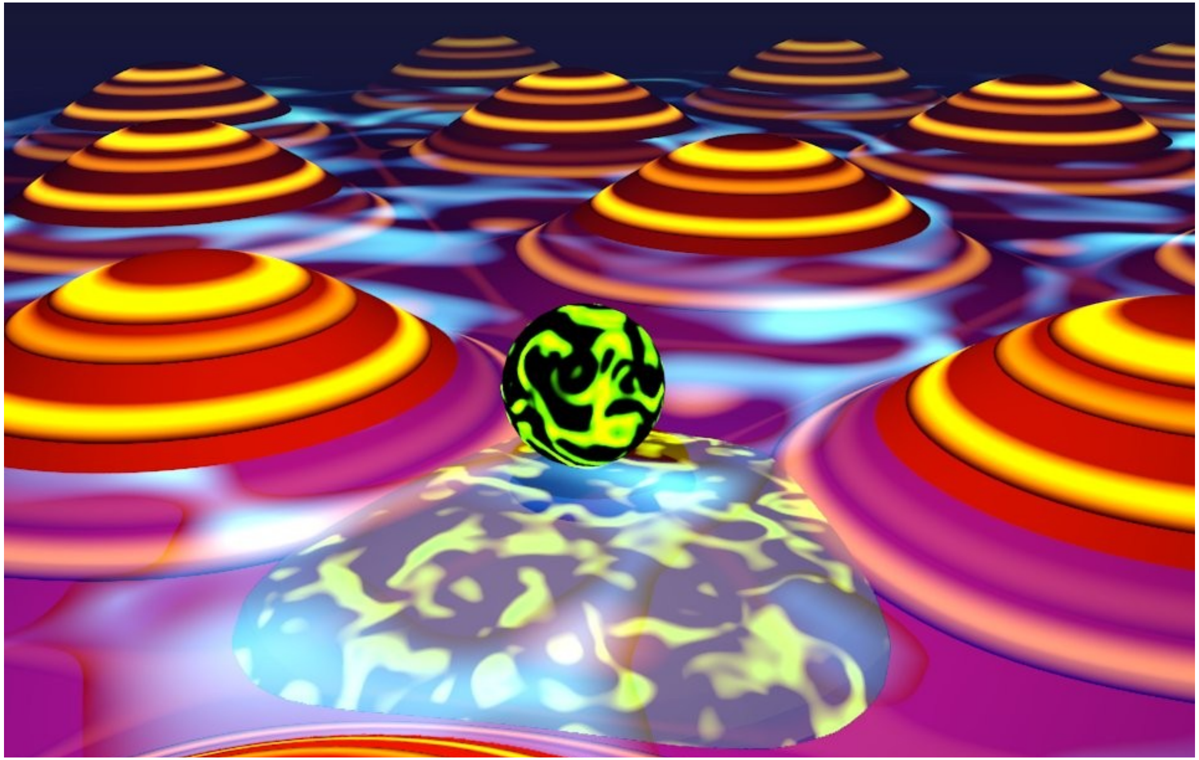
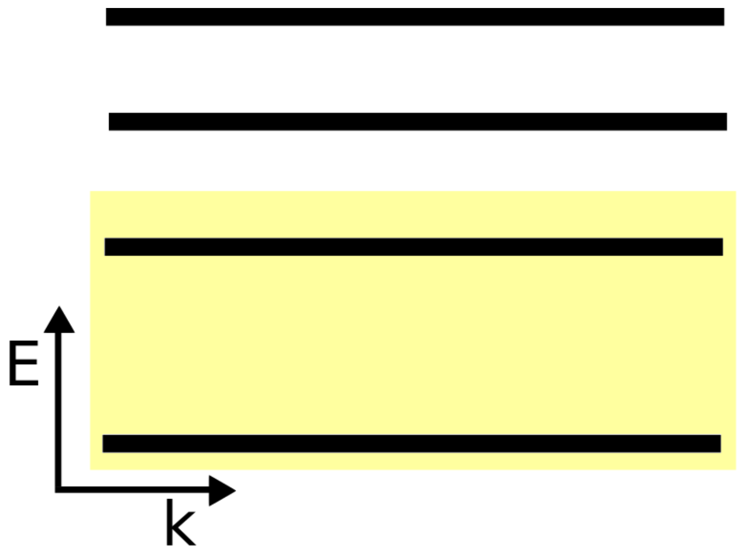
Fractionalization in quantum materials: The fractional quantum Hall effect



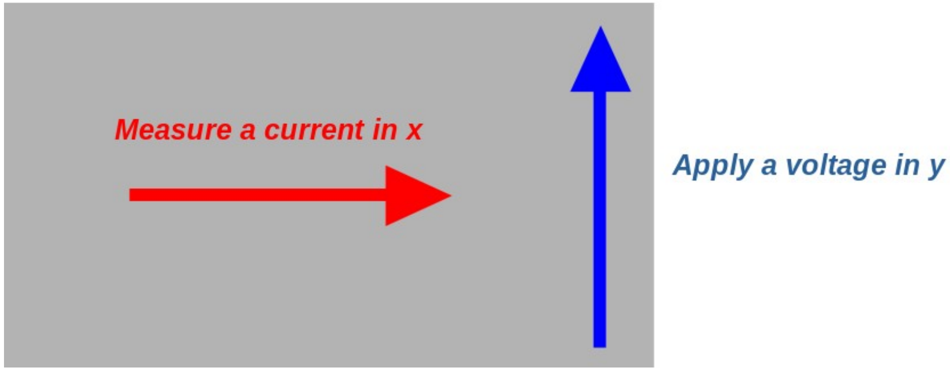
April 3rd 2023

A reminder from session 6

The quantum Hall state has flat bands



The Hall conductance is quantized



$$J_x = \sigma_{xy} V_y$$

$$\sigma_{xy} = \sum_{\alpha \in occ} \int \Omega_\alpha d^2\mathbf{k}$$

Today's plan

- The fractional quantum Hall effect
- Laughlin's wave function
- Fractional Chern insulators

What about interactions?

