Quantization of superconducting qubits

2. In the lecture, we saw the derivation of the quantization of Transmon qubit. In this task, we will fill some of the gaps to understand the derivation clearly.

A charge qubit provides an excellent anharmonicity to the energy levels, however the charge dispersion, i.e., the dependence of the energy on the gate charge, introduces a drastic charge noise. Thus, charge noise is the main source of decoherence in the charge qubit.

By adding an additional large shunt capacitance in parallel with the Josephson junction, we suppress the charge noise quite dramatically. This type of qubit is called transmon qubit. The Hamiltonian of the transmon qubit is of the same form as of the charge qubit,

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\varphi}. \tag{1}$$

However, for the transmon qubit the energy ratio is in the range $40 < \frac{E_J}{E_C} < 100$. The Josepshon coupling energy dominates the charging energy, thus suppressing the charge noise. Set the offset gate charge $n_g = 0$, since it does not matter how we bias the transmon with the gate charge.

Consider the following definitions:

$$\boxed{\begin{aligned} \hat{n} &= i n_{zpf}(\hat{c} + \hat{c}^\dagger) \\ \hat{\varphi} &= \varphi_{zpf}(\hat{c} + \hat{c}^\dagger), \end{aligned}} \quad \begin{aligned} (2) \\ (3) \end{aligned}$$

where,

$$\boxed{\begin{aligned} n_{zpf} &= \left(\frac{E_J}{32E_C}\right)^{\frac{1}{4}} \\ \varphi_{zpf} &= \left(\frac{2E_C}{E_J}\right)^{\frac{1}{4}} \end{aligned}}$$

corresponds to zero-point fluctuation in charge and phase states.

(a) Show that by plugging (2) and (3) in (1) and Taylor expanding cosine potential to φ^4 term, you obtain

$$\hat{H}_{tr} = \hbar\omega_q \hat{c}^\dagger\hat{c} + \frac{\delta}{2}\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c}. \qquad \omega_q = \omega_0 - \frac{E_c}{\hbar} \tag{4}$$

Please don't write down the steps from slides and fill in the details as clearly as possible, that is, workout the missing steps from the lecture slide. Explain with proper steps why only the term $\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c}$ survives from the term $(\hat{c}^\dagger - \hat{c})^4$.

(b) Express the Hamiltonian in (4) in qubit eigenbasis i.e. $\hat{J}|j\rangle = j|j\rangle, \hat{J} = \hat{c}^\dagger\hat{c}$ and use the completeness relation $\sum_{j=0}|j\rangle\langle j| = \hat{I}_Q$. Show that the qubit frequency has the analytical form:

$$\omega_j = j\omega_0 + \frac{\delta}{2}j(j-1). \tag{5}$$

This shows that the qubit frequency depends on the state j and is not evenly-spaced like quantum harmonic oscillator. The anharmonicity scales linearly with the eigenstate of the qubit.

(2) (a)

1° Expand $\cos\phi$ using Taylor expansion 4^{th} order term

$$\cos\phi \approx 1 + \frac{\phi^2}{2!} - \frac{\phi^4}{4!} + O(\phi^6)$$

2°: Plug cosine ϕ approx in eq^n (1): Use def^n (2-3).

3°. Invoke RWA: ignore fast oscillating terms (number of $\hat{c}^\dagger$ should be equal to # of $\hat{c}$)

$\Rightarrow$ Keep only $\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c}$.

$$\hat{c} = \hat{c}\,e^{-i\omega_0 t}$$
$$\hat{c}^\dagger = \hat{c}^\dagger e^{i\omega_0 t}$$

$$\omega_q = \frac{\sqrt{8E_J E_c}}{\hbar} - \frac{E_c}{\hbar}$$

Express Hamiltonian of transmon in qubit eigenbasis $\sum_j |j\rangle\langle j| = \hat{I}_Q$

