Public Economics 2
Problem Set 2
Suggested solutions

1. *(40pts)*

$$\begin{aligned} \text{cost of boat:} r &> 0 \\ \text{number of boats:} b &\geq 0 \\ \text{amount of fish:} f(b), f'(b) &> 0, f''(b) < 0 \\ \text{price of fish:} p &> 0 \end{aligned}$$

Each boat catches $\frac{f(b)}{b}$ fish.

1.1 (10pts) The profit of the individual boat fishing is:

$$\pi(b) = \frac{pf(b)}{b} - r \tag{1}$$

The number of boats fishing can be solved by setting the profit equation equal to zero. The number of boats fishing in equilibrium is:

$$b = \frac{pf(b)}{r} \tag{2}$$

We also see that at optimum,

$$\frac{f(b)}{b} = \frac{r}{p} \tag{3}$$

1.2 (10pts) The optimal level of boats is solved from the total profit function:

$$\pi_{tot}(b) = pf(b) - br \tag{4}$$

This is maximized when

$$pf'(b_{tot}) - r = 0 \tag{5}$$

In the social optimum,

$$f'(b_{tot}) = \frac{r}{p} \tag{6}$$

From equation (3) and equation (6) follow that

$$f'(b_{tot}) = \frac{f(b)}{b} \tag{7}$$

By the definition of the function f, this holds only if $b > b_{tot}$. More boats will fish on the lake than is socially optimal.

1.3 (10pts) The private decision to fish on the lake is only determined by the expected profit of the individual and does not take into account how the decision to fish affects the amount of fish in the lake. This is taken into account in the social optimum, which is why these two levels differ.

1.4 (10pts) The government needs to set a tax such that each fishing boat pays for the decrease in fish. This tax is set such that

$$t = \frac{pf(b)}{b} - pf'(b_{tot}) \tag{8}$$

2. (*60pts*)

2.1 (10pts) The budget constraints of the individual are:

$$\text{Period 1: } c_1 + s = 300$$
$$\text{Period 2: } c_2 = (1+r)s$$

Solve the maximization problem using the Lagrangian to find optimal savings:

$$\mathcal{L} = \frac{3}{5}ln(c_1) + \frac{2}{5}ln(c_2) - \lambda_1(c_1 + s - 300) - \lambda_2(c_2 - (1+r)s)$$

FOCs:

$$\frac{d}{dc_1} = \frac{3}{5}\frac{1}{c_1} - \lambda_1 = 0 \tag{9}$$
$$\frac{d}{dc_2} = \frac{2}{5}\frac{1}{c_2} - \lambda_2 = 0 \tag{10}$$
$$\frac{d}{ds} = -\lambda_1 + \lambda_2(1+r) = 0 \tag{11}$$
$$\frac{d}{d\lambda_1} = c_1 + s - 300 = 0 \tag{12}$$
$$\frac{d}{d\lambda_2} = c_2 - (1+r)s = 0 \tag{13}$$

From equation (1) and (2):

$$\frac{3}{5c_1} = \lambda_1 \;\; \& \;\; \frac{2}{5c_2} = \lambda_2$$

This can be plugged into equation (3), giving:

$$\begin{aligned}
&-\frac{3}{5c_1} + \frac{2}{5c_2}(1+r) = 0 \\
\Leftrightarrow &\frac{2}{5c_2}(1+r) = \frac{3}{5c_1} \\
\Leftrightarrow &\frac{2}{c_2}(1+r) = \frac{3}{c_1} \\
\Leftrightarrow &2(1+r) = \frac{3c_2}{c_1} \\
\Leftrightarrow &2c_1(1+r) = 3c_2 \\
\Leftrightarrow &c_2 = \frac{2}{3}c_1(1+r)
\end{aligned}$$

Fro, the second period budget constraint and the expression for c_2 follows that

$$s = \frac{2}{3}c_1$$

Therefore,

$$\begin{aligned}
&c_1 + \frac{2}{3}c_1 = 300 \\
\Leftrightarrow \; &\frac{5}{3}c_1 = 300 \\
\Leftrightarrow \; &c_1 = 300\frac{3}{5} = \frac{900}{5} = 180
\end{aligned}$$

and

$$s = \frac{2}{3}c_1 = \frac{2}{3}180 = \frac{360}{3} = 120$$

The personal savings of a patient individual are 120.

2.2 a. (2pts) This is a funded social security system.

b. (6pts) The new budget constraints of the individual under the social security system are:

$$\begin{aligned}\text{Period 1: } & c_1 + s = 300 - 50 \\ \text{Period 2: } & c_2 = (1+r)(s+50)\end{aligned}$$

The utility function is the same and we can see that the change in the budget constraints does not affect the first order conditions. Thus, as

$$c_2 = \frac{2}{3}c_1(1+r)$$

follows that

$$\frac{2}{3}c_1 = s + 50 \Leftrightarrow s = \frac{2}{3}c_1 - 50$$

From the budget constraint for period 1,

$$\begin{aligned}& c_1 + \frac{2}{3}c_1 - 50 = 300 - 50 \\ \Leftrightarrow\ & \frac{5}{3}c_1 = 300 \\ \Leftrightarrow\ & c_1 = \frac{900}{5} = 180\end{aligned}$$

The personal savings of a patient individual, given the new social security system, are

$$s = 300 - 50 - c_1 = 250 - 180 = 70$$

c. (2pts) For the rational individual, this social security system does not affect welfare. Consumption in both periods stays the same, as the savings decrease following the mandatory social security.

