Practical Quantum Computing

Week 2
Gates and Circuits

Computational Basis

We normally expand the wavefunction in terms of a basis of bit strings: the **computational basis**, aka the Z basis.

$$|\psi\rangle = \frac{i}{\sqrt{3}}|010\rangle + \frac{\sqrt{2}}{\sqrt{3}}|111\rangle$$

2^n amplitudes for n qubits.

Other bases are sometimes convenient, e.g., the X basis

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

States as Column Vectors

$$|0\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle \leftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

States as Column Vectors

$$|00\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |01\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

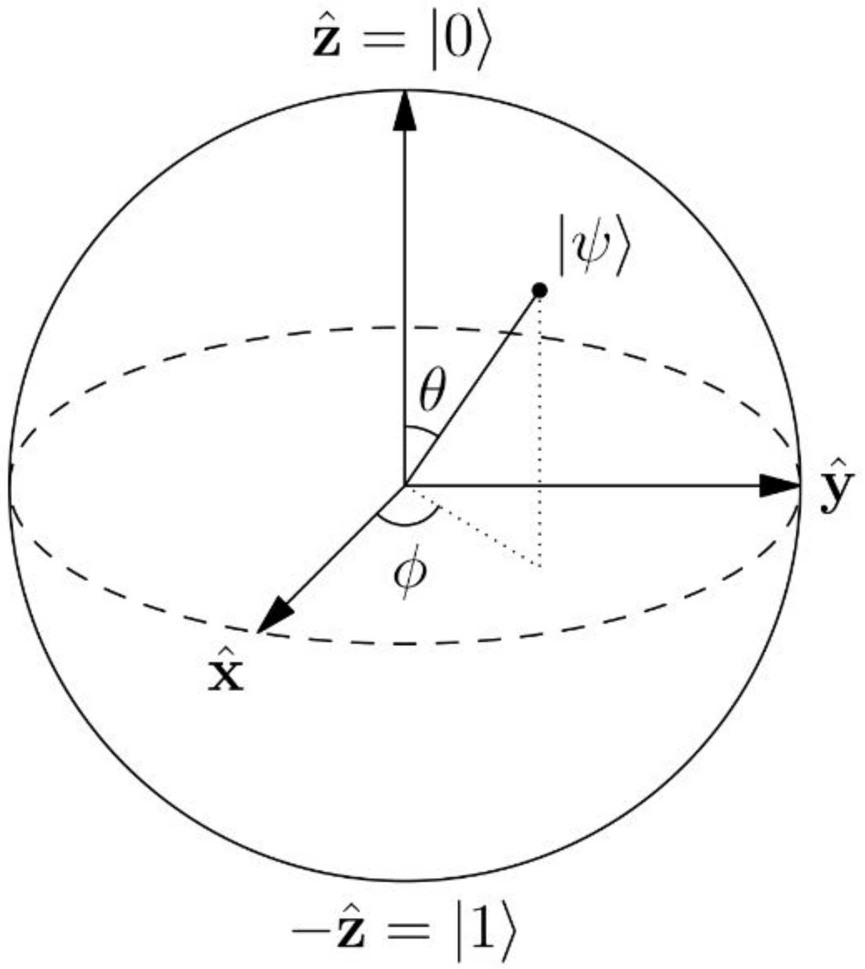
$$|10\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Bloch Sphere

Antipodal points = **orthogonal** states (perfectly distinguishable)

Rotations = **unitary** operations

No convenient analogue for multiple qubits, but still useful for a single qubit



Measurement and Born Rule

Quantum state is not directly observable --- sampling from the Born distribution is all we can do.

Quantum computer outputs 1s and 0s.

Probability = Absolute Value Squared of Amplitude.

Repeating a measurement **immediately** returns the **same** answer.

Must repeat the whole experiment to resample from the distribution.

Expectation Values

The averages of quantities can also be calculated from the wavefunction.

Expectation values are not directly observable: only recoverable after many measurements as the mean.