So far, we have focused on single particle Hamiltonians that can be easily diagonalized

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j \rightarrow \sum_k \epsilon_k \Psi_k^\dagger \Psi_k$$

<u>But what happens when we put interactions?</u>

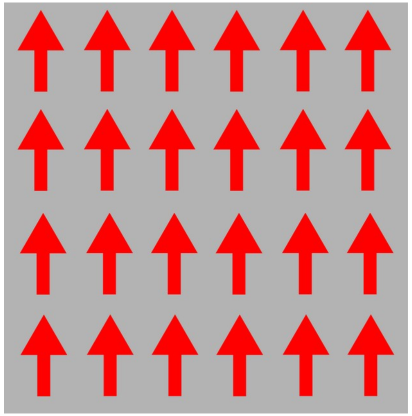
$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ijkl} V_{ijkl} c_i^\dagger c_j c_k^\dagger c_l$$

The role of electronic interactions

Electronic interactions are responsible for symmetry breaking

Broken time-reversal symmetry

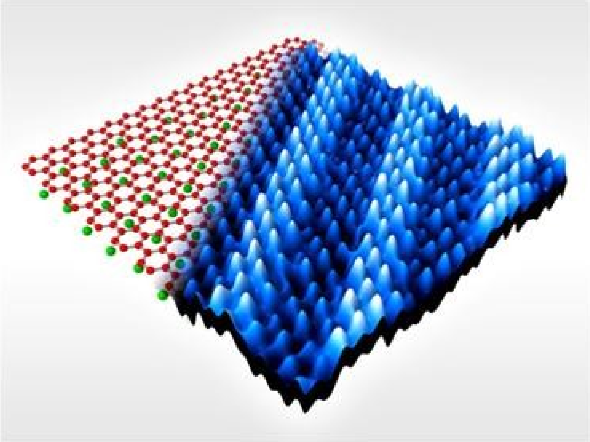
Classical magnets



$$\mathbf{M} \to -\mathbf{M}$$

Broken crystal symmetry

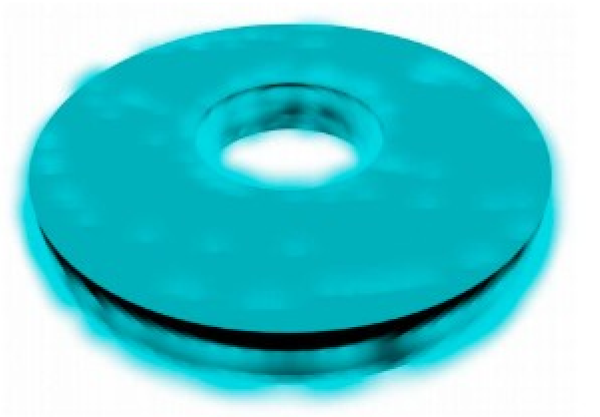
Charge density wave



$$\mathbf{r} \to \mathbf{r} + \mathbf{R}$$

Broken gauge symmetry

Superconductors



$$\langle c_\uparrow c_\downarrow \rangle \to e^{i\phi} \langle c_\uparrow c_\downarrow \rangle$$

Quantum matter with interactions

We can consider two broad groups of interacting quantum matter

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ijkl} V_{ijkl} c_i^\dagger c_j c_k^\dagger c_l$$

With a mean field description

$$H \approx \sum_{ij} \bar{t}_{ij} c_i^\dagger c_j + \sum_{ij} \Delta_{ij} c_i c_j$$

Approximate quadratic Hamiltonian
Effective single particle description

Weakly correlated matter

Without a mean field description

????

No good quadratic approximation
Requires exact solutions or numerical

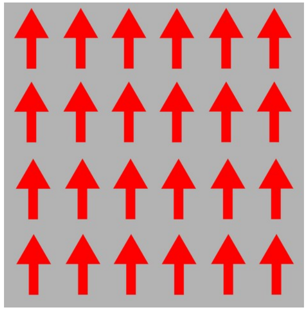
Strongly correlated matter

Correlations and mean field

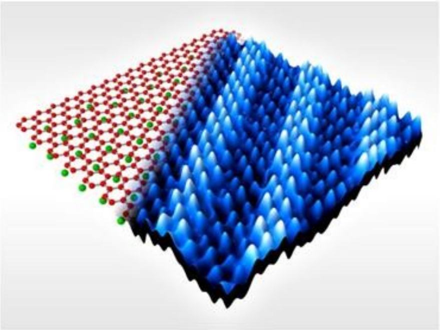
Many quantum states can be approximately described by mean field theories

$$H \approx \sum_{ij} \bar{t}_{ij} c_i^\dagger c_j + \sum_{ij} \Delta_{ij} c_i c_j + h.c.$$

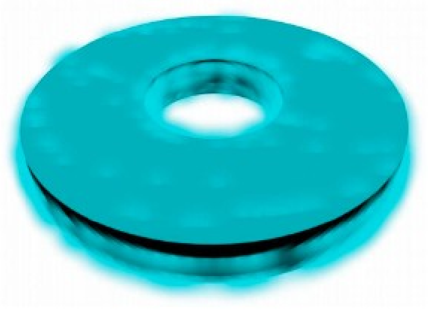
Magnets



Charge density waves



Superconductors

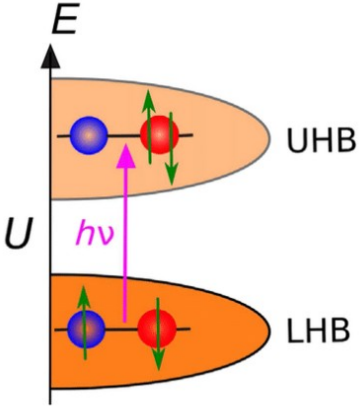


Strongly correlated matter

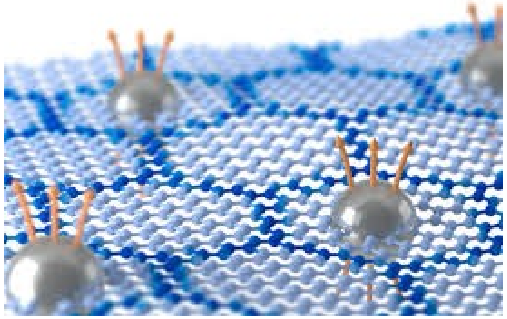
Some quantum states can only be described with the fully many-body Hamiltonian

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ijkl} V_{ijkl} c_i^\dagger c_j c_k^\dagger c_l$$

