Practical Quantum Computing

Lecture 10
Fault-Tolerance

based on Chapter 10 from *An Introduction to Quantum Computing*, Phillip Kaye, Raymond Laflamme, Michele Mosca, Oxford University Press

Classical Error Correction

- If individual steps in a computation succeed with **probability *p***, then the computation involving ***t* steps** will have a success probability that decreases exponentially as p^t
- **Error model == channel**, describes evolution/transformation of bits
  - simplest model is the bit flip channel: flipped with probability *p* and unaffected with *1-p*
  - **independent errors** vs. **correlated errors** between different bits

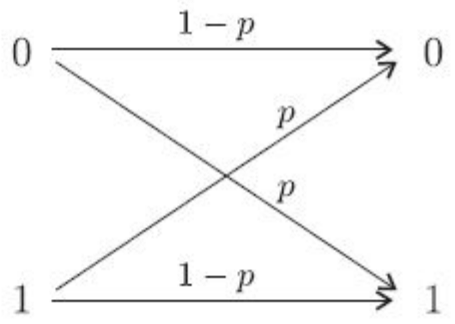
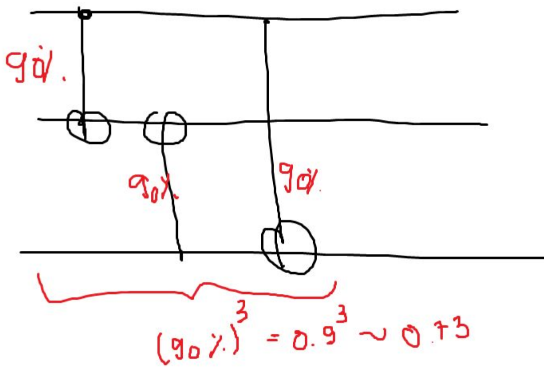


Fig. 10.1 The classical bit-flip channel.



Encoding

- Add extra bits to a *logical bit*
  - *definition diff. from the generally used one*
  - the result is called an *encoded bit*
  - the *set of codewords* is called a *code*
- **encoding** - process of mapping logical strings *b* to their respective codewords
- **recovery** - procedure for correcting errors and recovering the string *b*
  - **unambiguously** distinguish between codewords
  - **correctable errors** - set of errors for which recovery works

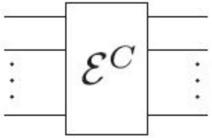


Fig. 10.2 A block representing the effect of errors on a register in a circuit diagram.

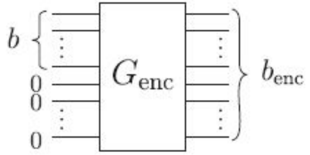


Fig. 10.3 Encoding operation taking a logical string b, and an ancilla of bits in the state 0, into the codeword b_{enc}.

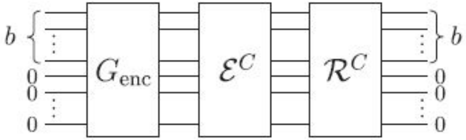
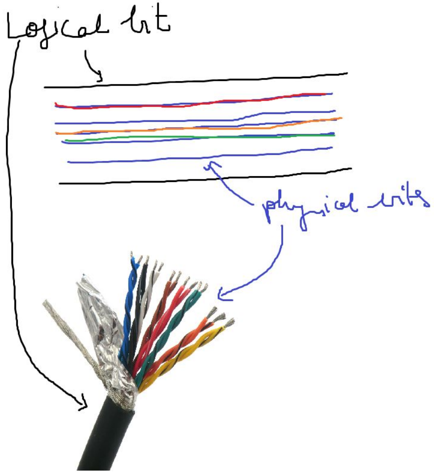
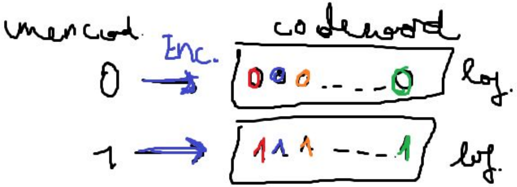


Fig. 10.4 After a codeword b_{enc} is subjected to some errors, the recovery operation $\mathcal{R}^C$ corrects these errors and recovers the logical string b.