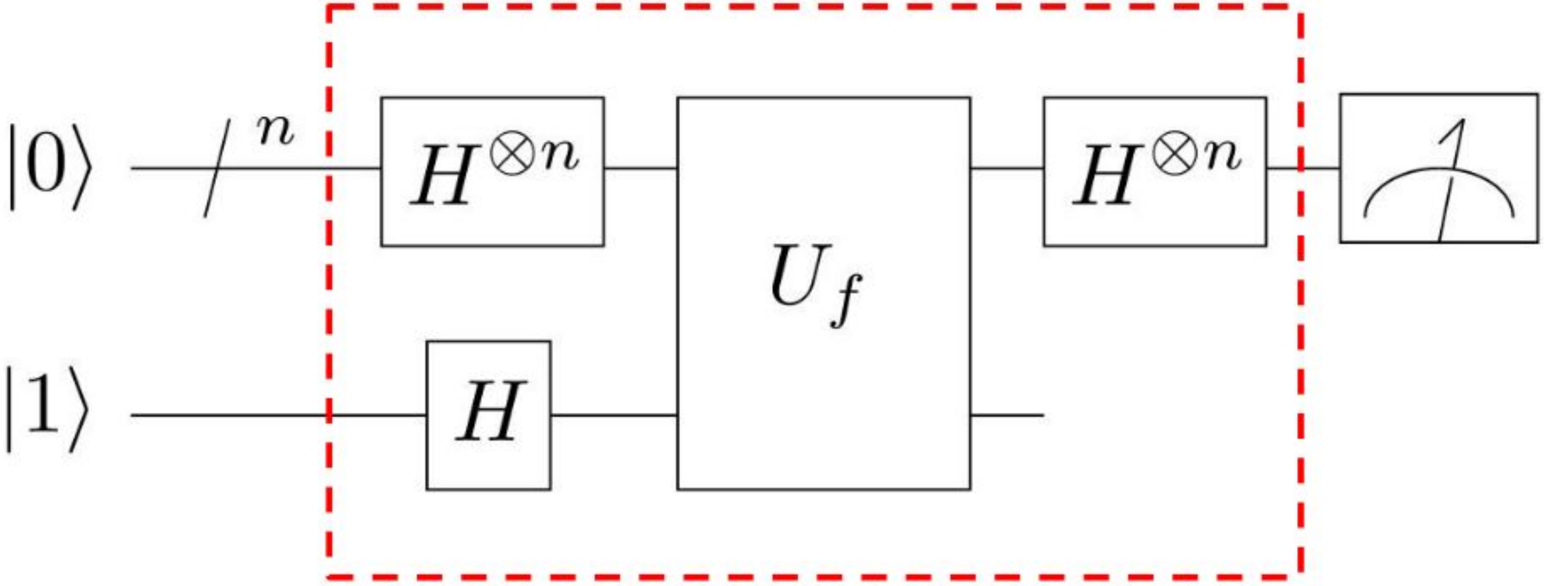
Consequence of the Born rule, not a separate axiom.

$$\langle X \rangle = \langle \psi | X | \psi \rangle = \text{ Expectation value}$$

Quantum Circuit = Time Evolution

We construct the time evolution operator from simple building blocks.

Those building blocks are the quantum gates.



Operators as Matrices

Just like how we represent states as column vectors, we can represent operators as matrices which act on those column vectors.

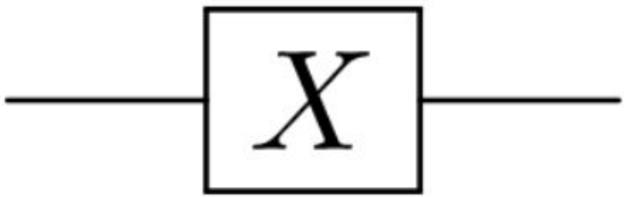
For example, the X operator: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pauli-X (NOT)