Mott insulators



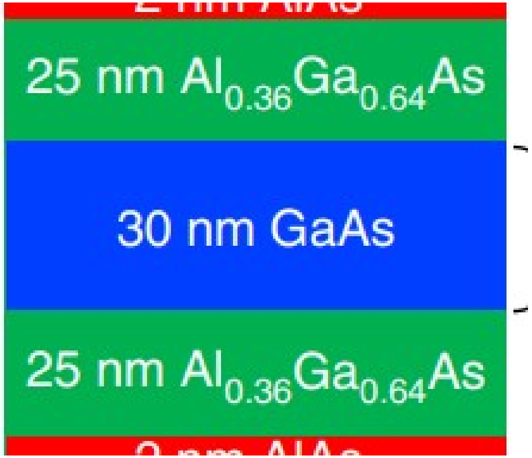
Fractional quantum Hall states



The fractional quantum Hall effect

Materials showing (fractional) quantum Hall effect

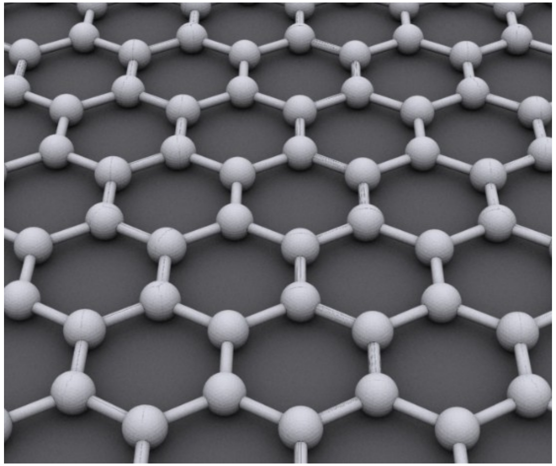
GaAs quantum wells



$$E \sim B$$

$$T \sim 1K$$

Graphene

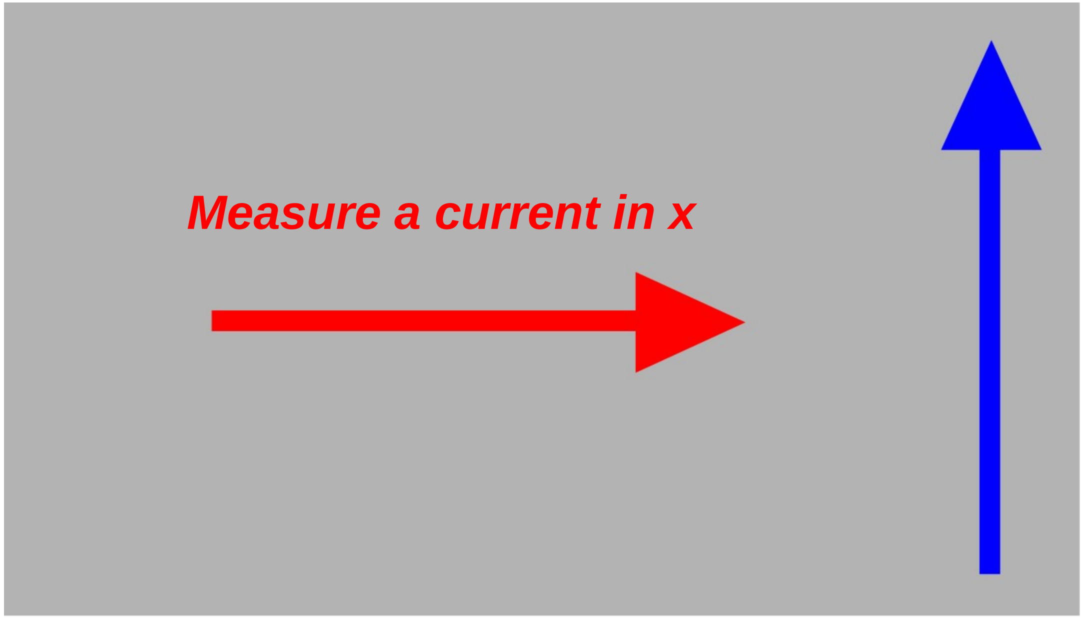


$$E \sim \sqrt{B}$$

$$T \sim 100K$$

The quantum Hall state

Take a two-dimensional material



Apply a voltage in y

Hall conductance $$J_x = \sigma_{xy} V_y$$

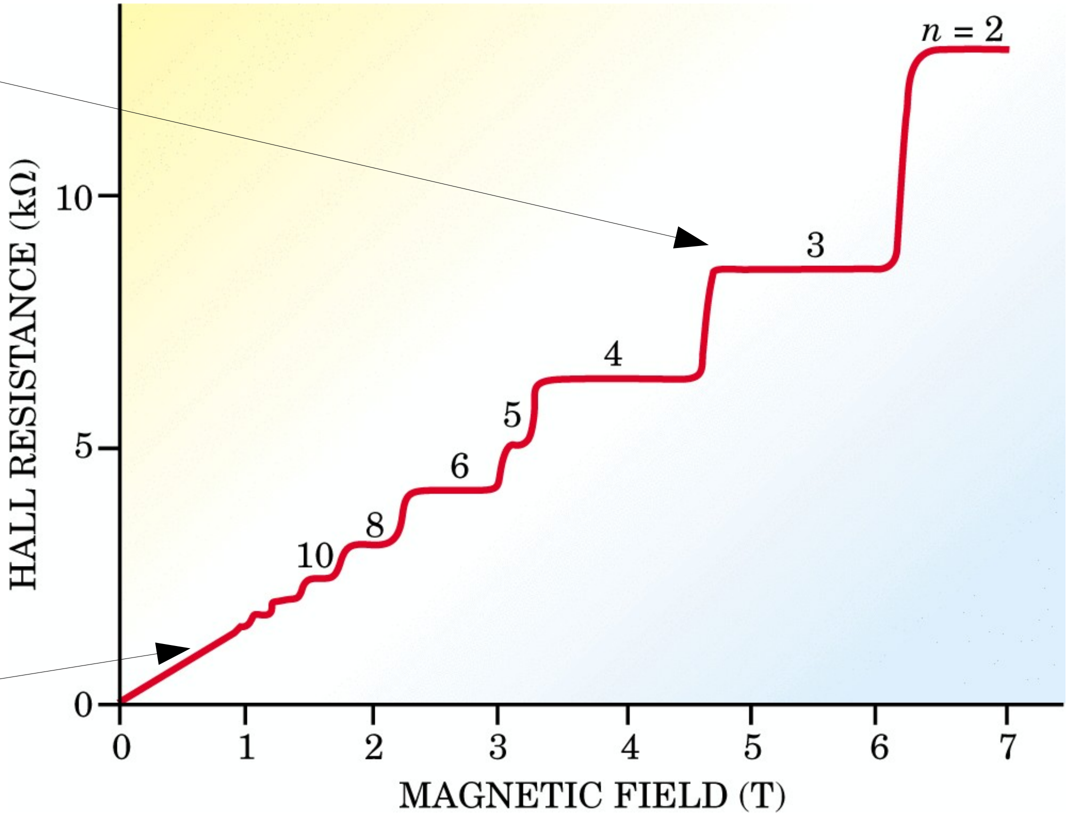
The quantum Hall state

Quantum regime

In the quantum Hall regime

$$\rho_{xy} \sim \sigma_{xy}^{-1}$$

Classical regime



The quantum Hall state

The quantum Hall conductivity $$\sigma_{xy} = \nu \frac{e^2}{h}$$

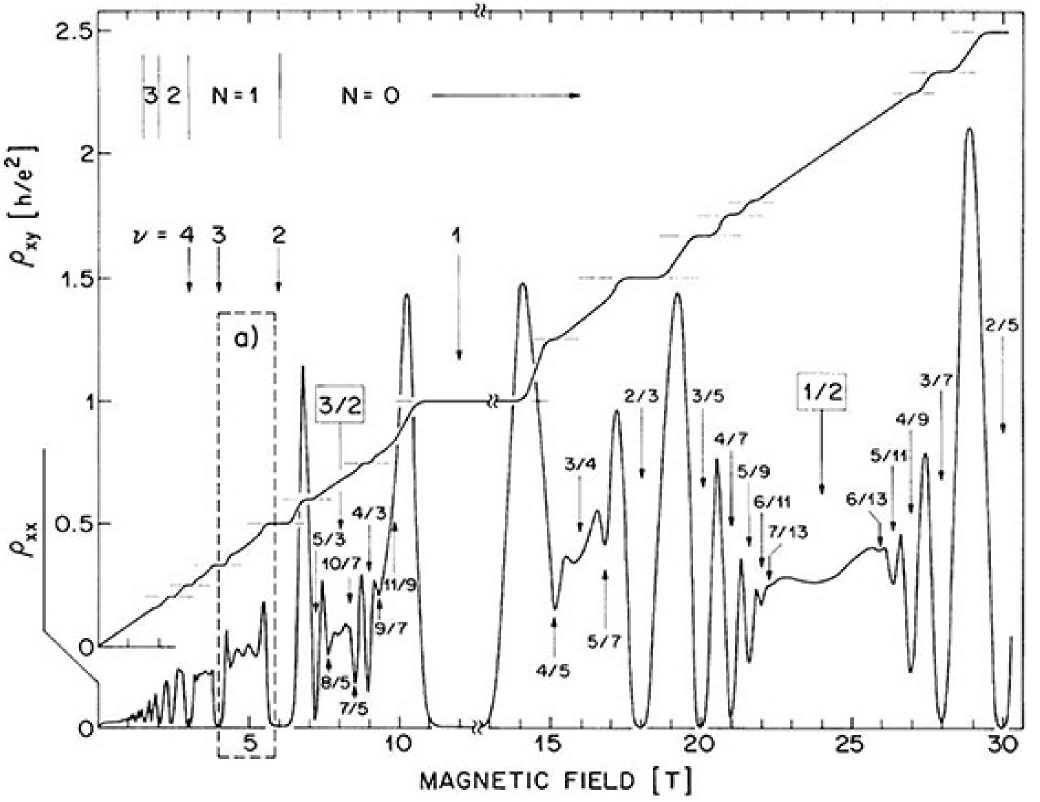
$$\nu \equiv C$$ **Chern number, integer for any band insulator**

<u>However, certain experiments show fractional values of ν</u>

<u>This is fully incompatible with any kind of single particle picture</u>

Fractional conductance

Ultra-clean samples show quantized fractional conductance



Integer VS fractional quantum Hall

- Fractional QH requires ultraclean samples
  - So that interactions overcome disorder
