Practical Quantum Computing

Lecture 04
Teleportation, Superdense Coding

with slides from Dave Bacon https://homes.cs.washington.edu/~dabacon/teaching/siena/

Measurement Rule

If we measure a quantum system whose $|v\rangle$ wave function is in the basis $|w_i\rangle$, then the probability of getting the outcome corresponding to $|w_i\rangle$ is given by

$$Pr(|w_i\rangle) = |\langle w_i|v\rangle|^2 = \langle v|w_i\rangle\langle w_i|v\rangle = \langle v|P_{w_i}|v\rangle$$

where

$$P_{w_i} = |w_i\rangle\langle w_i|$$

The new wave function of the system after getting the measurement outcome corresponding to $|w_i\rangle$ is given by

$$|v'\rangle = \frac{P_{w_i}|v\rangle}{\sqrt{Pr(|w_i\rangle)}}$$

Measuring One of Two Qubits

Suppose we measure the first of two qubits in the computational basis. Then we can form the two projectors:

$$P_0 \otimes I = |0\rangle\langle 0| \otimes I$$
$$P_1 \otimes I = |1\rangle\langle 1| \otimes I$$

$$I = |0\rangle\langle 0| + |1\rangle\langle 1|$$

If the two qubit wave function is $|v\rangle$ then the probabilities of these two outcomes are

$$Pr(0) = \langle v|P_0 \otimes I|v\rangle$$
$$Pr(1) = \langle v|P_1 \otimes I|v\rangle$$

And the new state of the system is given by either

$$|v'\rangle = \frac{P_0 \otimes I|v\rangle}{\sqrt{Pr(0)}} \qquad |v'\rangle = \frac{P_1 \otimes I|v\rangle}{\sqrt{Pr(1)}}$$

Instantaneous Communication?

Suppose two distant parties each have a qubit and their joint quantum wave function is

$$|v\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

If one party now measures its qubit, then…

$$P_0 \otimes I = |0\rangle\langle 0| \otimes I \qquad Pr(0) = \frac{1}{2} \qquad |v'\rangle = |0\rangle \otimes |0\rangle$$

$$P_1 \otimes I = |1\rangle\langle 1| \otimes I \qquad Pr(1) = \frac{1}{2} \qquad |v'\rangle = |1\rangle \otimes |1\rangle$$

The other parties qubit is now either the $|0\rangle$ or $|1\rangle$

Instantaneous communication? NO! These two results happen with probabilities.

Quantum Teleportation

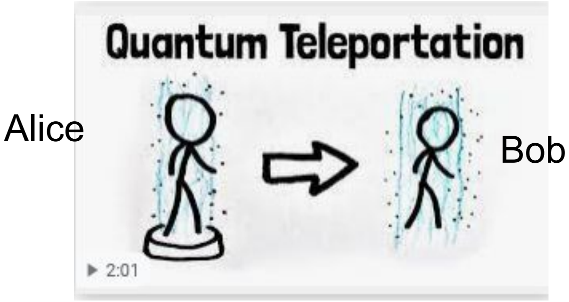
Alice wants to send her qubit to Bob.

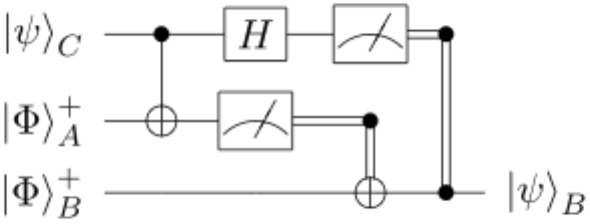
She does not know the wave function of her qubit.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Can Alice send her qubit to Bob using classical bits?

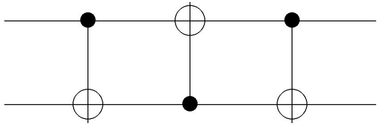
Since she doesn't know $|\psi\rangle$ and measurements on her state do not reveal $|\psi\rangle$, this task appears impossible.



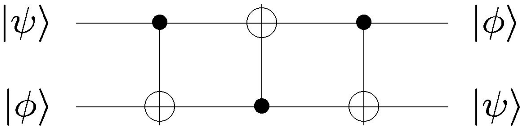


Deriving Quantum Teleportation

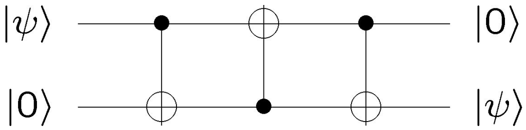
Our path: We are going to “derive” teleportation

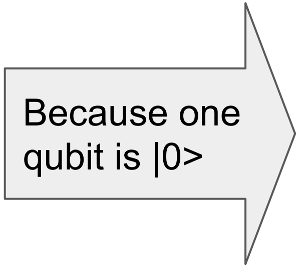


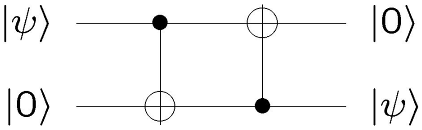
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Only concerned with from Alice to Bob transfer

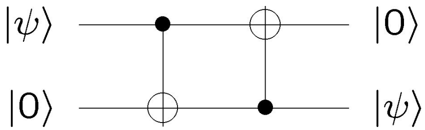


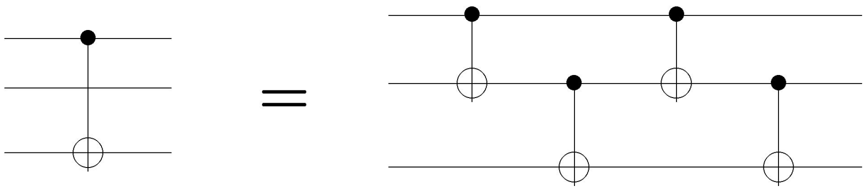


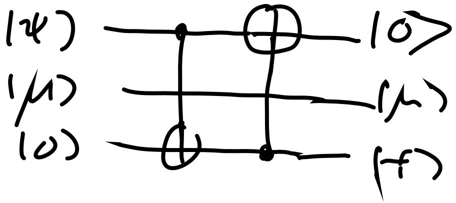


Deriving Quantum Teleportation

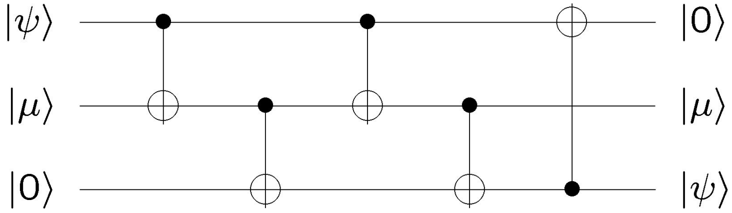
Need some way to get entangled states





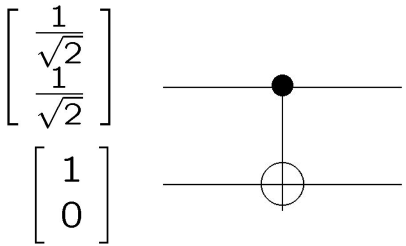


new equivalent circuit:

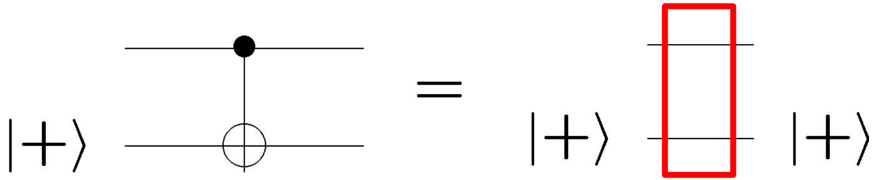


Deriving Quantum Teleportation

How to generate an entangled state:

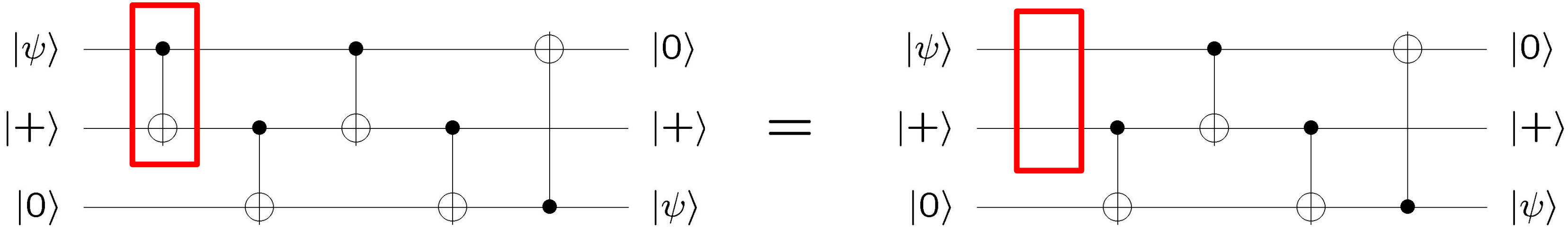


$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

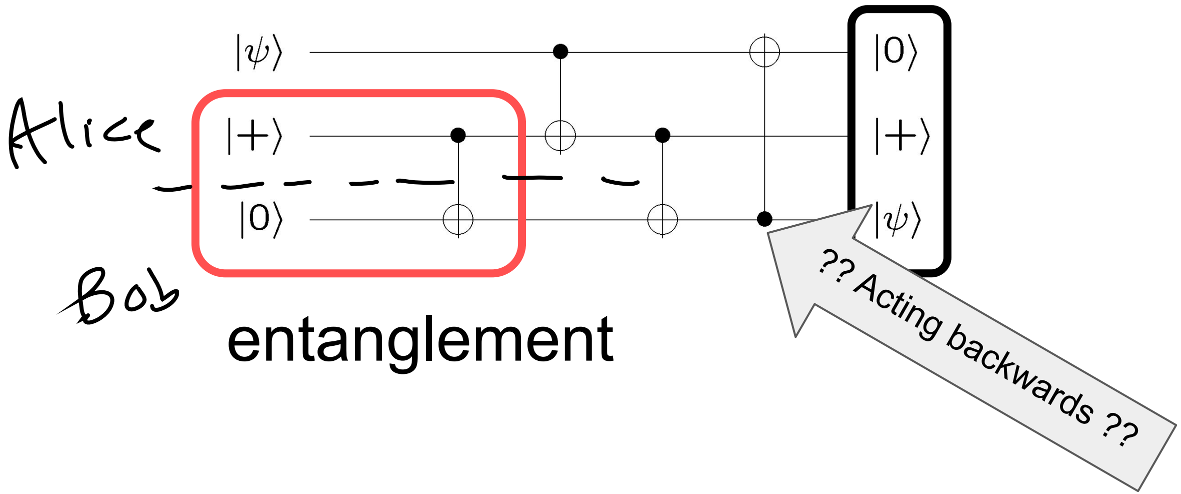


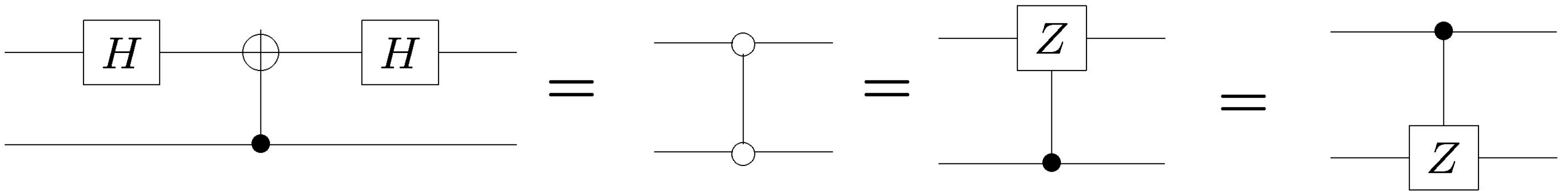
$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$X|+\rangle = |+\rangle$$