$$|0\rangle \mapsto |1\rangle$$
$$|1\rangle \mapsto |0\rangle$$

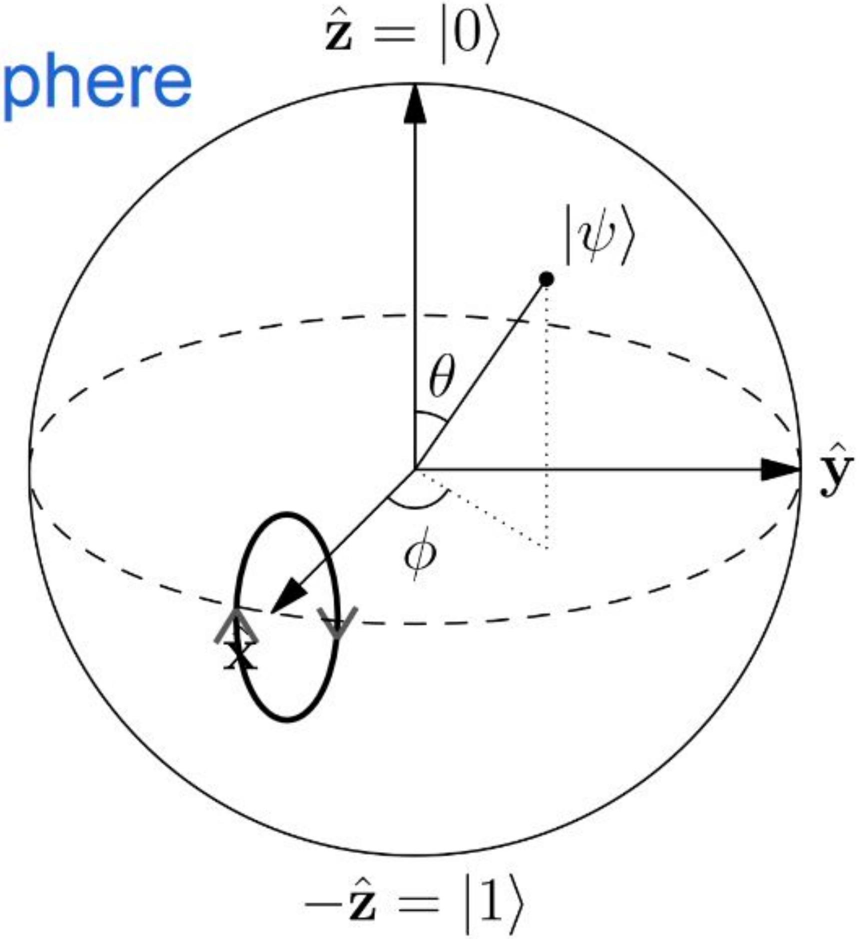
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



X Operator on the Bloch Sphere

Rotates around the X axis by 180°

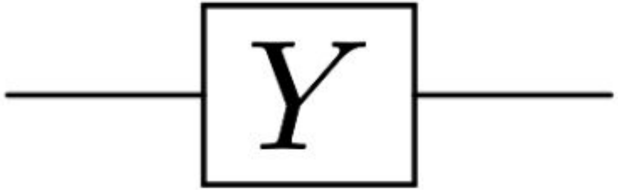
Clockwise or counterclockwise?