Classical Three-Bit Code

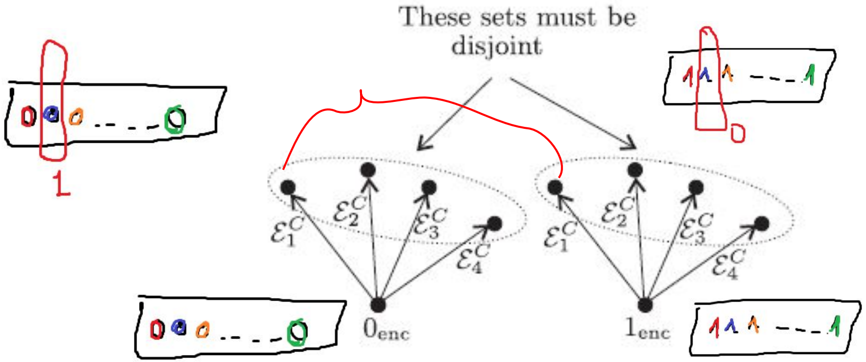


Fig. 10.5 The error correction condition for a code with two codewords, under an error model in which there are four possible errors $\mathcal{E}_1^C, \mathcal{E}_2^C, \mathcal{E}_3^C, \mathcal{E}_4^C$ affecting each codeword. The condition is that when any errors act on two distinct codewords, the resulting strings are never equal.

Condition for error correction can include identity I operation (leave unchanged)

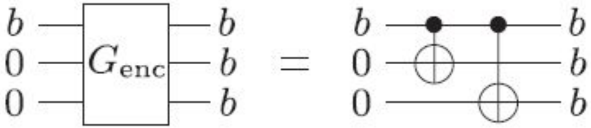
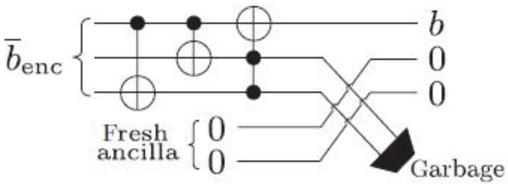
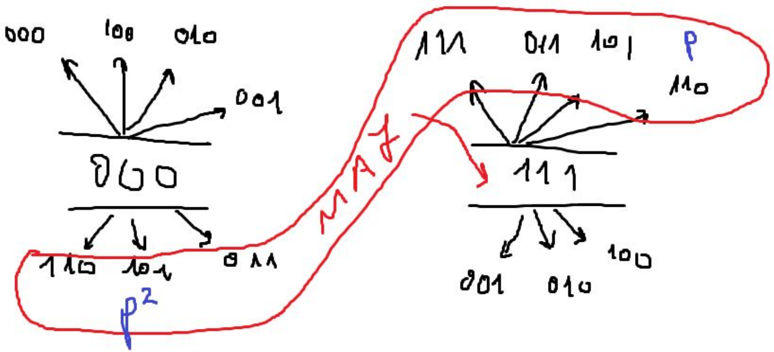


Fig. 10.6 A circuit for the encoding operation for the classical three-bit code. Recall the circuit symbol for the classical CNOT gate from Figure 1.3.

Three-Bit Code - Recovery

- Take a **majority vote** of the three bits
- Reset all the bits to the value resulting from the majority vote





Procedure to restore information in the first bit and use the last two bits to tell us on which bit the error occurred

- no need to learn the actual value of any bit, only the parities
- **error syndrome**: information about parities

Error Location	Final State, $\|data\rangle \|ancilla\rangle$
No Error	$\alpha \|000\rangle \|00\rangle + \beta \|111\rangle \|00\rangle$
Qubit 1	$\alpha \|100\rangle \|11\rangle + \beta \|011\rangle \|11\rangle$
Qubit 2	$\alpha \|010\rangle \|10\rangle + \beta \|101\rangle \|10\rangle$
Qubit 3	$\alpha \|001\rangle \|01\rangle + \beta \|110\rangle \|01\rangle$

Repetition code - Probability of error

- If individual steps in a computation succeeds with probability 1-*p*, then the computation involving *t* steps will have a failure probability that decreases exponentially as p^t

$$
\begin{aligned}
000 \rightarrow \{ & (000, (1-p)^3), \\
& (001, p(1-p)^2), (010, p(1-p)^2), (100, p(1-p)^2), \\
& (011, p^2(1-p)), (110, p^2(1-p)), (101, p^2(1-p)), \\
& (111, p^3)\}.
\end{aligned}
$$

- The probability of **two or more bits** of codeword being flipped is

$$3p^2(1-p) + p^3$$

Quantum Error Correction

- Errors occur on qubit when its evolution differs from the desired one
  - imprecise control
  - interaction of the qubits with the environment
- The generic evolution of a qubit interacting with an environment |E>
  - in the state |0> will yield a superposition

$$|0\rangle|E\rangle \mapsto \beta_1|0\rangle|E_1\rangle + \beta_2|1\rangle|E_2\rangle.$$