Deriving Quantum Teleportation





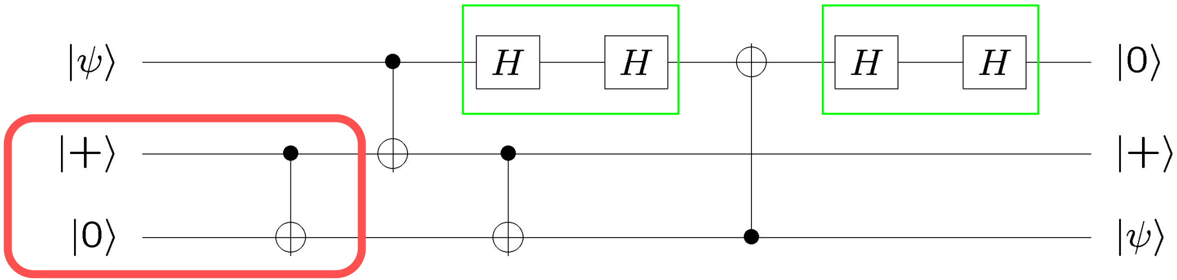
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$HXH = Z$$

$$H^2 = I$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

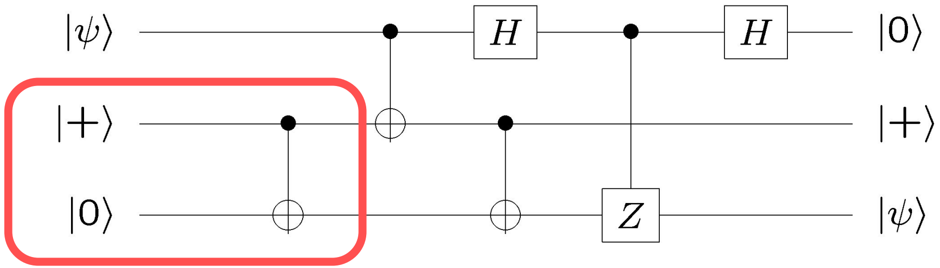
Deriving Quantum Teleportation



$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

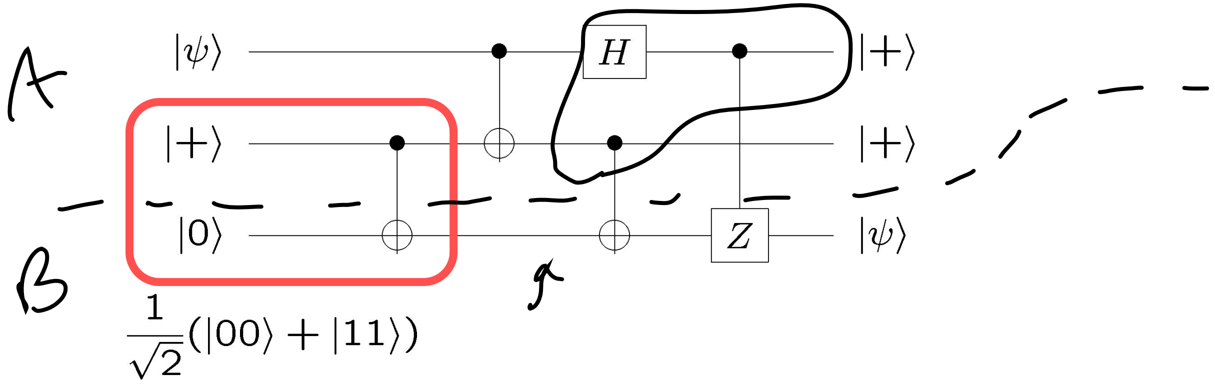
$H|\psi\rangle = |0\rangle$

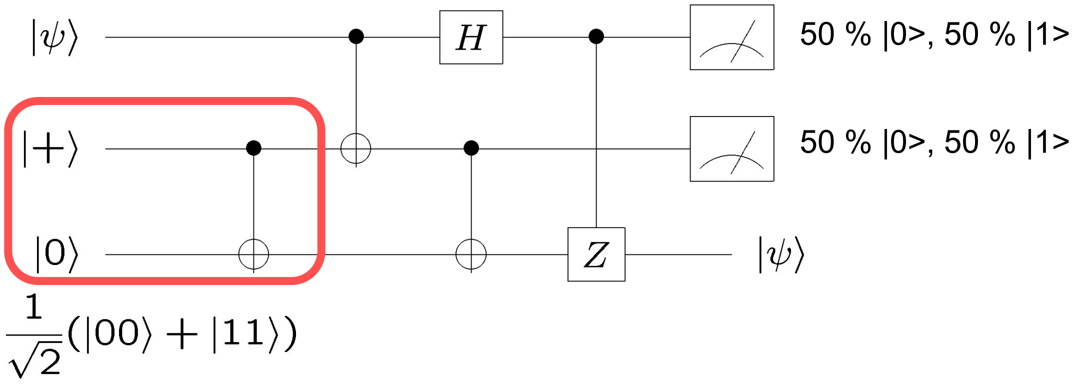
$H\,|+\rangle = |0\rangle$



$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

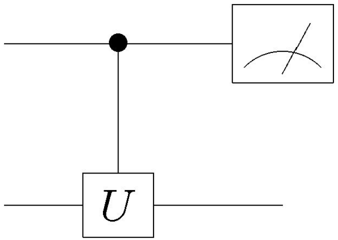
Deriving Quantum Teleportation

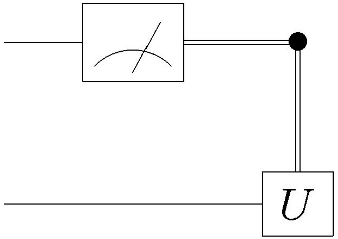




Measurements Through Control

Measurement in the computational basis commutes with a control on a controlled unitary.