Pauli-Y

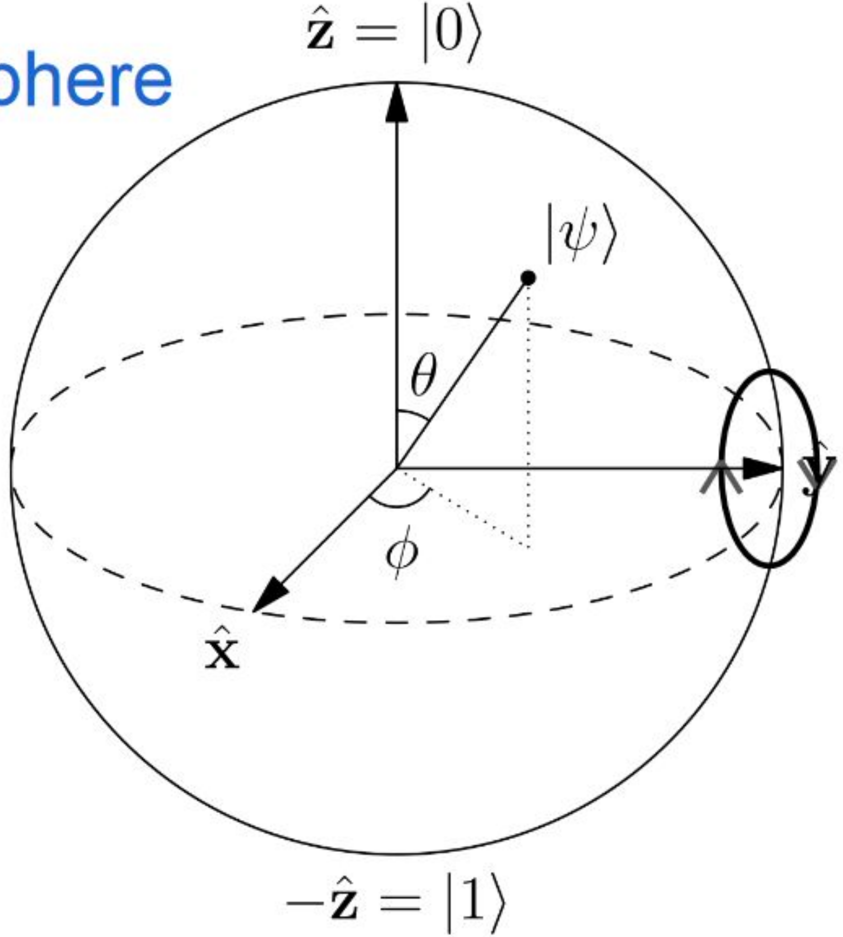
$$|0\rangle \mapsto i|1\rangle$$
$$|1\rangle \mapsto -i|0\rangle$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$



Y Operator on the Bloch Sphere

Rotates around the Y axis by 180°



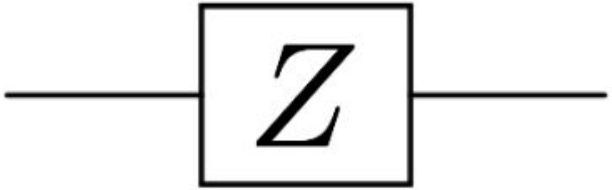
Pauli-Z (Phase Flip)