  - in the state |1> will yield a superposition

$$|1\rangle|E\rangle \mapsto \beta_3|1\rangle|E_3\rangle + \beta_4|0\rangle|E_4\rangle.$$

  - in general

$$(\alpha_0|0\rangle + \alpha_1|1\rangle)|E\rangle \mapsto \alpha_0\beta_1|0\rangle|E_1\rangle + \alpha_0\beta_2|1\rangle|E_2\rangle + \alpha_1\beta_3|1\rangle|E_3\rangle + \alpha_1\beta_4|0\rangle|E_4\rangle.$$

- Discrete set of errors
- Coherent / incoherent (environment is entangled with the state)

Discretisation of Errors

We can rewrite the state after the interaction as

$$
\begin{aligned}
\alpha_0\beta_1|0\rangle|E_1\rangle + \alpha_0\beta_2|1\rangle|E_2\rangle + \alpha_1\beta_3|1\rangle|E_3\rangle + \alpha_1\beta_4|0\rangle|E_4\rangle \\
= \tfrac{1}{2}\big(\alpha_0|0\rangle + \alpha_1|1\rangle\big)\big(\beta_1|E_1\rangle + \beta_3|E_3\rangle\big) \\
+ \tfrac{1}{2}\big(\alpha_0|0\rangle - \alpha_1|1\rangle\big)\big(\beta_1|E_1\rangle - \beta_3|E_3\rangle\big) \\
+ \tfrac{1}{2}\big(\alpha_0|1\rangle + \alpha_1|0\rangle\big)\big(\beta_2|E_2\rangle + \beta_4|E_4\rangle\big) \\
+ \tfrac{1}{2}\big(\alpha_0|1\rangle - \alpha_1|0\rangle\big)\big(\beta_2|E_2\rangle - \beta_4|E_4\rangle\big). \qquad (10.4.4)
\end{aligned}
$$

Let $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$. Then we have

$$\alpha_0|0\rangle - \alpha_1|1\rangle = Z|\psi\rangle \qquad (10.4.5)$$

phase error

bit error

$$\alpha_0|1\rangle + \alpha_1|0\rangle = X|\psi\rangle \qquad (10.4.6)$$

$$\alpha_0|1\rangle - \alpha_1|0\rangle = XZ|\psi\rangle \qquad (10.4.7)$$

bit and phase error

and the interaction between the state and the environment can be written as

$$
\begin{aligned}
|\psi\rangle|E\rangle \mapsto \tfrac{1}{2}|\psi\rangle\big(\beta_1|E_1\rangle + \beta_3|E_3\rangle\big) + \tfrac{1}{2}\big(Z|\psi\rangle\big)\big(\beta_1|E_1\rangle - \beta_3|E_3\rangle\big) \\
+ \tfrac{1}{2}\big(X|\psi\rangle\big)\big(\beta_2|E_2\rangle + \beta_4|E_4\rangle\big) + \tfrac{1}{2}\big(XZ|\psi\rangle\big)\big(\beta_2|E_2\rangle - \beta_4|E_4\rangle\big).
\end{aligned}
\qquad (10.4.8)
$$

Discretisation of Errors

$$|q\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \xrightarrow{\text{Error}} \frac{1}{2}\, i|q\rangle \boxed{\qquad\quad} + \frac{1}{2}\, X|q\rangle \boxed{\qquad\quad} + \frac{1}{2}\, Z|q\rangle \boxed{\qquad\quad} + \frac{1}{2}\, XZ|q\rangle \boxed{\qquad\quad}$$

After measurement: (prob. $\frac{1}{4}$)

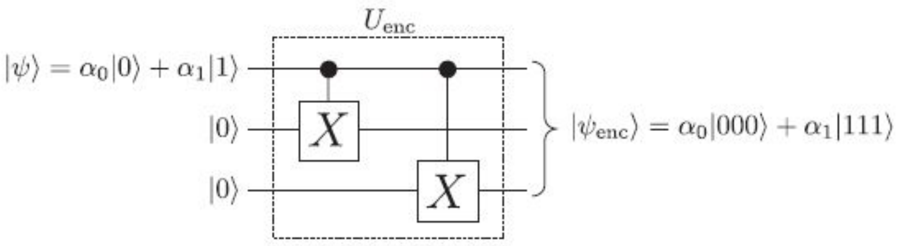
$i|q\rangle$, $X|q\rangle$, $Z|q\rangle$, $XZ|q\rangle$

Three-Qubit Code

- Encodes a single logical qubit into three physical qubits with the property that it can correct for a single bit flip error.
- The two logical basis states |0> and|1> are defined as