- Fractional QH usually requires higher magnetic fields and lower temperatures
  - Fractional interaction gaps are smaller than LL splitting

Properties of the fractional quantum Hall effect

- Strongly interacting state, no single-particle picture
- Featureless gas: no associated symmetry breaking
- Fractional excitations

Laughlin’s wavefunction

Landau levels in a nutshell

Band-structure in the quantum Hall state

The energy levels are

$$E \sim \left(n + \frac{1}{2}\right) B$$

For a Dirac equation they would be

$$E \sim \sqrt{nB}$$

Complex coordinates

We can define a new complex “spatial” coordinate for our wavefunctions

$$z_n = x_n + iy_n$$

So that the many-body wavefunction is now a function of “complex coordinates”

$$\Psi(x_1, y_1, x_2, y_2, \ldots\ldots, x_n, y_n) \rightarrow \Psi(z_1, z_2, \ldots\ldots, z_n)$$

The Landau levels in the symmetric gauge

Take the symmetric gauge

$$\mathbf{A} = \frac{B}{2}(-y, x, 0) \qquad \nabla \times \mathbf{A} = (0, 0, B)$$

Landau level wavefunctions $$\Psi_m(z) \sim z^m e^{-|z|^2/(4\ell^2)}$$

$$z = x + iy \qquad \ell \sim 1/\sqrt{B}$$

Now that we have the single-particle wavefunctions, we “only” have to solve the many-body interacting problem