(b) $$\hat{I}_Q \cdot \hat{H}_{tr} \cdot \hat{I}_Q = \sum_k |k\rangle\langle k| \left[\left(\hbar\omega_0 \hat{c}^\dagger\hat{c} + \frac{\delta}{2}\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c}\right]\sum_{j=0}|j\rangle\langle j|\right.$$

→ Use the following:

- $\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c} = \hat{c}^\dagger(\hat{c}\hat{c}^\dagger - 1)\hat{c} = \hat{c}^\dagger\hat{c}\hat{c}^\dagger\hat{c} - \hat{c}^\dagger\hat{c}$

  with $[c, \hat{c}^\dagger] = \hat{c}\hat{c}^\dagger - \hat{c}^\dagger\hat{c} = 1 \Rightarrow \hat{c}\hat{c}^\dagger - 1 = \hat{c}^\dagger\hat{c}$

- $\hat{J}|j\rangle = j|j\rangle$ & $\hat{J}^2|j\rangle = j^2|j\rangle$

- $\langle k|j\rangle = \delta_{kj}$ in $\sum_{j,k}|k\rangle\langle k|\hat{H}_{tr}|j\rangle\langle j|$

→ In the end, you should get

$$\hat{I}_Q \hat{H}_{tr} \hat{I}_Q = \sum_j \hbar\omega_j |j\rangle\langle j|, \text{ with } \omega_j \text{ as in (5).}$$

(3) (a)

1° Dipole interaction

$\hat{H}_{int} = C_G \hat{V}_R \otimes \hat{V}_Q$

$\hat{V}_q = -\frac{2e\hat{n}}{C_\Sigma}$; $\hat{n} = i n_{zpf}(\hat{c} - \hat{c}^\dagger)$

$\hat{V}_R = \frac{\hat{q}}{C_r} = \frac{i}{C_r} q_{zpf}(\hat{a} - \hat{a}^\dagger)$; $q_{zpf} = \sqrt{\frac{\hbar\omega_r C_r}{2}}$

$\hat{H} = \hat{H}_R + \hat{H}_Q + \hat{H}_{int}$

$\hbar g(\hat{a} - \hat{a}^\dagger)(\hat{c} - \hat{c}^\dagger)$

(b) $(a + \hat{a}^\dagger) \otimes (c + \hat{c}^\dagger)$ $\quad \hat{a}\hat{c}^\dagger + \hat{a}^\dagger\hat{c}$

$\Rightarrow \hat{a}\hat{c}^\dagger + \hat{a}^\dagger\hat{c} + \hat{a}^\dagger\hat{c}^\dagger + \hat{a}\hat{c}$

(Invoke RWA) $\hat{a} = \hat{a}e^{-i\omega_r t}$

$\hat{c}^\dagger = \hat{c}^\dagger e^{i\omega_q t}$

$\hat{a}\hat{c}^\dagger = \hat{a}\hat{c}^\dagger e^{i(\omega_q - \omega_r)t}$ ✓

$\hat{a}\hat{c} = \hat{a}\hat{c}\, e^{-i(\omega_q + \omega_r)t}$ ✗

Jaynes-Cummings Hamiltonian

3. In this exercise, we study a coupled interaction between the resonator and the transmon qubit.

(a) Form the total Hamiltonian of the transmon–resonator system, which contains the uncoupled diagonalized Hamiltonians of the resonator and the qubit and the interaction Hamiltonian. The interaction between the resonator and the qubit is modeled by the interation Hamiltonian,

$$\hat{H}_{int} = C_G \hat{V}_Q \otimes \hat{V}_R. \tag{6}$$

where, the voltage operators take the form $\hat{V}_q = -\frac{2e}{C_\Sigma}\hat{n}$ and $\hat{V}_r = \frac{\hat{q}}{C_r}$ for the qubit and the resonator respectively. Here, $C_\Sigma = C_G + C_J + C_S$ is the sum capacitance.