Diagonal in the computational basis

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

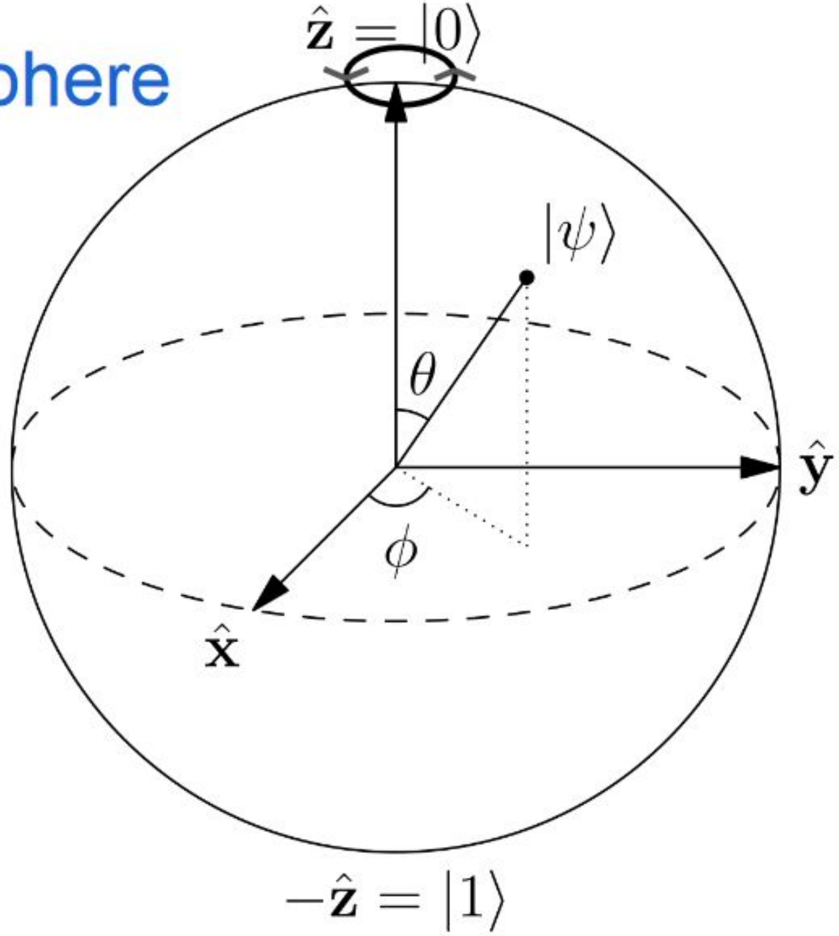
$$|0\rangle \mapsto |0\rangle$$

$$|1\rangle \mapsto -|1\rangle$$



Z Operator on the Bloch Sphere

Rotates around the Z axis by 180°



Hadamard Gate

$$|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

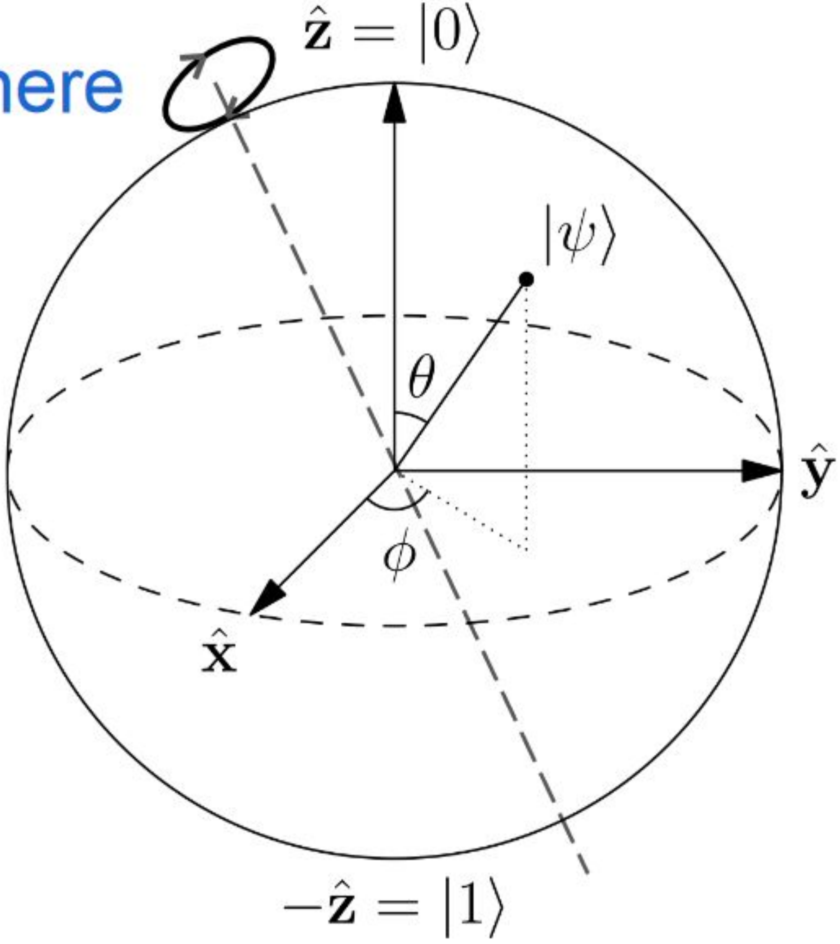
$$H = \frac{1}{\sqrt{2}}(X + Z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