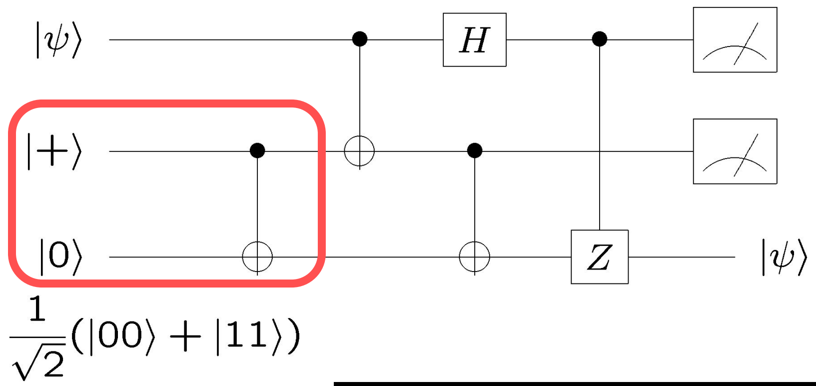


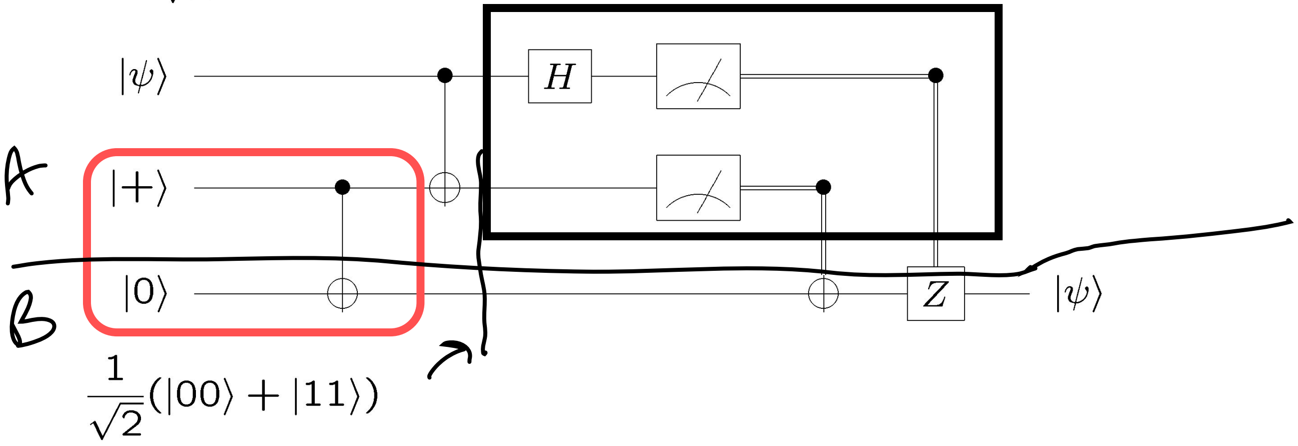


$$(P_0 \otimes I)C_U = C_U(P_0 \otimes I)$$
$$(|0\rangle\langle 0|\otimes I)(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U) = (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U)(|0\rangle\langle 0|\otimes I)$$

$$(P_1 \otimes I)C_U = C_U(P_1 \otimes I)$$
$$(|1\rangle\langle 1|\otimes I)(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U) = (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U)(|1\rangle\langle 1|\otimes I)$$

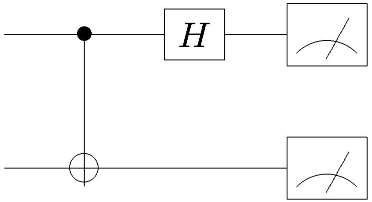
Deriving Quantum Teleportation



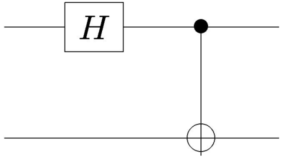


Bell Basis Measurement

Unitary followed by measurement in the computational basis is a measurement in a different basis.



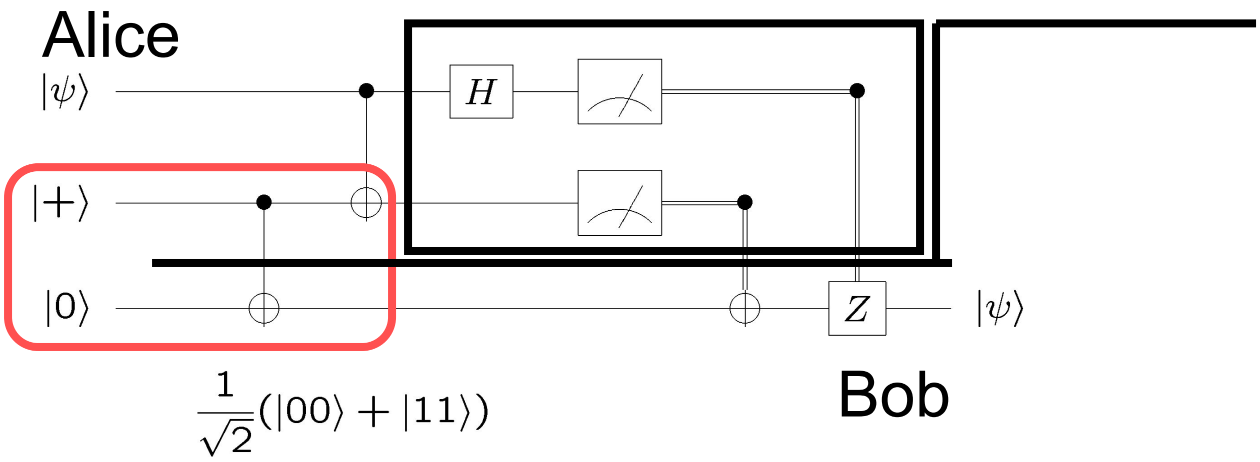
Run circuit backward to find basis:



$$|00\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|01\rangle \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|10\rangle \to \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|11\rangle \to \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Teleportation

1. Initially Alice has $|\psi\rangle$ and they each have one of the two qubits of the entangled wave function $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
2. Alice measures $|\psi\rangle$ and her half of the entangled state in the Bell Basis.
3. Alice send the two bits of her outcome to Bob who then performs the appropriate X and Z operations to his qubit.

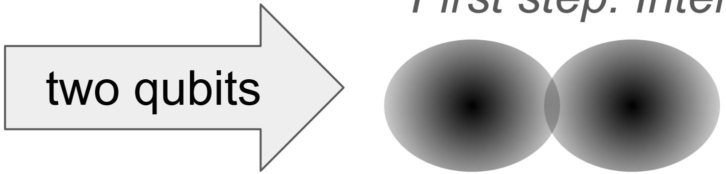


Teleportation

Alice and Bob each have a qubit

- and the wave function of their two qubits is entangled
- we can't think of Alice's qubit as having a particular wave function
- we have to talk about the "global" two qubit wave function.