(b) Invoke rotating wave approximation, and obtain the Jaynes-Cummings Hamiltonian

$$\hat{H}_{JC} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hbar g(\hat{\sigma}_- \otimes \hat{a}^\dagger + \hat{\sigma}_+ \otimes \hat{a}), \tag{7}$$

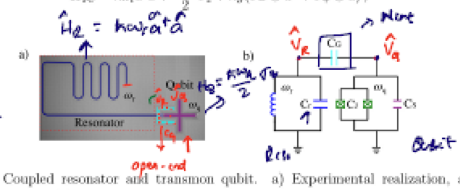


Figure 1: Coupled resonator and transmon qubit. a) Experimental realization, and b) Lumped-circuit model.

where $\hat{\sigma}_- = |0\rangle\langle 1|$ and $\hat{\sigma}_+ = |1\rangle\langle 0|$.

(c) Solve the Jaynes-Cummings Hamiltonian for the case i) $\Delta = \omega_r - \omega_q = 0$, ii) $\Delta = \omega_r - \omega_q = 0.5\,\text{GHz}$, and iii) $\Delta = \omega_r - \omega_q = 1\,\text{GHz}$. Initially, there are $n+1$ photons in the resonator and the qubit is in the ground state. Plot the probability density of the transmon being in the excited states for n = 1, 10 and 100 photons. $g = 100\,\text{MHz}$.

As usual, the state of the cavity mode can be written in terms of the number-state basis, $\{|n\rangle\}$, and the qubit state in terms of the computational basis, $\{|0\rangle, |1\rangle\}$. The overall state of the system can therefore be described as a tensor product of these two subsystems, e.g., $|n, 0\rangle = |n\rangle \otimes |0\rangle$.

Bonus!

(A) Show that the following relations hold:

$$H_{JC}|n,0\rangle = \hbar g\sqrt{n}\,|n-1,1\rangle$$
$$H_{JC}|n,1\rangle = \hbar g\sqrt{n+1}\,|n+1,0\rangle$$

(B) Consider now the case where the cavity starts in the vacuum state ($|0\rangle$) and the qubit is initially in the excited state ($|1\rangle$). Use the previous results to show that:

$$H_{JC}|0,1\rangle = \hbar g\,|1,0\rangle$$
$$H_{JC}^2|0,1\rangle = (\hbar g)^2\,|0,1\rangle$$

(C) Because the interaction Hamiltonian is time-independent, the Schrödinger equation can be solved in the usual way to give the unitary evolution operator:

$$U(t) = \exp\left(-\frac{it}{\hbar}H_{JC}\right) = \sum_{j=0}^{\infty}\left(-\frac{it}{\hbar}\right)^j \frac{H_{JC}^j}{j!}$$

Given the initial state $|\psi(t)\rangle = |0,1\rangle$, use Taylor expansion of the unitary evolution operator to show that the state of the overall system after time t will be:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \cos(gt)\,|0,1\rangle - i\sin(gt)\,|1,0\rangle$$

Aside:

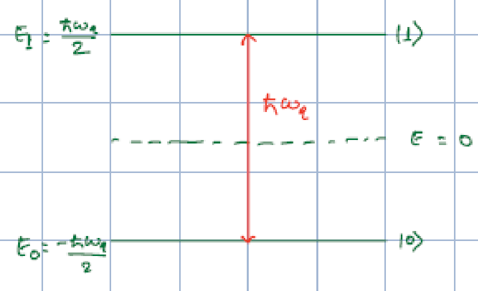
$\hat{H}_{tr} = \hbar\omega_q \hat{c}^\dagger\hat{c} + \frac{\delta}{2}\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c} \longrightarrow \hat{H}_Q = \frac{\hbar\omega_q}{2}\hat{\sigma}_z$

Two Level Approximation (TLA)

$\hat{H}_{tr} = \sum_j E_j |j\rangle\langle j| \xrightarrow{TLA} \sum_{j=0}^{1} E_o |j\rangle\langle j|$

