3. Osittaisintegrointi

$$\int u'(t)v(t)\,dt = u(t)v(t) - \int u(t)v'(t)\,dt$$

a) $\int x\sin(x)\,dx$

Valitaan
$u'(x) = \sin(x) \Leftarrow u(x) = -\cos(x)$
$v(x) = x \Rightarrow v'(x) = 1$

$= -x\cos(x) + \int \cos(x)\,dx$

$= \underline{\underline{-x\cos(x) + \sin(x) + C}}$, $C \in \mathbb{R}$

b) $\int x^2 \sin(x)\,dx$

Valitaan
$u'(x) = \sin(x) \Leftarrow u(x) = -\cos(x)$
$v(x) = x^2 \Rightarrow v'(x) = 2x$

$= -x^2\cos(x) + \int 2x\cos(x)\,dx$

Valitaan
$u'(x) = \cos(x) \Leftarrow u(x) = \sin(x)$
$v(x) = 2x \Rightarrow v'(x) = 2$

$= -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx$

$= \underline{\underline{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}}$, $C \in \mathbb{R}$

c) $\int \ln^2 x\,dx$

Valitaan
$u'(x) = 1 \Leftarrow u(x) = x$
$v(x) = \ln^2 x \Rightarrow v'(x) = 2\ln x \cdot \frac{1}{x}$

$= x\ln^2 x - \int 2x\ln x \frac{1}{x}\,dx = x\ln^2 x - 2\int \ln x\,dx$

$\Bigg\{$ Otetaanpa $\int \ln x\,dx$ lähempään tarkasteluun.

$\frac{d}{dx}\ln x = \frac{1}{x}$, mutta $\frac{d}{dx} x\ln x = \ln x + 1$. Täten

$\int \ln x\,dx = x\ln x - x + C$, $C \in \mathbb{R}$ $\Bigg\}$

$= x\ln^2 x - 2\int \ln x\,dx = \underline{\underline{x\ln^2 x - 2x\ln x + 2x + C}}$, $C \in \mathbb{R}$

3d. $\int \frac{x}{e^x}\,dx = \int x e^{-x}\,dx$

Valitaan
$u'(x) = e^{-x} \Leftarrow u(x) = -e^{-x}$
$v(x) = x \Rightarrow v'(x) = 1$

$= -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + C$

$= -\frac{x}{e^x} - \frac{1}{e^x} + C$

$= \underline{\underline{\frac{-x-1}{e^x} + C}}, \quad C \in \mathbb{R}$

Differentiaali- ja integraalilaskenta 1

Malliratkaisut, Loppuviikko 4, teht 4-5

4 a. $\int_0^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx = \lim_{c \to 0} \int_c^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}} dx$

$= \lim_{c \to 0} \Big/_c^{\pi} -2e^{-\sqrt{x}}$

$= \lim_{c \to 0} \left(-2e^{-\sqrt{\pi}} + 2e^{-\sqrt{c}}\right)$

$= -2e^{-\sqrt{\pi}} + 2e^{-\sqrt{0}}$

$= -2e^{-\sqrt{\pi}} + 2$

b) $\int_0^1 \frac{\ln x}{x} dx = \lim_{c \to 0} \int_c^1 \frac{\ln x}{x} dx$

$= \lim_{c \to 0} \Big/_c^1 \frac{(\ln x)^2}{2}$

$= \lim_{c \to 0} \left(\frac{(\ln 1)^2}{2} - \frac{(\ln c)^2}{2}\right)$

$= 0 - \frac{(\ln 0)^2}{2}$

$= -\infty \Rightarrow$ ei suppene

4. c. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx = \lim_{c\to\pi/2} \int_{\pi/4}^{c} \frac{\sin x}{\sqrt{\cos x}}\,dx$

$$= \lim_{c\to\pi/2} \Big/_{\pi/4}^{c} -2\sqrt{\cos x}$$

$$= \lim_{c\to\pi/2} \left(-2\sqrt{\cos c} + 2\sqrt{\cos\left(\tfrac{\pi}{2}\right)}\right)$$

$$= 0 + 2\sqrt{\tfrac{1}{\sqrt{2}}}$$

$$= \underline{\underline{\frac{2}{\sqrt[4]{2}} = 2^{3/4}}}$$

d. $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx = \lim_{c\to\infty} \int_{1}^{c} \frac{1}{\sqrt{x^2+1}}\,dx$

$$= \lim_{c\to\infty} \Big/_{1}^{c} \sinh^{-1}(x)$$

$$= \lim_{c\to\infty} \left(\sinh^{-1}(c) - \sinh^{-1}(1)\right)$$

$$= \underline{\underline{\infty \Rightarrow \text{ei suppene}}}$$

Huomio: $\sinh: \mathbb{R}\to\mathbb{R}$ on aidosti kasvava bijektio, joka kahden eksponenttitermin summana pätee $\lim_{x\to\infty} \sinh(x) = \infty$. Tällöin myös tämän käänteisfunktiolle $\lim_{x\to\infty} \sinh^{-1}(x) = \infty$.