H Operator on the Bloch Sphere

Rotates around the “X+Z” axis by 180°

Exchanges X with Z

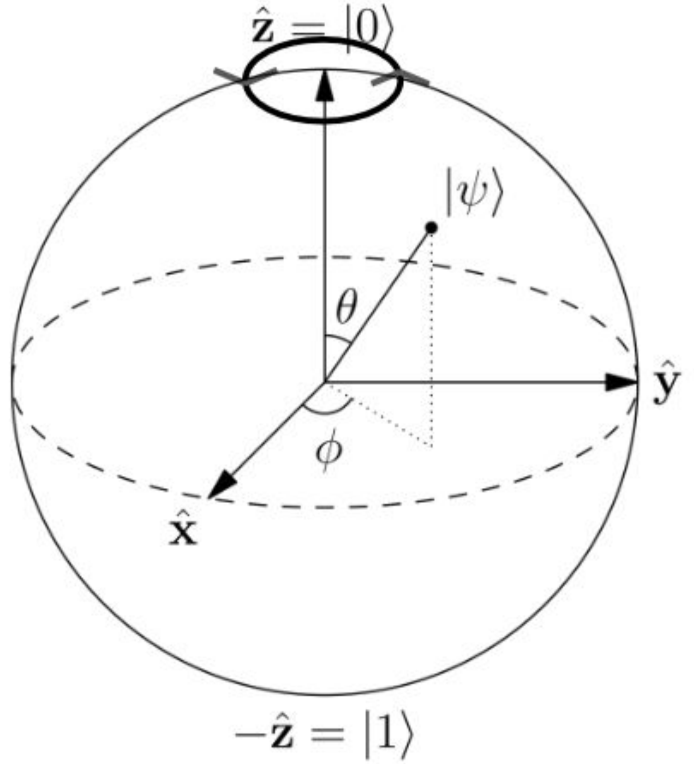
$$HZH = X$$



Arbitrary Rotations of the Bloch Sphere

Exponentiate the Z operator to rotate by an arbitrary angle around the Z axis.

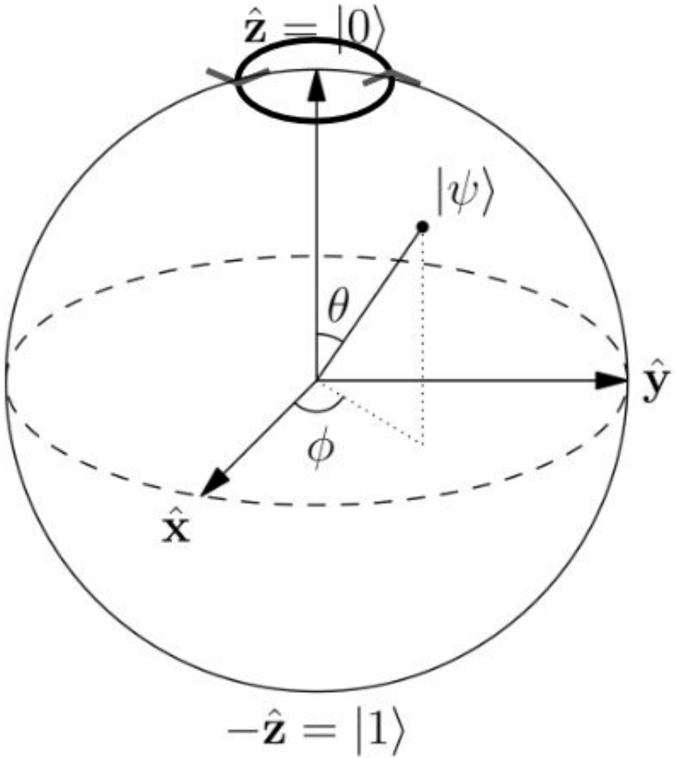
$$e^{-i\pi x Z} = \begin{pmatrix} e^{-i\pi x} & 0 \\ 0 & e^{i\pi x} \end{pmatrix}$$



Arbitrary Rotations of the Bloch Sphere

What about a ½ rotation? Shouldn't that just be Z?

$$e^{-i\pi Z/2} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$



Arbitrary Rotations of the Bloch Sphere

Quarter-rotation and eighth-rotation have names (up to overall phase).

$$
\begin{aligned}
e^{-i\pi Z/4} &= \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \\
&= e^{-i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\
&= e^{-i\pi/4} S
\end{aligned}
$$

$$
\begin{aligned}
e^{-i\pi Z/8} &= \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \\
&= e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \\
&= e^{-i\pi/8} T
\end{aligned}
$$

Arbitrary Rotations of the Bloch Sphere

Trick for exponentiating certain operators. Works because $Z^2 = 1$.