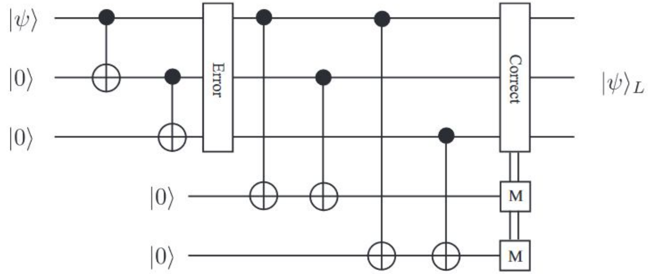
$$|0\rangle_L = |000\rangle\,, \qquad |1\rangle_L = |111\rangle\,,$$

$$\begin{aligned} \alpha\,|0\rangle + \beta\,|1\rangle \rightarrow \alpha\,|0\rangle_L + \beta\,|1\rangle_L \\ = \alpha\,|000\rangle + \beta\,|111\rangle \\ = |\psi\rangle_L\,. \end{aligned}$$



- Note that the no-cloning theorem applies

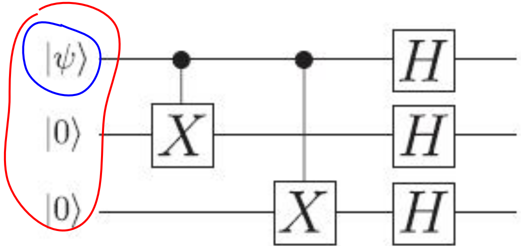
$$\begin{aligned} \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\ \neq \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)^{\otimes 3}. \end{aligned}$$



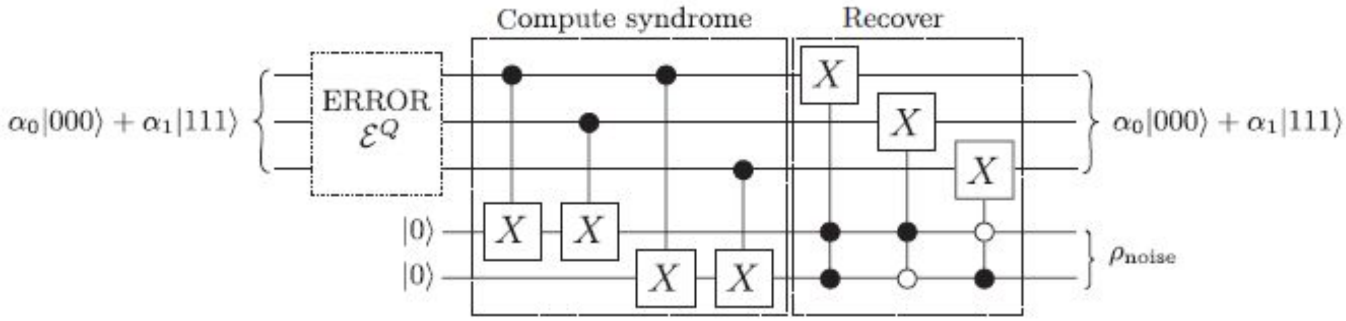
Three-Qubit Code for Phase Errors

- We can adapt the three-qubit bit-flip code to correct phase-flip errors.
  - There are no phase errors in classical digital information
  - It is easy to transform a phase-flip error into a bit-flip error
  - Specifically, consider the Hadamard basis states

$$|+\rangle \equiv \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
$$|-\rangle \equiv \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$



$$\alpha_0|0\rangle + \alpha_1|1\rangle \mapsto \alpha_0|000\rangle + \alpha_1|100\rangle \mapsto \alpha_0|{+}{+}{+}\rangle + \alpha_1|{-}{-}{-}\rangle.$$



