Exercise Session 5

Problem 1:

A radiowave propagates down through seawater, with electric field function $\mathbf{E}(z) = a_x E_0 \exp(-jk_c z)$ Here $E_0 = 10\,\mathrm{V/m}$ is the magnitude of the electric field at the surface, and k_c is the (complex) propagation factor. The (real part of the) permittivity of seawater is $\varepsilon' = 80\varepsilon_0$ and its conductivity $\sigma = 3\,\mathrm{S/m}$. The frequency of the wave is 1 MHz.

i. What is the average power density of the wave (in $\mathrm{W/m^2}$) at the surface ($z = 0$)?
ii. What is the average power density of the wave (in $\mathrm{W/m^2}$) two meters below the surface ($z = 2\,\mathrm{m}$)?

The time-average Poynting vector — in Cheng's notation $\mathscr{P}_{\mathrm{av}}$ — is the real part of the complex Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}^*/2$ (where $\mathbf{E}$ and $\mathbf{H}$ are the time-harmonic (complex) electric and magnetic fields).

Solution:

Since $\frac{\sigma}{\omega\epsilon} = \frac{3}{2\pi 1\cdot 10^6 (\frac{1}{36\pi}\cdot 10^{-7})80} = 675 \gg 1$ we can use the formulas for good conductors:

The attenuation constant:

$\alpha = \sqrt{\pi f \mu\sigma} = \sqrt{\pi 10^6 4\pi \cdot 10^{-7} 3} \approx 3.44\ \mathrm{Np/m}$

The phase constant:

$\beta = \sqrt{\pi f \mu\sigma} \approx 3.44\ \mathrm{rad/m}$

The intrinsic impedance:

$$\eta_c = (1+\mathrm{j})\sqrt{\frac{\pi f \mu}{\sigma}} = (1+\mathrm{j})\sqrt{\frac{\pi 10^6 4\pi \cdot 10^{-7}}{3}} \approx (1+\mathrm{j})1.15\,\Omega$$

The average power density of the wave is given by the time-average Poynting vector:

$$\begin{aligned}\mathscr{P}_{\mathrm{av}}(z) &= \tfrac{1}{2}\Re\left\{\mathbf{E}\times\mathbf{H}^*\right\} = \tfrac{1}{2}\Re\left\{\left(E_0 e^{-\alpha z} e^{-\mathrm{j}\beta z}\mathbf{a}_x\right)\times\left(\tfrac{E_0}{\eta_c} e^{-\alpha z} e^{-\mathrm{j}\beta z}\mathbf{a}_y\right)^*\right\}\\ &= \tfrac{1}{2}\Re\left\{\left(E_0 e^{-\alpha z} e^{-\mathrm{j}\beta z}\mathbf{a}_x\right)\times\left(\tfrac{E_0^*}{\eta_c^*} e^{-\alpha z} e^{+\mathrm{j}\beta z}\mathbf{a}_y\right)\right\}\\ &= \tfrac{1}{2}\Re\left\{\tfrac{|E_0|^2}{\eta_c^*} e^{-2\alpha z}\mathbf{a}_x\times\mathbf{a}_y\right\} = \Re\left\{\tfrac{1}{\eta_c^*}\right\}\tfrac{|E_0|^2}{2} e^{-2\alpha z}\mathbf{a}_z\end{aligned}$$

i. The average power density at the surface is:

$\mathscr{P}_{\mathrm{av}}(0) = \Re\left\{\frac{1}{\eta_c^*}\right\}\frac{|E_0|^2}{2} e^{-2\alpha\cdot 0} = 0.435\cdot\frac{|10|^2}{2} \approx 22\ \mathrm{W/m^2}$

ii. What is the average power density of the wave (in $\mathrm{W/m^2}$) two meters below the surface ($z = 2$ m)?

$\mathscr{P}_{\mathrm{av}}(2) = \Re\left\{\frac{1}{\eta_c^*}\right\}\frac{|E_0|^2}{2} e^{-2\alpha\cdot 2} = 0.435\cdot\frac{|10|^2}{2} e^{-2\cdot 3.44\cdot 2} \approx 23\ \mu\mathrm{W/m^2}$

Problem 2:

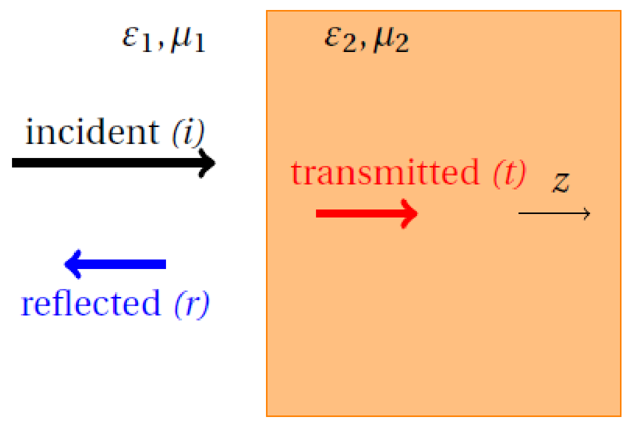
Consider plane wave reflection from the planar interface between two lossless materials. Assume that the wave arrives from free space (air, $\varepsilon_1 = \varepsilon_0$ and $\mu_1 = \mu_0$) into a non-magnetic dielectric materials ($\varepsilon_2 = 4\varepsilon_0$ and $\mu_2 = \mu_0$). The wave has a normal incidence (incidence angle $\theta_1 = 0$). The peak value of the incident electric field is E_0.

i. Compute the power density of the incident wave (= amplitude of the Poynting vector $\mathbf{S}_i = \mathbf{a}_z S_i$).

ii. Compute the power density of the reflected wave (= amplitude of the Poynting vector $\mathbf{S}_r = -\mathbf{a}_z S_r$).

iii. Compute the power density of the transmitted wave (= amplitude of the Poynting vector $\mathbf{S}_t = \mathbf{a}_z S_t$).

iv. Do your results satisfy the energy balance (that the powers of the reflected and transmitted fields match that of the incident wave)?



Solution:

Let's assume an incident electric field of $\mathbf{E}_i(z) = E_0 e^{-jkz}\mathbf{a}_x$ which has the corresponding incident magnetic field of $\mathbf{H}_i(z) = \frac{E_0}{\eta_0} e^{-jkz}\mathbf{a}_y$. The plane wave has a normal incidence $\theta_1 = 0°$.

i. The complex Poynting vector is:

$$\mathbf{S}_i = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\left(E_0 e^{-jkz}\mathbf{a}_x\right) \times \left(\frac{E_0}{\eta_0} e^{-jkz}\mathbf{a}_y\right)^* = \frac{1}{2}\left(E_0 e^{-jkz}\mathbf{a}_x\right) \times \left(\frac{E_0^*}{\eta_0} e^{+jkz}\mathbf{a}_y\right) = \frac{|E_0|^2}{2\eta_0} \underbrace{e^{-jkz} e^{+jkz}}_{=1} \overbrace{\mathbf{a}_x \times \mathbf{a}_y}^{=\mathbf{a}_z}$$

$$= \frac{|E_0|^2}{2\eta_0}\mathbf{a}_z$$

Thus the amplitude is $S_i = \frac{|E_0|^2}{2\eta_0}$.

ii. The reflection coefficient is:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{1}{\sqrt{4}}\eta_0 - \eta_0}{\frac{1}{\sqrt{4}}\eta_0 + \eta_0} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3}$$

Since we are dealing with power density, the reflection coefficient has to be squared. Thus we get the Poynting vector for the reflected wave:

$$\mathbf{S}_r = -\left(-\frac{1}{3}\right)^2 \frac{|E_0|^2}{2\eta_0}\mathbf{a}_z$$

Which gives us the amplitude of $S_r = \frac{1}{9}\frac{|E_0|^2}{2\eta_0}$

iii. The transmission coefficient is:

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{\frac{2}{\sqrt{4}}\eta_0}{\frac{1}{\sqrt{4}}\eta_0 + \eta_0} = \frac{1}{\frac{1}{2} + 1} = \frac{2}{3}$$

Again, since we are dealing with power density, the transmission coefficient has to be squared. Thus we get the Poynting vector for the transmitted wave:

$$\mathbf{S}_t = \left(\frac{2}{3}\right)^2 \frac{|E_0|^2}{2\eta_2}\mathbf{a}_z$$

Which gives us the amplitude of $S_t = \frac{4}{9}\frac{|E_0|^2}{2\eta_2}$

iv. The net power flow should be the same on both sides of the boundary:

$$\mathbf{S}_i + \mathbf{S}_r = \frac{|E_0|^2}{2\eta_0}\mathbf{a}_z - \frac{1}{9}\frac{|E_0|^2}{2\eta_0}\mathbf{a}_z = \left(1 - \frac{1}{9}\right)\frac{|E_0|^2}{2\eta_0}\mathbf{a}_z = \frac{4 \cdot 2}{9}\frac{|E_0|^2}{2\eta_0}\mathbf{a}_z = \frac{4}{9}\frac{|E_0|^2}{2\eta_2}\mathbf{a}_z = \mathbf{S}_t$$

Thus, the energy balance is satisfied.