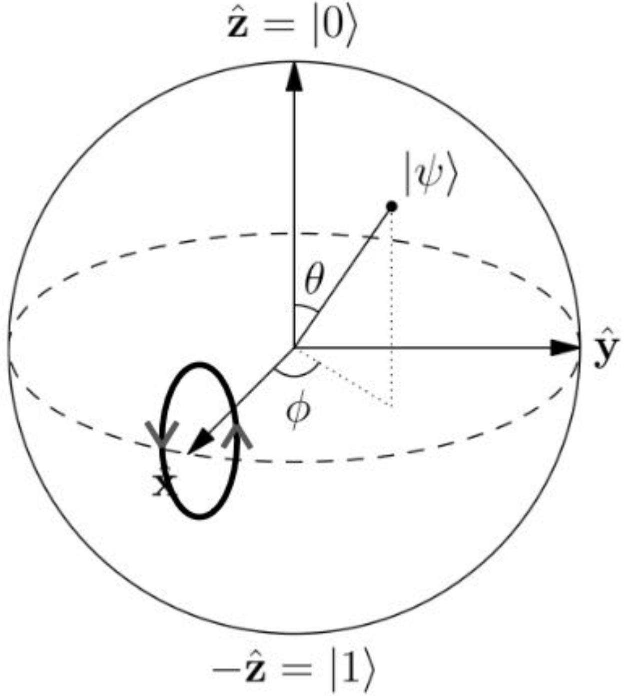
Similar formula for other Pauli matrices.

$$
\begin{aligned}
e^{-i\pi x Z} &= cos\pi x - i(\sin \pi x)Z \\
&= \begin{pmatrix} e^{-i\pi x} & 0 \\ 0 & e^{i\pi x} \end{pmatrix}
\end{aligned}
$$

Arbitrary Rotations of the Bloch Sphere

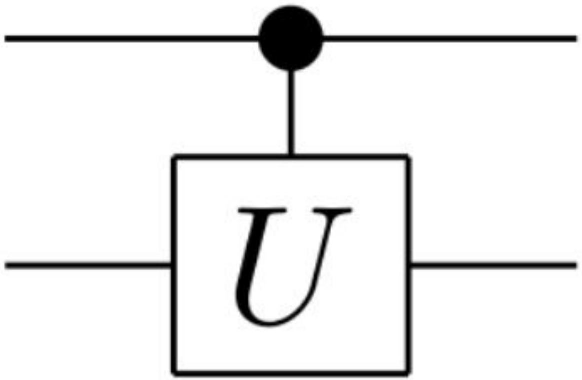
Exponentiate the X operator to rotate by an arbitrary angle around the X axis, similarly with Y

$$e^{-i\pi x X} = \begin{pmatrix} \cos \pi x & -i \sin \pi x \\ -i \sin \pi x & \cos \pi x \end{pmatrix}$$



Controlled Gates

Acts as unitary operator U on the target qubit when the control qubit is in the |1> state.



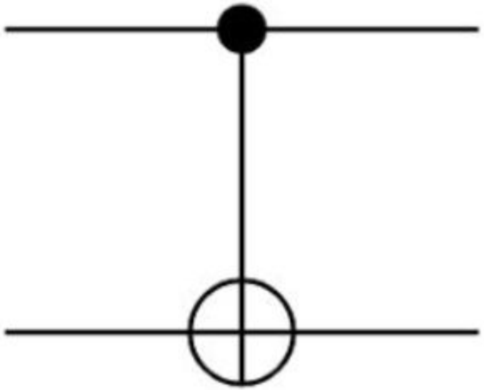
$$\mathrm{CU} = \begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{U}} \\ 0 & 0 & & \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

Controlled NOT (CNOT)

If the control bit is |0>, the target bit is left unchanged.

