l+\alpha_L) = u'(w-l+\delta_L)$, aka full insurance regardless of investigation or not. Which would not result in a separating equilibrium.

3) $\pi_L < 1$

The FOC for π_L is not straight forward to interpret, but: If $\pi_L = 1$, then an H-type claiming to be an L-type, if he makes a claim, will be investigated for sure, and found to be lying. Therefore they will end up with $w-l+\gamma$ if there is no accident, and because $\gamma = 0$, they will end up with $w-l$ if there is an accident. They would do better buying no insurance at all, and therefore better still by taking the full, fair contract intended for their type. But that would mean that the incentive constraint of the H-type would not be binding. That would imply $\lambda_H = 0$, which would lead to a contradiction as proved above.

4) Show that $w-l+\delta_L > w-\beta_L > w-l+\alpha_L$

Using a similar argument to 2) (g), the FOCs imply that in equilibrium

$$u'(w-l+\delta_L) < u'(w-\beta_L) < u'(w-l+\alpha_L)$$

which in turn implies

$$w-l+\delta_L > w-\beta_L > w-l+\alpha_L$$

f) (2pts) Give an intuition for the inequalities in e)(4).

The inequality in e)(4) implies that the L type will get less than full insurance if there is no investigation and will get ***more*** *than full insurance if there is an investigation and they are found to be telling the truth. This makes sense from the perspective that the insurer wants to reward truth-telling and punish lying. That means in the event of an investigation giving L types a bonus for telling the truth ($\delta_L > \alpha_L$) and a penalty to H types for lying ($\gamma_L = 0$), which is worse than no insurance because they get no payout in the bad state and still had to purchase the contract in the good state. It should be noted that $\delta_L > \alpha_L$ comes out of the math and it would probably seem odd to implement this in the real world. L types would always ask for an investigation H types would never ask to be investigated (which actually sounds like a good mechanism if investigations are 100% accurate)*

g) (2pts) Are type-L's better off in this equilibrium compared to the classic Rothschild-Stiglitz equilibrium? If so how?

L type's expected payout in the bad state in expectation is almost certainly larger than the payoff in the classic model.

$$\underbrace{\pi_L u'(w - l + \delta) + (1 - \pi_L)u(w - l + \alpha_L)}_{\text{bad-state utility with investigation}} \quad > \quad \underbrace{u(w - l + \alpha_L)}_{\text{classic RS bad-state utility}}$$

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So the L-type receives the same payout as the RS model if not investigated and a larger payout if investigated. This is because the insurance company has more instruments screen out and punish the H-types. Which translates to the L-type contract the H-types will be just indifferent to can offer more insurance