The Landau levels in the symmetric gauge

$$\Psi_m(z) \sim z^m e^{-|z|^2/(4\ell^2)}$$

Eigenstates of the angular momentum $\hat{L}\Psi_m = \hbar m \Psi_m$

$$|\Psi_m|^2 \sim r^{2m} \exp\left(-r^2/\left(2\ell^2\right)\right)$$

Maximum of the function at a radius

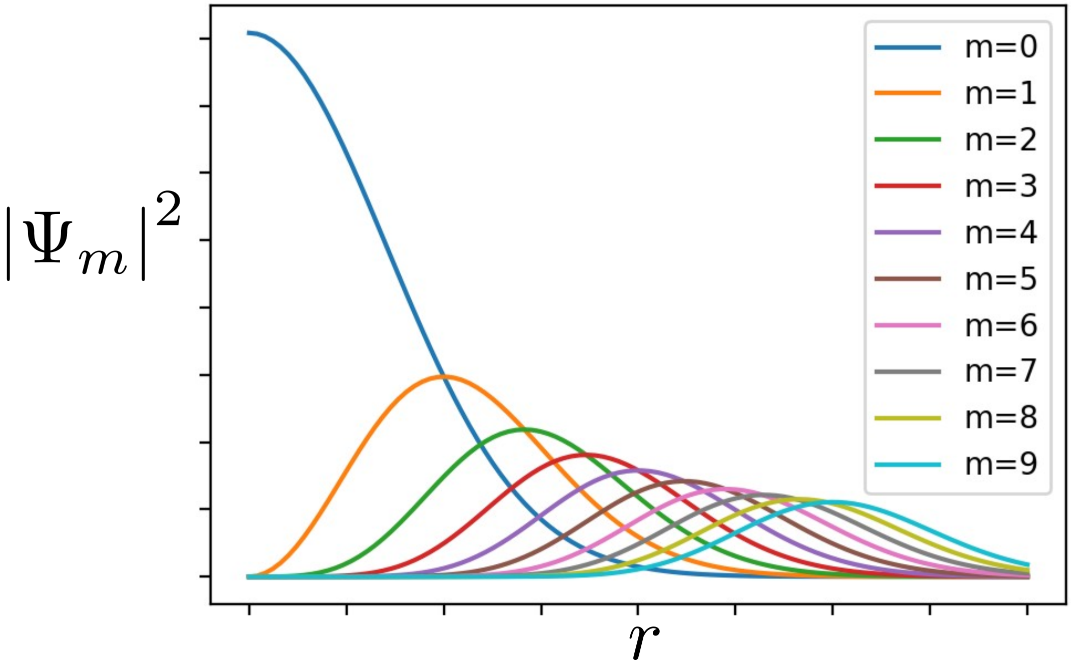
$$r = \ell\sqrt{2m}$$

The Landau levels in the symmetric gauge

$$\Psi_m(z) \sim z^m e^{-|z|^2/(4\ell^2)}$$

Maximum of the function at a radius

$$r = \ell\sqrt{2m}$$



The area enclosed by the Landau level is $\pi r^2 = 2\pi m\ell^2 \sim m/B$

Single-particle wavefunction

Landau level wavefunction

$$\Psi_m(z) \sim z^m e^{-|z|^2/(4\ell^2)}$$

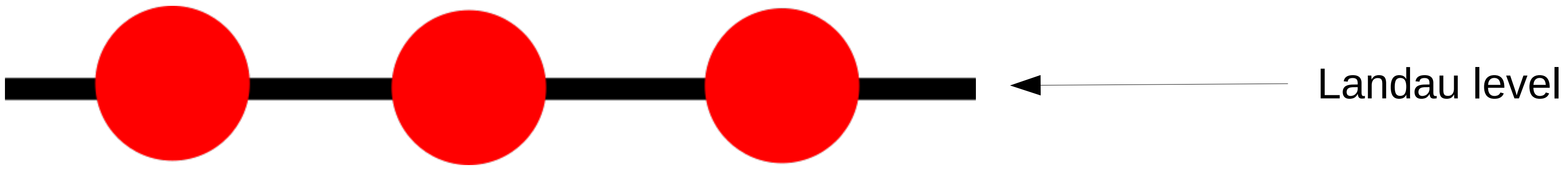
(another) Landau level wavefunction

$$\Psi(z) \sim p(z) e^{-|z|^2/(4\ell^2)}$$

Polynomial

This is for a single electron, how do we extend it to many-electrons?

The filled lowest Landau level



How to build the many-body wavefunction:
- Take all the single particle states
- Make them antisymmetric

The filled lowest Landau level

Many-body wavefunction of the filled lowest Landau level

$$\Psi_{\mathrm{LLL}}(z_1, \cdots, z_{N_\mathrm{p}}) \propto \prod_{i<j}^{N_\mathrm{p}} (z_i - z_j) \exp\left[-\sum_{l}^{N_\mathrm{p}} |z_l|^2/(4\ell^2)\right]$$

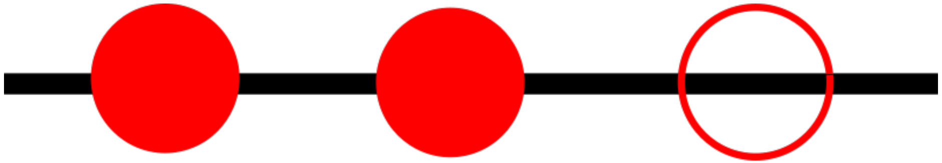
Fully antisymmetrical
(Pauli's principle)

Landau level like

$\ell \sim 1/\sqrt{B}$ $\qquad z_n = x_n + iy_n$

Towards fractional filling

Let us take a fractional filling



Single particle wavefunction for the lowest Landau level

$$\Psi(z) \sim p(z)e^{-|z|^2/(4\ell^2)}$$

Polynomial

How do we write a wavefunction for fractional filling?

$$\Psi(z_1, z_2, \ldots, z_n)$$

The Laughlin’s wave function

Let’s generalize the Many-body LL wavefunction

$$\Psi^{(m)}_{\text{Laughlin}} = \prod_{i<j} (z_i - z_j)^m \prod_{i=1}^{N} e^{-|z_i|^2/\left(4\ell^2\right)}$$

$$\ell \sim 1/\sqrt{B}$$

Notice the m in the exponent

This wavefunction describes a Landau level with

$1/m$ electronic density

excitations with charge $1/m$

Guessing a wavefunction

Solving a problem is much easier when you know the solution

We will take the filled wavefunction as starting point

$$\Psi_{\mathrm{LLL}}(z_1, \cdots, z_{N_{\mathrm{p}}}) \propto \prod_{i<j}^{N_{\mathrm{p}}} (z_i - z_j) \exp\left[-\sum_{l}^{N_{\mathrm{p}}} |z_l|^2/(4\ell^2)\right]$$

Guessing a wavefunction

Solving a Landau level with fractional filling

- What do we need
  - A many body wavefunction
  - That looks like a Landau level
  - That fulfills Pauli's exclusion principle
  - With the coordinates of the electrons highly correlated, avoiding each other (repulsive interactions)

The Laughlin's wave function

Let's generalize the Many-body LL wavefunction

$$\Psi^{(m)}_{\text{Laughlin}} = \prod_{i<j} (z_i - z_j)^m \prod_{i=1}^{N} e^{-|z_i|^2/\left(4\ell^2\right)}$$

$$\ell \sim 1/\sqrt{B}$$

Notice the m in the exponent

(m should be odd for electrons)

This wavefunction describes Landau levels with

$1/m$ electronic density

excitations with charge $1/m$

The Laughlin's wave function

$$\Psi^{(m)}_{\text{Laughlin}} = \prod_{i<j} \left(z_i - z_j\right)^m \prod_{i=1}^{N} e^{-|z_i|^2/\left(4\ell^2\right)}$$

Maximum power of the wavefunction

$$N_\phi = m(N-1)$$

Leading to a filling fraction

$$\nu = N/N_\phi \to 1/m$$

Quasiholes

From the many-body wavefunction, we can write down the form of a quasihole excitations with charge $1/n$

$$\Psi_{qh}(\mathbf{0}) = \left[\prod_{i=1}^{N} z_i\right] \Psi_{\text{Laughlin}}$$

(excitation at r=0)

$$\Psi_{qh}(w) = \left[\prod_{i=1}^{N} (z_i - w)\right] \Psi_{\text{Laughlin}}$$

(excitation at r=w)

Quasiholes

$$\Psi_{qh}(\mathbf{0}) = \left[\prod_{i=1}^{N} z_i\right] \Psi_{\text{Laughlin}}$$

$$m = 1/\nu$$

$$\Psi^{(m)}_{\text{Laughlin}} = \prod_{i<j} (z_i - z_j)^m \prod_{i=1}^{N} e^{-|z_i|^2/\left(4\ell^2\right)}$$