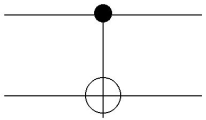
First step: Interact and entangle



Second step: Separate (physically)

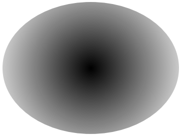
$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|0\rangle$$

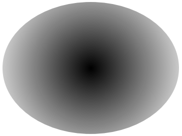


$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Teleportation

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice does not know the wave function

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

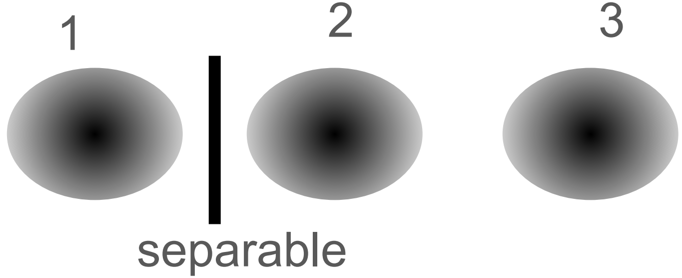
We have three qubits whose wave function is

qubit 1 $\rightarrow$ $|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\leftarrow$ qubit 2 and qubit 3

Separable, Entangled, 3 Qubits

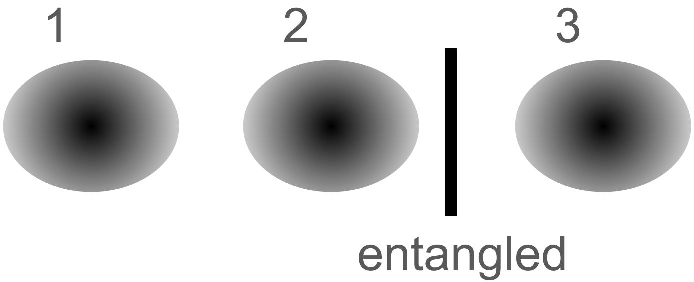
If we consider $(\alpha|0\rangle+\beta|1\rangle)\otimes\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) = \frac{\alpha}{\sqrt{2}}|000\rangle+\frac{\alpha}{\sqrt{2}}|011\rangle+\frac{\beta}{\sqrt{2}}|100\rangle+\frac{\beta}{\sqrt{2}}|111\rangle$

- qubit 1 as one subsystem
- qubits 2 and 3 as another subsystem
- then the state is separable across this divide

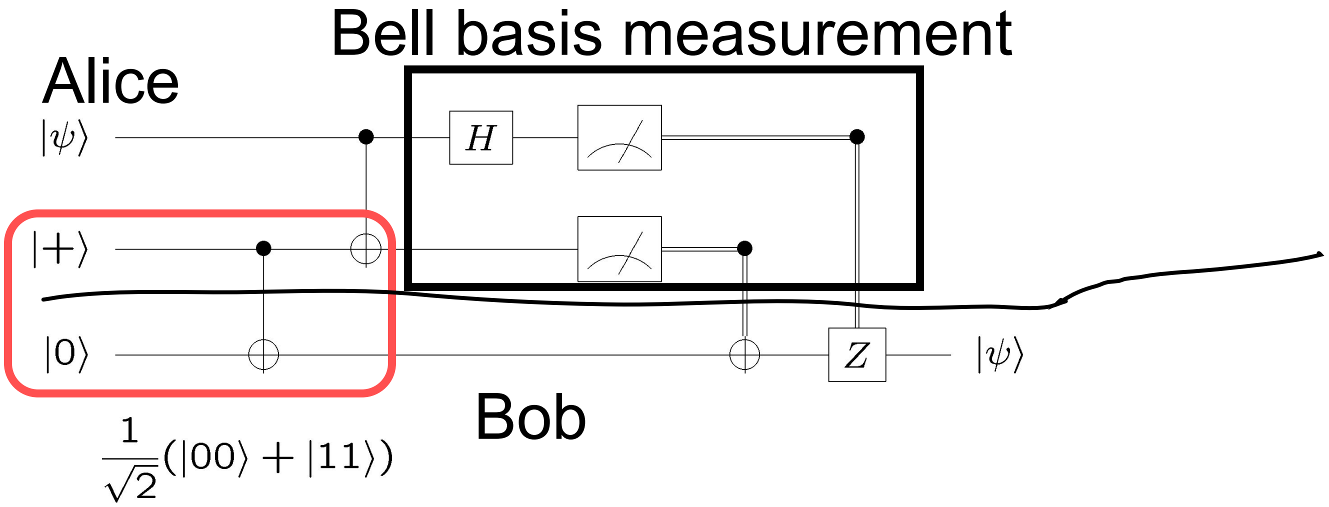


If we consider $\neq (a_{00}|00\rangle+a_{01}|01\rangle+a_{10}|10\rangle+a_{11}|11\rangle)\otimes(b_0|0\rangle+b_1|1\rangle)$

- qubits 1 and 2 as one system
- qubits 3 as one subsystem
- then the state is entangled across this divide.



Teleportation



$$|\psi\rangle_1 \otimes \frac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23})$$

$$= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

Teleportation

$$|\psi\rangle_1 \otimes \frac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23}) = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

Express this state in terms of Bell basis for first two qubits.

Bell basis

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Computational basis

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle)$$
$$|10\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)$$

Teleportation

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$\boxed{\begin{aligned} |00\rangle &= \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle) \\ |01\rangle &= \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle) \\ |11\rangle &= \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle) \\ |10\rangle &= \frac{1}{\sqrt{2}}(|\Psi_+\rangle - |\Psi_-\rangle) \end{aligned}}$$

$$\frac{\alpha}{\sqrt{2}}|\boxed{00}0\rangle + \frac{\alpha}{\sqrt{2}}|\boxed{01}1\rangle + \frac{\beta}{\sqrt{2}}|\boxed{10}0\rangle + \frac{\beta}{\sqrt{2}}|\boxed{11}1\rangle$$

$$\boxed{\begin{aligned} = \quad & \frac{1}{2}(\alpha(|\Phi_+\rangle + |\Phi_-\rangle) \otimes |0\rangle + \alpha(|\Psi_+\rangle + |\Psi_-\rangle) \otimes |1\rangle \\ & +\beta(|\Psi_+\rangle - |\Psi_-\rangle) \otimes |0\rangle + \beta(|\Phi_+\rangle - |\Phi_-\rangle) \otimes |1\rangle \end{aligned}}$$

$$\begin{aligned} = \quad & \frac{1}{2}\Big[|\Phi_+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |\Phi_-\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) \\ & +|\Psi_+\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |\Psi_-\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)\Big] \end{aligned}$$