$\Rightarrow \hat{H}_Q = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1|$

$= -\frac{\hbar\omega_q}{2}|0\rangle\langle 0| + \frac{\hbar\omega_q}{2}|1\rangle\langle 1|$



$$= \frac{\hbar\omega_q}{2}\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) \Rightarrow \boxed{\hat{H}_Q = \frac{\hbar\omega_q}{2}\hat{\sigma}_z}$$

$\hat{c} \to \hat{\sigma}_-$ and $\hat{c}^+ \to \hat{\sigma}_+$ (for TLA)

$$\hat{\sigma}_- = |0\rangle\langle 1| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

$$\hat{\sigma}_+ = |1\rangle\langle 0| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$$

$$i\frac{\partial}{\partial t}|\psi\rangle^{in} = H_I^{int}|\psi\rangle^{in}$$

(c) $$\hat{H}_{jc} = \underbrace{\hbar\omega_r \hat{a}^+\hat{a} + \frac{\hbar\omega_q}{2}\hat{\sigma}_z}_{\hat{H}_0 = \hat{H}_R + \hat{H}_Q} + \underbrace{\hbar g\left(\hat{a}\hat{\sigma}_+ + \hat{a}^+\hat{\sigma}_-\right)}_{\hat{H}_{int}}$$

1° Schrödinger picture: $|\psi(t)\rangle$ evolve (determined by H) & $\langle\hat{O}\rangle$ constant

2° Heisenberg picture: $|\psi\rangle$ const. & $\langle O(t)\rangle$ evolve (determined by H)

3° Dirac (interaction) picture: $|\psi^I\rangle$ evolution determined by $\hat{H}_{int}$

and $\langle\hat{O}(t)\rangle$ determined by H_0

$$\hat{O}(t) = e^{i\hat{H}_0 t/\hbar}$$

Interaction Hamiltonian in interaction picture

$$\hat{H}_I^{in}(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{H}_I e^{-\frac{i}{\hbar}H_0 t}$$ $\left[\hat{a} \propto e^{-i\omega_r t};\ \hat{\sigma}_- \propto e^{-i\omega_q t}\right]$

$$|\psi^{in}(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi(t)\rangle$$: state ket evolution in interaction picture.

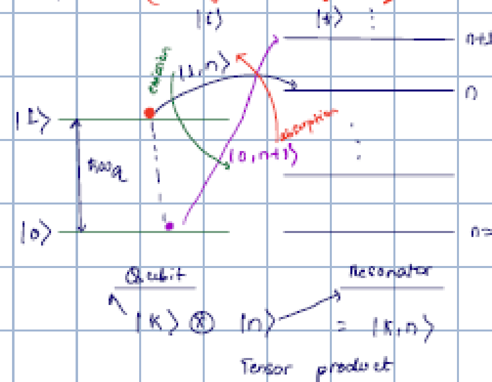
$$\hat{H}_I^{in}(t) = \hbar g\left(\hat{\sigma}_-\hat{a}^+ e^{i\Delta t} + \hat{\sigma}_+\hat{a}e^{-i\Delta t}\right)$$: $\Delta = \omega_r - \omega_q$

Time evolution

$$i\hbar \frac{\partial}{\partial t} |\psi^{in}(t)\rangle = \hat{H}_I^{in}(t) |\psi^{in}(t)\rangle$$

Complete Basis set

$$\{ |0, n+1\rangle, |1, n\rangle \}$$



where, we force the Ansatz

$$|\psi^{in}\rangle = c_i |i\rangle + c_f |f\rangle$$

$$i, f \in \{ |0, n+1\rangle, |1, n\rangle \}$$

In general,

$$i\hbar \left(\dot{c}_i(t) |i\rangle + \dot{c}_f(t) |f\rangle \right) =$$

$$\hbar g \left(\hat{\sigma}_- \hat{a}^+ e^{i\Delta t} + \hat{\sigma}_+ \hat{a}\, e^{-i\Delta t} \right) \left(c_i |i\rangle + c_f |f\rangle \right)$$