Increase the power of each filled orbital

Leaving an empty orbital with charge

$$e^* = \nu e$$

Quasiholes

Multiple quasiholes (M)

$$\Psi_{qhs}\left(w_1,\ldots,w_M\right)=\left[\prod_{\alpha=1}^{M}\prod_{i=1}^{N}\left(z_i-w_\alpha\right)\right]\Psi_{\mathrm{Laughlin}}$$

Some comments:
- Inserting n quasiholes is like putting a single hole
- It is a zero energy eigenstate (for the exact Hamiltonian with this wavefunction)
- **They have fractional braiding statistics**

Quasielectrons

$$\Psi_{qe}(\mathbf{0}) = \left(\prod_{i=1}^{N} \frac{\partial}{\partial z_i}\right] \phi\right) \prod_{i=1}^{N} e^{-|z_i|^2/\left(4\ell^2\right)}$$

$$m = 1/\nu$$

$$\Psi^{(m)}_{\text{Laughlin}} = \phi \prod_{i=1}^{N} e^{-|z_i|^2/\left(4\ell^2\right)}$$

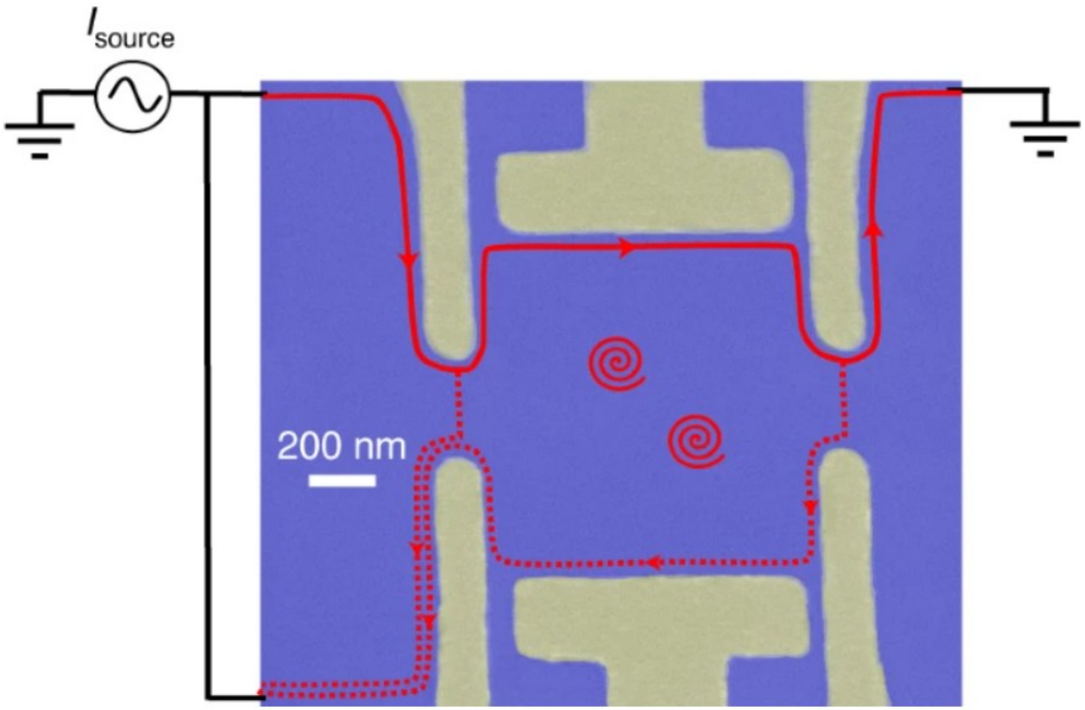
Decrease the power of each filled orbital