Teleportation

$$\boxed{\begin{aligned}|\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\ |\Phi_-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\\ |\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\\ |\Psi_-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\end{aligned}}$$

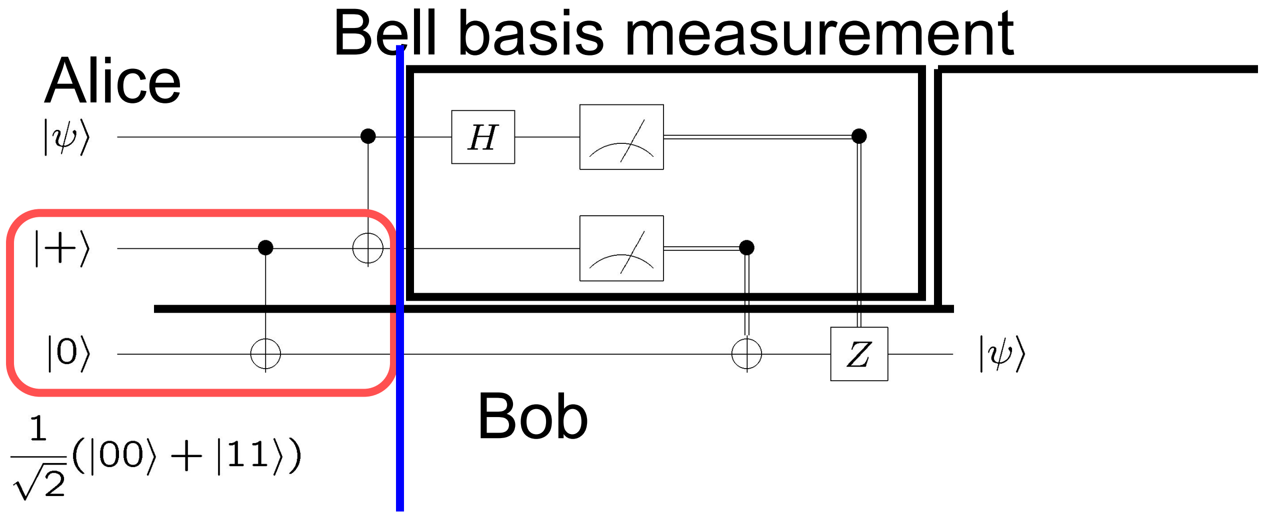
$$\begin{aligned}|00\rangle &= \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle)\\ |01\rangle &= \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle)\\ |11\rangle &= \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)\\ |10\rangle &= \frac{1}{\sqrt{2}}(|\Psi_+\rangle - |\Psi_-\rangle)\end{aligned}$$

$$\frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

$$\begin{aligned}= \ &\frac{1}{2}(\alpha(|\Phi_+\rangle + |\Phi_-\rangle) \otimes |0\rangle + \alpha(|\Psi_+\rangle + |\Psi_-\rangle) \otimes |1\rangle\\ &+\beta(|\Psi_+\rangle - |\Psi_-\rangle) \otimes |0\rangle + \beta(|\Phi_+\rangle - |\Phi_-\rangle) \otimes |1\rangle\end{aligned}$$

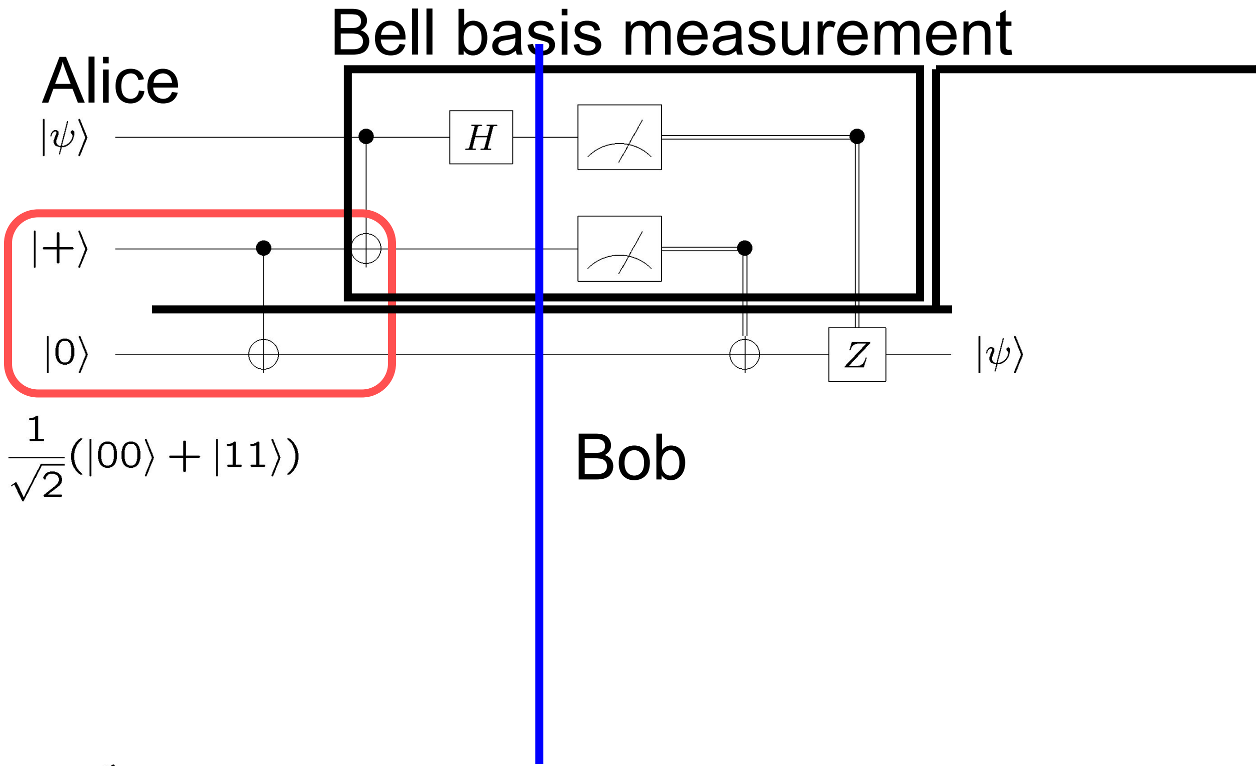
$$\boxed{\begin{aligned}= \ &\frac{1}{2}\Big[|\Phi_+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |\Phi_-\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)\\ &+|\Psi_+\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |\Psi_-\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)\Big]\end{aligned}}$$