⋮

$$\Rightarrow \quad \dot{c}_i(t) |i\rangle = -ig\sqrt{n+1}\, e^{i\Delta t} c_f |i\rangle$$

$$\dot{c}_f(t) |f\rangle = -ig\sqrt{n+1}\, e^{-i\Delta t} c_i |f\rangle \Rightarrow \begin{bmatrix} \dot{c}_i \\ \dot{c}_f \end{bmatrix} = \begin{bmatrix} 0 & -ig\sqrt{n+1}\, e^{i\Delta t} \\ -ig\sqrt{n+1}\, e^{-i\Delta t} & 0 \end{bmatrix} \begin{bmatrix} c_i \\ c_f \end{bmatrix}$$

You can choose to solve this numerically or analytically. I would suggest to solve it analytically to get a feel for it

(i) For case $\Delta = 0$;

$$\begin{cases} \dot{c}_i(t) = -ig\sqrt{n+1}\, c_f(t) \\ \dot{c}_f(t) = -ig\sqrt{n+1}\, c_i(t) \end{cases} \quad \frac{d}{dt}$$

take d/dt of $\dot{c}_f$ and plug it in $\dot{c}_i$

then solve 2nd order ode.

$$\ddot{c}_f(t) = -ig\sqrt{n+1}\, \dot{c}_i$$

$$= -ig\sqrt{n+1}\left(-ig\sqrt{n+1}\right) c_f$$

$$\boxed{\ddot{c}_f(t) = -g^2 (n+1)\, c_f}$$

$$\ddot{x} + \frac{k}{m} x = 0 \qquad \omega = \sqrt{\frac{k}{m}}$$

$$x = A\cos(\omega t + \phi) + B\sin(\omega t)$$

$$-i\omega t + \phi$$

(ii) for case $\Delta \neq 0$; $\begin{cases} \dot{c}_i = -ig\sqrt{n+1}\, e^{i\Delta t}\, c_f & \to (a) \\ \dot{c}_f = -ig\sqrt{n+1}\, e^{-i\Delta t}\, c_i \end{cases}$

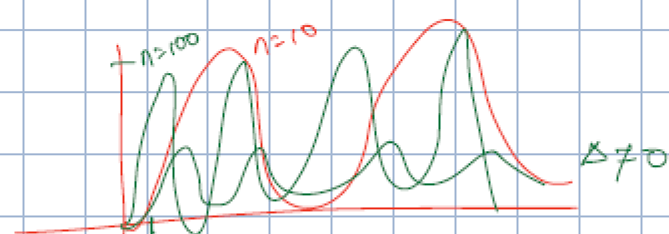
Assume the solution of the form : $c_i = e^{i\Delta t} \xi(t)$ then take the derivative $\frac{d}{dt} c_i = \frac{d}{dt}\left(e^{i\Delta t} \xi(t)\right)$! product rule.

plug $\dot{c}_i$ in (a) and solve.

$\frac{\Delta^2}{\sqrt{\Delta^2 + 4g^2}} \cos^2(gt\sqrt{n+1})$

In the end plot : $P_1(t) = |c_f|^2$ for $\Delta = 0$ & $\Delta \neq 0$ and $n = 1, 10, \& 100$.

Observe the nuances in the results and write it down.