Filling an orbital with charge

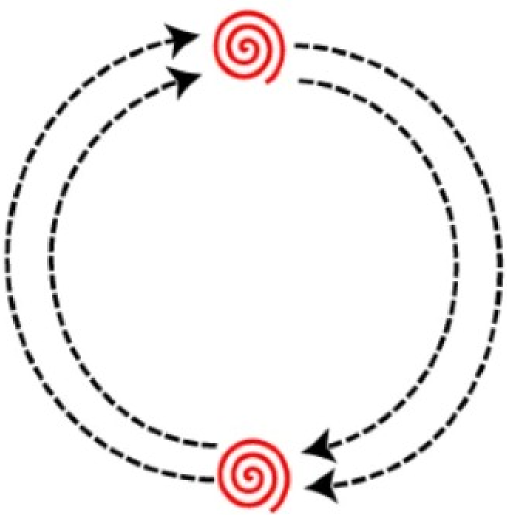
$$e^* = \nu e$$

Fractional statistics from interference

The fractional statistics of FQH states can be seen from interference

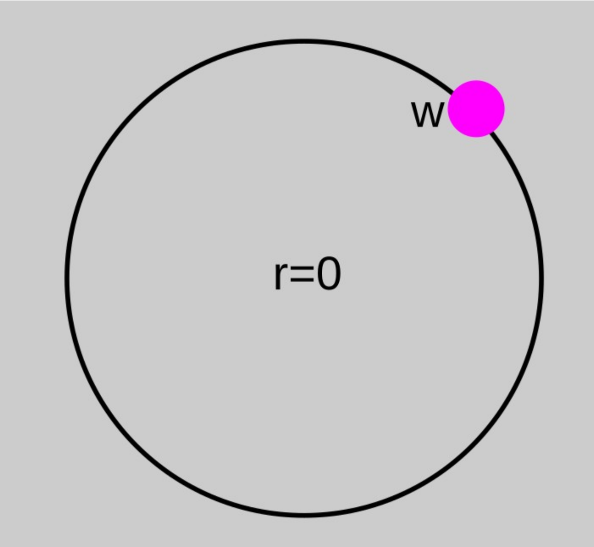


Different paths interfere reflecting the fractional (anyon) statistics



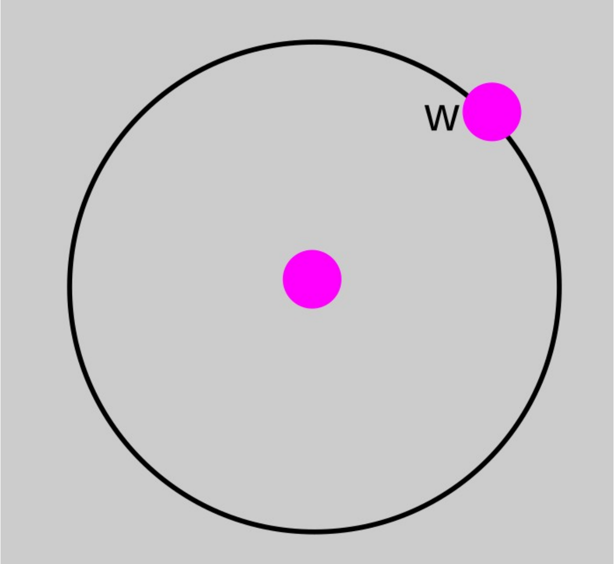
Fractional statistics in a nutshell

Wind a quasihole around the origin



$$\Delta\gamma = \gamma_{AB}$$

Wind a quasihole around a quasihole



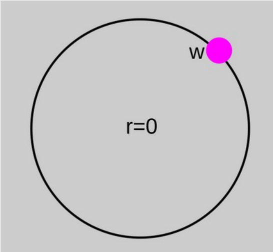
$$\Delta\gamma = \gamma_{AB} + \gamma_{\text{statistical}}$$

Aharonov-Bohm phase for charge e/m

$$\gamma_{\text{statistical}} = 2\pi/m$$

Fractional statistics in a nutshell

Recall the definition of a Berry phase



$$\Delta\gamma = -i\int_0^{2\pi} d\theta \left\langle \Psi(\theta) \left| \partial_\theta \right| \Psi(\theta) \right\rangle$$

Take a wavefunction with a single quasihole $\Psi(w) \sim \left[\prod_{i=1}^{N} (z_i - w)\right] \Psi^{(m)}_{\mathrm{Laughlin}}$

$w = |w|e^{i\theta}$ And wind it around the origin, obtaining the phase

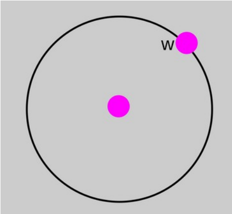
$$\Delta\gamma = 2\pi(1/m)\Phi/\phi_0 = \gamma_{AB}$$

Aharonov-Bohm phase for charge e/m

Enclosed flux

Quantum flux

Fractional statistics in a nutshell



Recall the definition of a Berry phase

$$\Delta\gamma = -i\int_0^{2\pi} d\theta \, \langle \Psi(\theta) \, | \partial_\theta | \, \Psi(\theta) \rangle$$

Take a wavefunction with two quasihole

$$\Psi(w) \sim \left[\prod_{i=1}^{N} (z_i)\right] \left[\prod_{i=1}^{N} (z_i - w)\right] \Psi^{(m)}_{\text{Laughlin}}$$

$$w = |w| e^{i\theta}$$

And wind it around the origin, obtaining the phase

$$\Delta\gamma = \gamma_{AB} + \gamma_{\text{statistical}}$$

Aharonov-Bohm phase

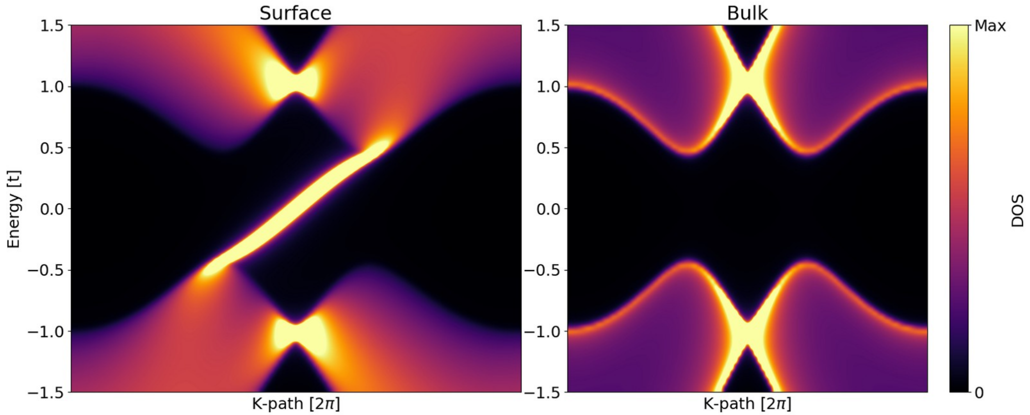
Statistical phase

$$\gamma_{\text{statistical}} = 2\pi/m$$

Fractional Chern insulators

Fractional quantum Hall effect without Landau levels

- In session 4 & 5, we saw that we could have quantum Hall effect without Landau levels
- So, can we have fractional quantum Hall effect without Landau levels?

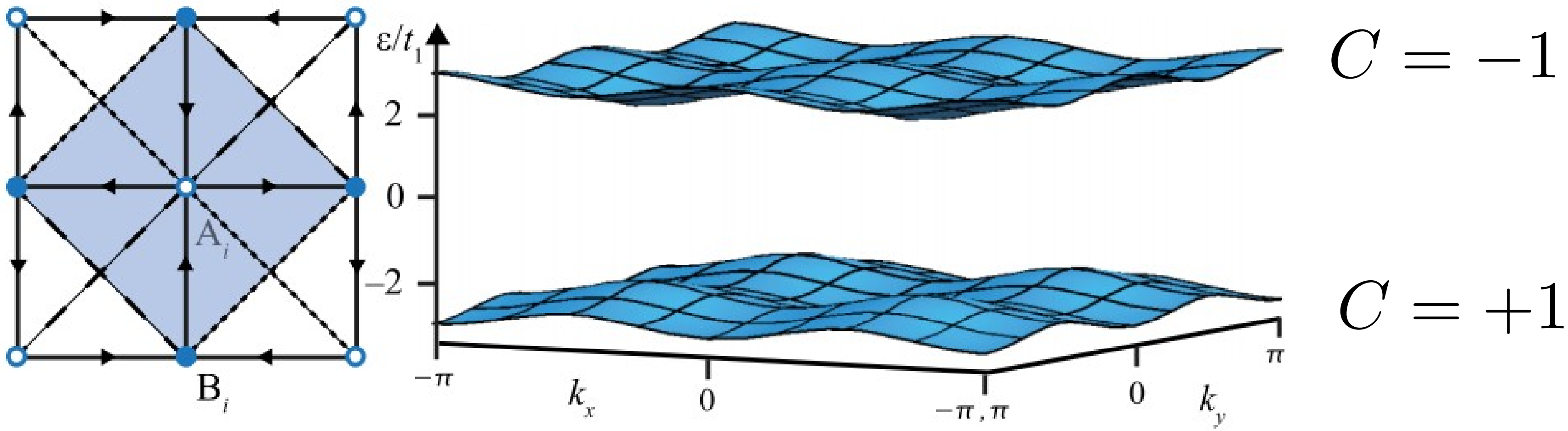


Fractional quantum Hall effect without Landau levels

- The ingredients for fractional Chern insulators: emulating Landau levels
  - Topologically non-trivial bands (with finite Chern number)
  - Flat bands (leading to strong interactions)