If the control bit is |1> then the target bit is flipped.

$$\mathrm{CNOT} = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

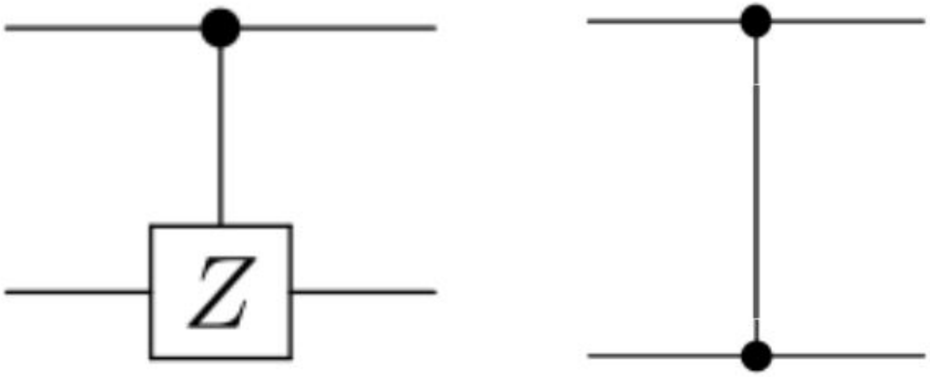


Controlled Z (CZ)

Acts as Z on the target qubit when the control bit is |1>.

CZ is symmetric between the two qubits --- it doesn't matter which bit is the control!

$$\begin{array}{cc} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \mathrm{CZ} = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$



Controlled Rotation

Acts as Z rotation on the target qubit when the control bit is |1>.

How does this compare with CZ?

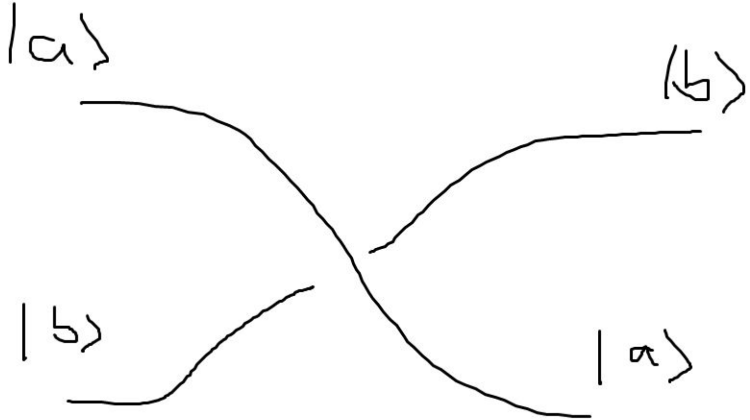
$$\mathrm{CR} = \begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

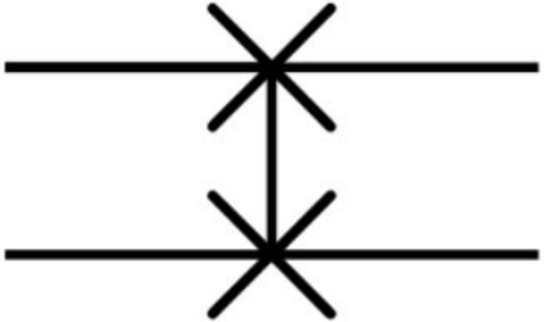
SWAP

Exchanges the states of two qubits.

Equivalent to “crossing the wires.”

$$\mathrm{SWAP} = \begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$





Tensor Product Gates

$$Z \otimes I = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$$I \otimes Z = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$$Z \otimes Z = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

Tensor Product Gates

$$Z \otimes I \otimes I \qquad \mathrm{diag}(+1, +1, +1, +1, -1, -1, -1, -1)$$

$$I \otimes Z \otimes I \qquad \mathrm{diag}(+1, +1, -1, -1, +1, +1, -1, -1)$$

$$I \otimes I \otimes Z \qquad \mathrm{diag}(+1, -1, +1, -1, +1, -1, +1, -1)$$