Teleportation



$$\frac{1}{2}\Big[|\Phi_+\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |\Phi_-\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3)$$
$$+|\Psi_+\rangle_{12}(\alpha|1\rangle_3 + \beta|0\rangle_3) + |\Psi_-\rangle_{12}(\alpha|1\rangle_3 - \beta|0\rangle_3)\Big]$$

Teleportation



$$\frac{1}{2}\left[|00\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |10\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3)\right.$$
$$\left.+|01\rangle_{12}(\alpha|1\rangle_3 + \beta|0\rangle_3) + |11\rangle_{12}(\alpha|1\rangle_3 - \beta|0\rangle_3)\right]$$

Teleportation

Given the wave function

$$\frac{1}{2}[|00\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |10\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3)$$
$$+|01\rangle_{12}(\alpha|1\rangle_3 + \beta|0\rangle_3) + |11\rangle_{12}(\alpha|1\rangle_3 - \beta|0\rangle_3)]$$

Measure the first two qubits in the computational basis

$$M_{00} = |00\rangle\langle 00| \otimes I \qquad M_{01} = |01\rangle\langle 01| \otimes I$$
$$M_{10} = |10\rangle\langle 10| \otimes I \qquad M_{11} = |11\rangle\langle 11| \otimes I$$

Equal ¼ probability for all four outcomes and new states are:

$$|00\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) \qquad |10\rangle_{12} \otimes (\alpha|0\rangle_3 - \beta|1\rangle_3)$$
$$|01\rangle_{12} \otimes (\alpha|1\rangle_3 + \beta|0\rangle_3) \qquad |11\rangle_{12} \otimes (\alpha|1\rangle_3 - \beta|0\rangle_3)$$

Teleportation

If the bits sent from Alice to Bob are 00, do **nothing**

$$|00\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) = |00\rangle_{12} \otimes |\psi\rangle_3$$

If the bits sent from Alice to Bob are 01, apply a **bit flip**

$$(I_4 \otimes X)|01\rangle_{12} \otimes (\alpha|1\rangle_3 + \beta|0\rangle_3) = |01\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta_1|1\rangle_3)$$

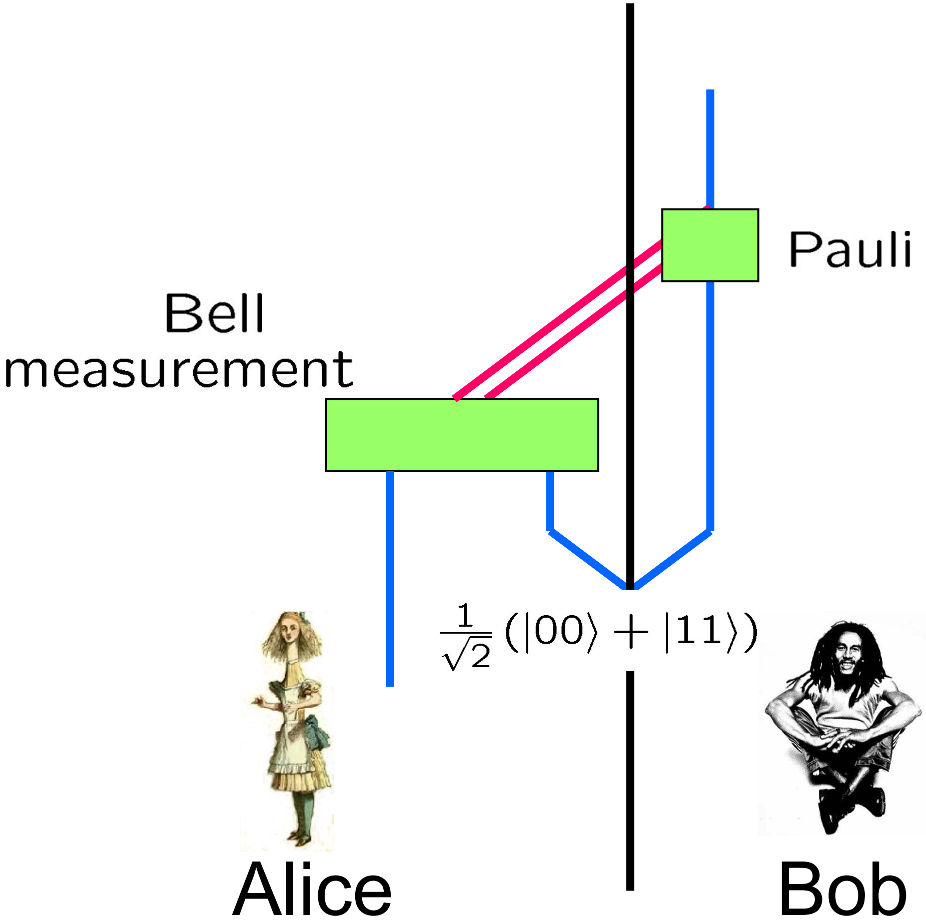
If the bits sent from Alice to Bob are 10, apply a **phase flip**

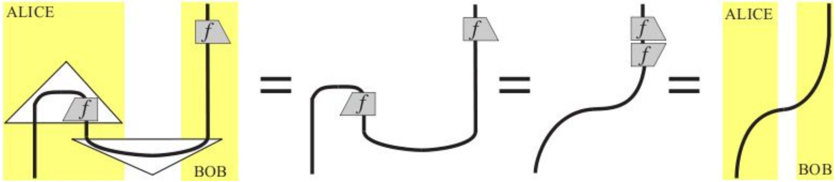
$$(I_4 \otimes Z)|10\rangle_{12} \otimes (\alpha|0\rangle_3 - \beta|1\rangle_3) = |10\rangle \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) = |10\rangle_{12} \otimes |\psi\rangle_3$$