Exact calculations of fractional Chern insulators

Let us take a model with topological flat bands

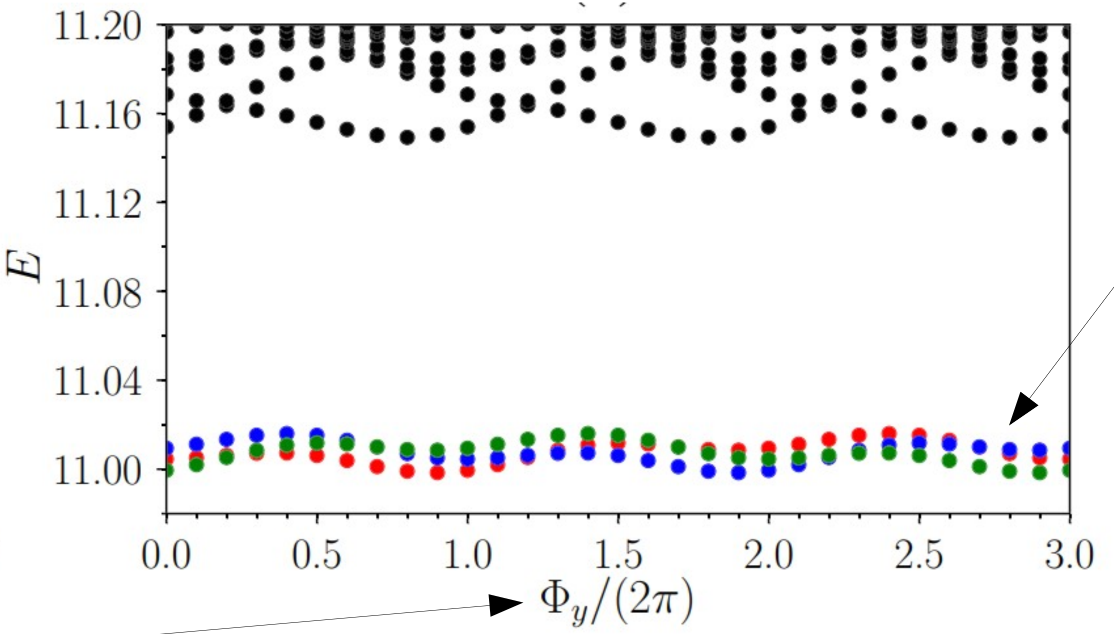


$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ijkl} V_{ijkl} c_i^\dagger c_j c_k^\dagger c_l$$

How do we see if it has a fractional Chern insulating state?

Laughlin's argument for conductance quantization

Inserting a flux pumps an integer number of electrons (Hall conductance)



Three-fold Degeneracy (hallmark of FQH)

Many-body energy

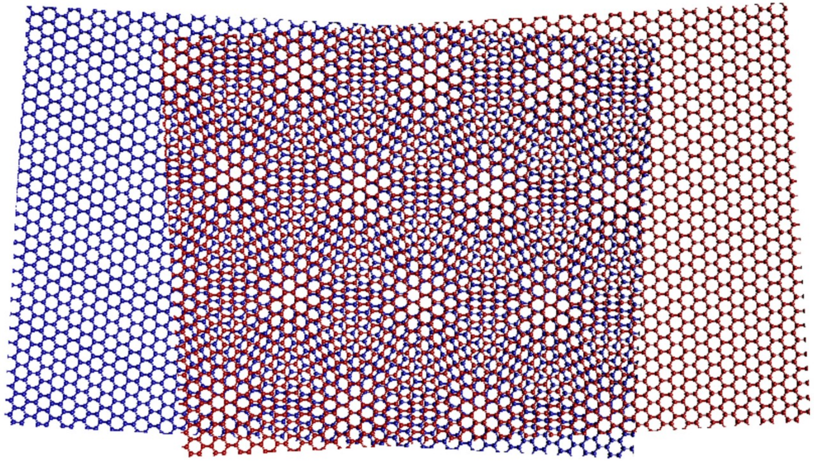
Flux

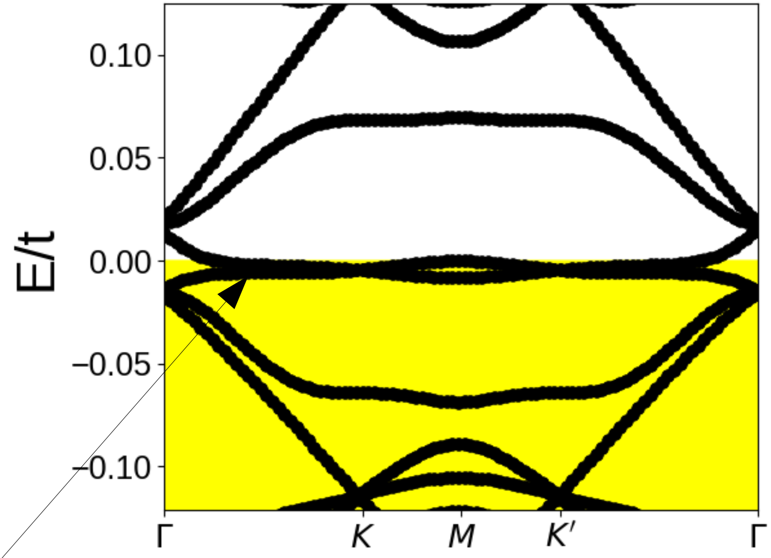
$$\xrightarrow{\text{insert } \phi_0} |GS_1\rangle \xrightarrow{\text{insert } \phi_0} |GS_2\rangle \xrightarrow{\text{insert } \phi_0} |GS_3\rangle \xrightarrow{\text{insert } \phi_0} |GS_1\rangle \xrightarrow{\text{insert } \phi_0}$$

Materials potentially showing fractional Chern insulators

Twisted 2D materials

```
from pyqula import specialhamiltonian # special Hamiltonians library
h = specialhamiltonian.twisted_bilayer_graphene(13,ti=0.4) # TBG Hamiltonian
(k,e) = h.get_bands(num_bands=20) # compute band structure
```





(pseudo) Landau levels arising from an elastic gauge field

Fractional quantum Hall observed at 5 T in twisted graphene bilayers

Take home

- Interactions in Landau levels lead to fractional quantum Hall states
- Excitations in FQH are fractional (non-integer charge)
- Fractional quantum Hall effect can (theoretically) exist without Landau levels