Vaihtoehtoisesti rajan voi nähdä funktion $\sinh^{-1}$ lausekkeesta:

$y = \sinh(x) \Leftrightarrow y = \frac{1}{2}(e^x - e^{-x}) \overset{\cdot 2e^x}{\Leftrightarrow} 2ye^x = e^{2x} - 1 \Leftrightarrow (e^x)^2 - 2y(e^x) - 1 = 0$

$\Leftrightarrow e^x = \dfrac{2y \overset{+}{(-)} \sqrt{4y^2+4}}{2}$ | $e^x > 0$ kaikilla $x \in \mathbb{R}$, joten miinusmerkki ei kelpaa.

$\Leftrightarrow e^x = y + \sqrt{y^2+1}$

$\Leftrightarrow x = \sinh^{-1}(y) = \ln\left(y + \sqrt{y^2+1}\right) \xrightarrow{y\to\infty} \infty.$

5. Integraalin muuttujanvaihto

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

kääntyvälle ja derivoituvalle funktiolle $g\ [a,b] \longrightarrow \mathbb{R}$

a) $\int_0^{\pi/3} 3\sin^2(3x)\,dx = \int_0^{\pi} \sin^2(y)\,dy$

Nyt $g(x) = 3x$, $g'(x) = 3$, $f(x) = \sin^2(g(x))$

$$\Rightarrow \int_0^{\pi/3} 3\sin^2(3x)\,dx = \int_0^{\pi/3} \sin^2(g(x))\,g'(x)\,dx$$

$$= \int_{g(0)}^{g(\frac{\pi}{3})} \sin^2(y)\,dy \quad \left|\begin{array}{l} g(\frac{\pi}{3}) = 3\cdot\frac{\pi}{3} = \pi \\ g(0) = 3\cdot 0 = 0 \end{array}\right.$$

$$= \underline{\underline{\int_0^{\pi} \sin^2(y)\,dy}}$$

b) $\int_1^e (\ln x)^3\,dx = \int_0^1 y^3 e^y\,dy$

Nyt $g(y) = e^y$, $g'(y) = e^y$, $g(0) = 1$, $g(1) = e$

$$\Rightarrow \int_0^1 y^3 e^y\,dy = \int_0^1 \ln^3 g(y)\,g'(y)\,dy \quad \left|\begin{array}{l} = \int_0^1 \ln^3 e^y \cdot e^y\,dy \\ = \int_0^1 (\ln e^y)^3\, e^y\,dy \\ = \int_0^1 y^3 e^y\,dy \end{array}\right\} \text{mistä } \ln^3\text{ ?? tässä avattuna}$$

$$= \int_{g(0)}^{g(1)} \ln^3 x\,dx$$

$$= \underline{\underline{\int_1^e \ln^3 x\,dx}}$$

5 c) $\int_1^2 2\ln(x^2+1)\,dx = \int_1^4 \frac{\ln(y+1)}{\sqrt{y}}\,dy$

Nyt $g(y)=\sqrt{y}$, $g'(y)=\frac{1}{2\sqrt{y}}$, $g(1)=1$, $g(4)=2$

$$\int_1^4 \frac{\ln(y+1)}{\sqrt{y}}\,dy = 2\int_1^4 \ln(g(y)^2+1)\,g'(y)\,dy$$

$$= 2\int_{g(1)}^{g(4)} \ln(x^2+1)\,dx$$

$$= \underline{\underline{\int_1^2 2\ln(x^2+1)\,dx}}$$

$$= 2\int_1^4 \ln((\sqrt{y})^2+1)\cdot\frac{1}{2\sqrt{y}}\,dy$$

$$= \int_1^4 \frac{2\ln(y+1)}{2\sqrt{y}}\,dy$$

$$= \int_1^4 \frac{\ln(y+1)}{\sqrt{y}}\,dy$$

d) $\int_0^\pi x\cos(\pi-x)\,dx = \int_0^\pi (\pi-y)\cos y\,dy$

Nyt $g(x)=\pi-x$, $g'(x)=-1$, $g(0)=\pi$, $g(\pi)=0$

$$\int_0^\pi x\cos(\pi-x)\,dx = \int_0^\pi (\pi-(\overbrace{\pi-x}^{g(x)}))\cos(\overbrace{\pi-x}^{g(x)})\cdot(-g'(x))\,dx$$

$$= -\int_{g(0)}^{g(\pi)} (\pi-y)\cos y\,dy$$

$$= -\int_\pi^0 (\pi-y)\cos y\,dy$$

$$= \underline{\underline{\int_0^\pi (\pi-y)\cos y\,dy}}$$