2.3 a. (6pts)

As the utility function of the impatient individual is different, the optimization must be redone, following the method in exercise 2.1. The budget constraints do not change.

$$\mathcal{L} = \frac{9}{10}ln(c_1) + \frac{1}{10}ln(c_2) - \lambda_1(c_1 + s - 300) - \lambda_2(c_2 - (1+r)s)$$

FOCs:

$$\frac{d}{dc_1} = \frac{9}{10}\frac{1}{c_1} - \lambda_1 = 0 \tag{14}$$
$$\frac{d}{dc_2} = \frac{1}{10}\frac{1}{c_2} - \lambda_2 = 0 \tag{15}$$
$$\frac{d}{ds} = -\lambda_1 + \lambda_2(1+r) = 0 \tag{16}$$
$$\frac{d}{d\lambda_1} = c_1 + s - 300 = 0 \tag{17}$$
$$\frac{d}{d\lambda_2} = c_2 - (1+r)s = 0 \tag{18}$$

Substituting equation (6) and (7) into equation (8) gives:

$$\begin{aligned}
&-\frac{9}{10c_1} + \frac{1}{10c_2}(1+r) = 0 \\
\Leftrightarrow &\frac{1}{10c_2}(1+r) = \frac{9}{10c_1} \\
\Leftrightarrow &c_1(1+r) = 9c_2 \\
\Leftrightarrow &c_2 = \frac{1}{9}c_1(1+r) \\
\Leftrightarrow &s = \frac{1}{9}c_1
\end{aligned}$$

Plug this into equation (9):

$$\begin{aligned}
c_1 + \frac{1}{9}c_1 &= 300 \\
\Leftrightarrow c_1 &= 300\frac{9}{10} = 270
\end{aligned}$$

The optimal consumption and savings are:

$$\begin{aligned}
c_1^* &= 300\frac{9}{10} = 270 \\
s^* &= \frac{1}{9}c_1 = 30 \\
c_2^* &= 30(1+r)
\end{aligned}$$

b. (6pts) The budget constraints are now the same as in 2.2.b. From first order conditions we therefore get

$$s = \frac{1}{9}c_1 - 50$$

and following that,

$$\begin{aligned}
c_1 + \frac{1}{9}c_1 - 50 &= 250 \\
\Leftrightarrow c_1 &= 300\frac{9}{10} = 270
\end{aligned}$$

We have no restrictions on borrowing, which allows for

$$s = \frac{1}{9} \cdot 270 - 50 = 30 - 50 = -20$$

Optimal allocations are:

$$\begin{aligned} c_1^* &= 270 \\ c_2^* &= 30(1+r) \\ s^* &= -20 \end{aligned}$$

c. (4pts) If borrowing is not possible, the impatient individual will have to decrease its consumption to the nearest feasible level. The optimal allocations will be:

$$\begin{aligned} c_1^* &= 250 \\ c_2^* &= 50(1+r) \\ s^* &= 0 \end{aligned}$$

d. (2pts) This social security system does not affect individual welfare, as long as borrowing is allowed.

e. (2pts) For the impatient individual, the impact is zero given working credit markets. When borrowing is not allowed, we see a decrease in welfare of the individual.

2.4 a. (2pts) The direct reallocation of money makes it an unfunded system.

b. (6pts) The budget constraints of the individuals in this system are:

$$\begin{aligned} &\text{Period 1: } c_1 + s = 300 - 50 \\ &\text{Period 2: } c_2 = (1+r)s + 50 \end{aligned}$$

The optimization problem is otherwise the same as in exercise 2.1 and 2.2. Thus we can solve the optimal savings from

$$c_2 = \frac{2}{3} c_1 (1+r)$$

and the budget constraints.

$$\begin{aligned} &\frac{2}{3} c_1 (1+r) = (1+r)s + 50 \\ &\Leftrightarrow \frac{2}{3} c_1 = s + \frac{50}{1+r} \\ &\Leftrightarrow c_1 = \frac{3}{2} \left(s + \frac{50}{1+r} \right) \end{aligned}$$

Plugging this into the budget constraint of the first period gives

$$\begin{aligned} &\frac{3}{2} \left(s + \frac{50}{1+r} \right) + s = 250 \\ &\Leftrightarrow s + \frac{2}{3} s + \frac{50}{1+r} = \frac{500}{3} \\ &\Leftrightarrow \frac{5}{3} s = \frac{500}{3} - \frac{50}{1+r} \\ &\Leftrightarrow s = 100 - \frac{30}{1+r} \end{aligned}$$

Assuming that $r > 0$, individuals save a little less during this system, than in the unfunded system where $s^* = 70$, as they get the social security benefit without returns.

c. (2pts) This system does not directly acquire national savings, as the money is not saved but transferred immediately to a consumer.

2.5 a. (2pts) When the baby boom generation is young and work, the benefits of the older generation will be doubled compared to the usual.

b. (2pts) On the other hand, when the baby boom generation is old, they will receive half of the original benefit.

c. (2pts) The government could move to a partly funded system, where the social security payments that excess 50$ would be saved for the baby boom generation.

2.6 a. (2pts) The funded system, as everyone pays for their own pension benefits and different cohort sizes do not affect sizes of payments.

b. (2pts) This would mean that either the baby boomers would get no benefits, or the social security payment has to be doubled for the current working generation, to pay for the baby boom generation and the future benefits for the working generation. Neither of this is politically feasible, which is why nations do not switch today.