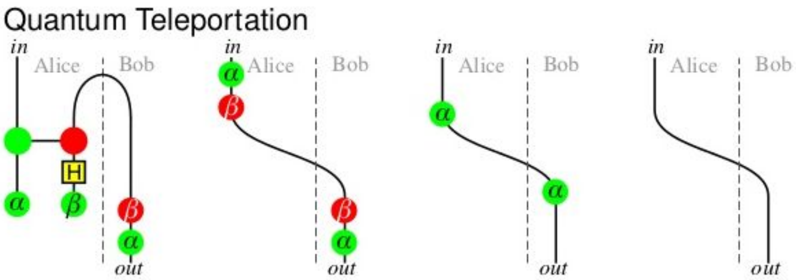
If the bits sent from Alice to Bob are 11, apply **a bit & phase flip**

$$(I_4 \otimes Z)(I_4 \otimes X)|11\rangle_{12} \otimes (\alpha|1\rangle_3 - \beta|0\rangle_3) = (I_4 \otimes Z)|11\rangle_{12} \otimes (\alpha|0\rangle_3 - \beta|1\rangle_3)$$
$$= |11\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) = |11\rangle_{12} \otimes |\psi\rangle_3$$

Teleportation







Teleportation and Superdense Coding

$$\boxed{1 \text{ qubit} = 1 \text{ ebit} + 2 \text{ bits}}$$

Teleportation says we can replace transmitting a qubit with a shared entangled pair of qubits plus two bits of classical communication.

Superdense Coding

Next we will see that

$$\boxed{2 \text{ bits} = 1 \text{ qubit} + 1 \text{ ebit}}$$

Superdense Coding

Suppose Alice and Bob each have one qubit and the joint two qubit wave function is the entangled state

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice wants to send two bits to Bob. Call these bits b_1 and b_2.

Alice applies the following operator to her qubit:

Bob then measures in the Bell basis to determine the two bits.

$\boxed{\text{2 bits = 1 qubit + 1 ebit}}$

Bell Basis

The four Bell states

- can be turned into each other
- using operations on only one of the qubits:

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$(X \otimes I)|\Phi_+\rangle = (X \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = |\Psi_+\rangle$$

$$(Z \otimes I)|\Phi_+\rangle = (Z \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi_-\rangle$$

$$(ZX \otimes I)|\Phi_+\rangle = (ZX \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle) = |\Psi_-\rangle$$

Superdense Coding

Initially: $$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice applies the following operator to her qubit: $Z^{b_2}X^{b_1}$

$$(Z^{b_2}X^{b_1} \otimes I)|\Phi_+\rangle$$

Bob can uniquely:

- determine which of the four states he has
- figure out Alice's two bits!

$$b_1 = 0, b_2 = 0 \qquad |\Phi_+\rangle$$

$$b_1 = 0, b_2 = 1 \qquad (Z \otimes I)|\Phi_+\rangle = |\Phi_-\rangle$$

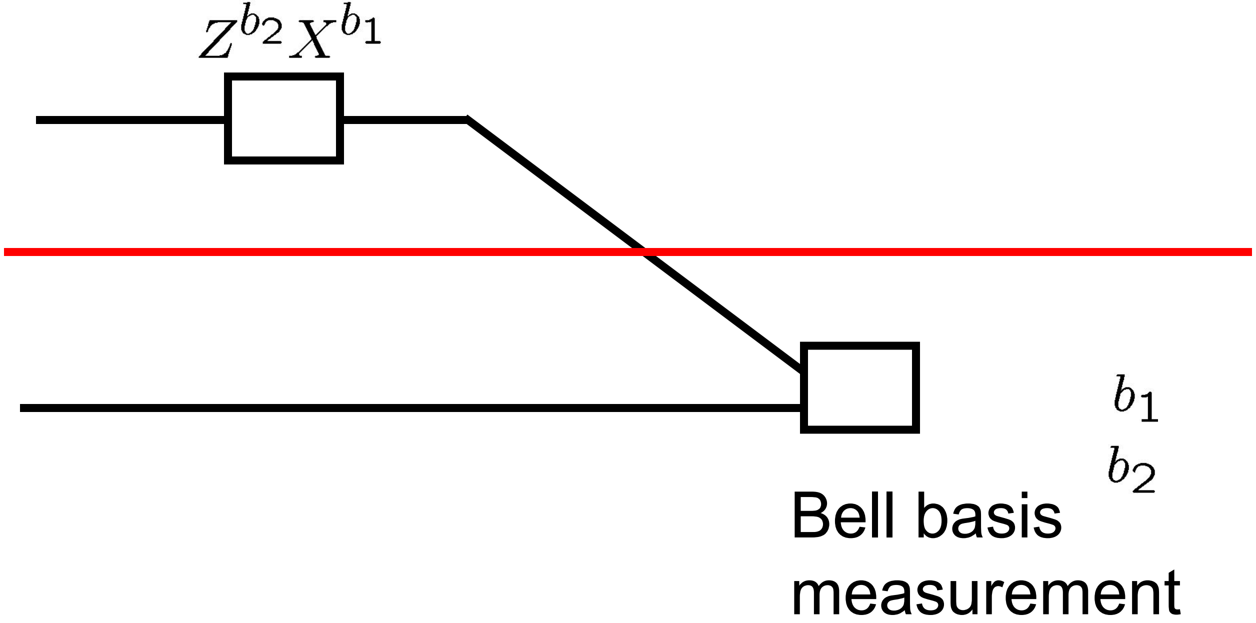
$$b_1 = 1, b_2 = 0 \qquad (X \otimes I)|\Phi_+\rangle = |\Psi_+\rangle$$

$$b_1 = 1, b_2 = 1 \qquad (ZX \otimes I)|\Phi_+\rangle = |\Psi_-\rangle$$

Superdense Coding

b_1
b_2

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



Teleportation and Superdense Coding

$$\boxed{\text{1 qubit = 1 ebit + 2 bits}}$$

Teleportation says we can replace transmitting a qubit with a shared entangled pair of qubits plus two bits of classical communication.

$$\boxed{\text{2 bits = 1 qubit + 1 ebit}}$$

Superdense coding. We can send two bits of classical information if we share an entangled state and can communicate one qubit of quantum information.