(2)

sphere coordinate : (r, θ, φ)

(θ, φ)

$\alpha = r_\alpha e^{i\theta_\alpha}$

$\beta = r_\beta e^{i\theta_\beta}$

$|\alpha|^2 + |\beta|^2 = 1$

$\langle\psi|\sigma_x|\psi\rangle = \left[\langle 0|\cos\frac{\theta}{2} + e^{-i\varphi}\sin\frac{\theta}{2}\langle 1|\right] \sigma_x \left| \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right.$

$= \cos^2\frac{\theta}{2}\langle 0|\sigma_x|0\rangle + \cdots$

$\langle 0|\sigma_x|1\rangle + \cdots$

$\langle 1|\sigma_x|0\rangle + \cdots$

$\langle 1|\sigma_x|1\rangle$

and we $\langle 0|1\rangle = 0 = \langle 1|0\rangle$.

Bloch sphere representation

2. As we have seen in the lecture, evolution of a qubit, especially during gate operations, can be visualized as rotations around the Bloch sphere. While we denote an arbitrary state of a qubit using

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle\,, \tag{4}$$

(a) show that it can be expressed as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\,, \tag{5}$$

where θ, φ are rotation angles on the Bloch sphere as shown in figure 1.

Hint: Express α and β in polar form of a complex number and use the probability condition $|\alpha|^2 + |\beta|^2 = 1$.

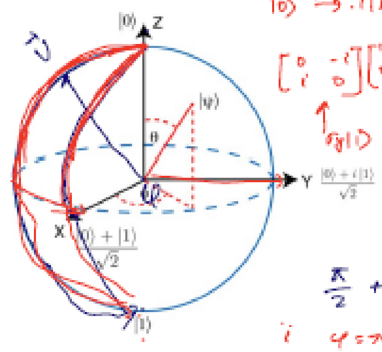
(b) Any point on a Bloch sphere can be represented by Bloch vector $\vec{V} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}$. This is just the position of a Bloch vector on a unit sphere in spherical coordinate. Show that the expectation value of the Pauli operators is given by the Bloch vector. That is, show

$$\vec{V} = \langle\vec{\sigma}\rangle = \begin{bmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{bmatrix} \tag{6}$$

$|0\rangle \to i|1\rangle$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i\begin{bmatrix} 0 \\ 1 \end{bmatrix} = i|1\rangle$

$\sigma_y|1\rangle$



$\frac{\pi}{2} + 90°$

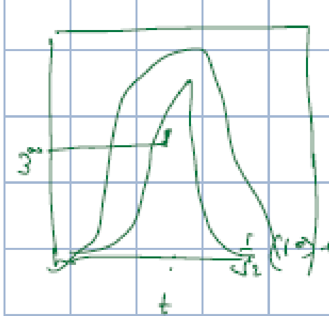
$i \quad \varphi = 90°$

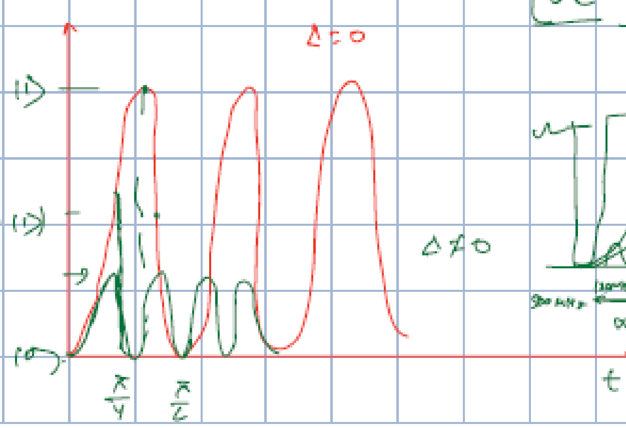
Figure 1: Bloch Sphere

Hint: Evaluate the expectation value for Pauli operator for example, $\langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle$ for an arbitrary state $|\psi\rangle$ using equation (5).

This shows us that we can describe any arbitrary state of a qubit using rotation angles (θ, φ) on a Bloch sphere. For single-qubit gates, the amplitude and duration of the microwave pulse determines the rotation angle θ and the pulse phase determines the angle φ.

$\Delta\omega\,\Delta t \geq \frac{\hbar}{2}$





$\Delta \neq 0$