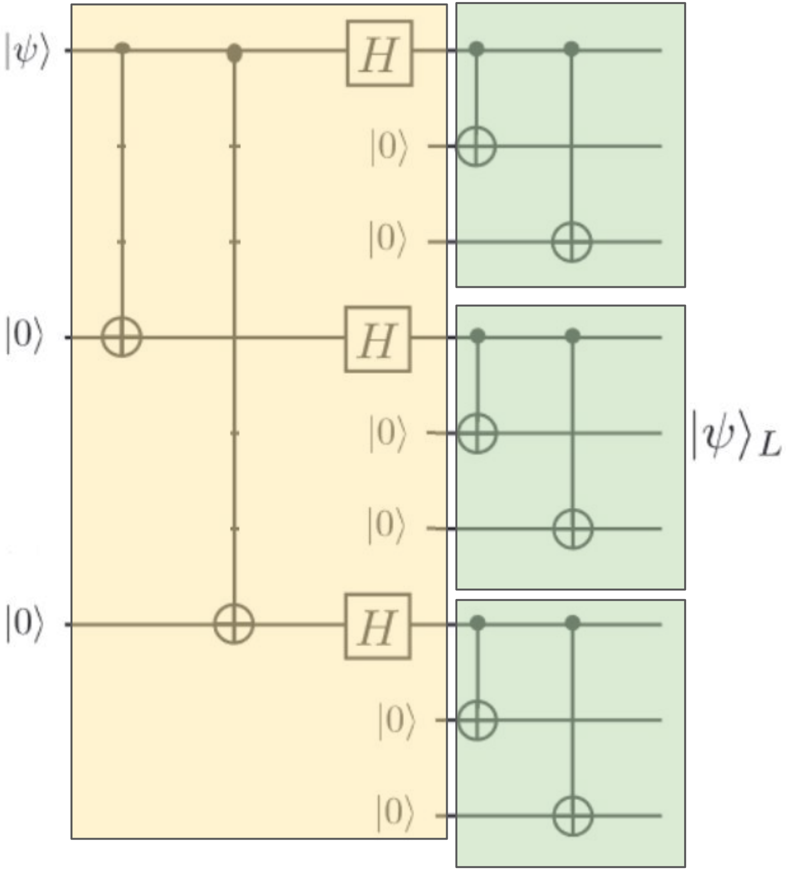
Shor’s Code - The 9 qubit code

Combine the three-qubit bit-flip and phase-flip codes can be combined to give a nine qubit code which corrects

- bit or phase flip errors on one of the nine qubits
- a simultaneous bit and phase flip on the same qubit
- correct for a generic one-qubit error

Encoding for the Shor code works in two stages

1. phase flip encoder
2. bit flip encoder



Shor's Code - The 9 qubit code

after 1st stage

$$|0\rangle \mapsto |{+}{+}{+}\rangle,$$
$$|1\rangle \mapsto |{-}{-}{-}\rangle.$$

$$|+\rangle \mapsto \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$
$$|-\rangle \mapsto \tfrac{1}{\sqrt{2}}(|000\rangle - |111\rangle).$$

$$|0\rangle \mapsto \tfrac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle),$$
$$|1\rangle \mapsto \tfrac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).$$

after 2nd stage

codewords

bit-flip errors will be identified by two parities for each triplet of the qubits

$$
\begin{aligned}
&Z\otimes Z\otimes I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes I\,,\\
&Z\otimes I\otimes Z\otimes I\otimes I\otimes I\otimes I\otimes I\otimes I\,,\\
&I\otimes I\otimes I\otimes Z\otimes Z\otimes I\otimes I\otimes I\otimes I\,,\\
&I\otimes I\otimes I\otimes Z\otimes I\otimes Z\otimes I\otimes I\otimes I\,,\\
&I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes Z\otimes Z\otimes I\,,\\
&I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes Z\otimes I\otimes Z\,.
\end{aligned}
$$

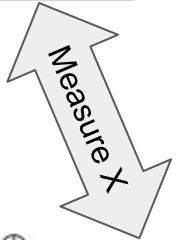
phase-flip errors are identified by the parities of the signs that appear in each triplet

$$
\begin{aligned}
&X\otimes X\otimes X\otimes X\otimes X\otimes X\otimes I\otimes I\otimes I\,,\\
&I\otimes I\otimes I\otimes X\otimes X\otimes X\otimes X\otimes X\otimes X\,.
\end{aligned}
$$

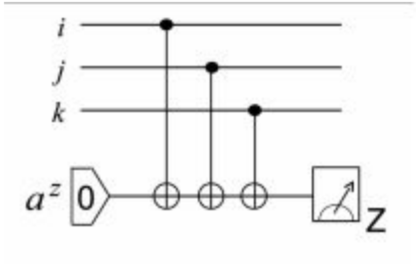
Shor's Code - Parities and Corrections

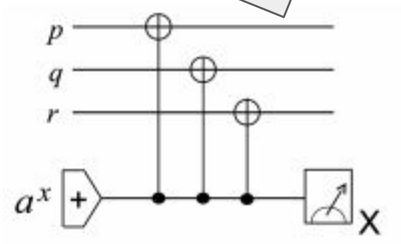
Phase flip errors on the nine-qubit state can be determined by measuring the parities

$$X \otimes X \otimes X \otimes X \otimes X \otimes X \otimes I \otimes I \otimes I \,,$$
$$I \otimes I \otimes I \otimes X \otimes X \otimes X \otimes X \otimes X \otimes X \,.$$



Circuits for **measuring parities**





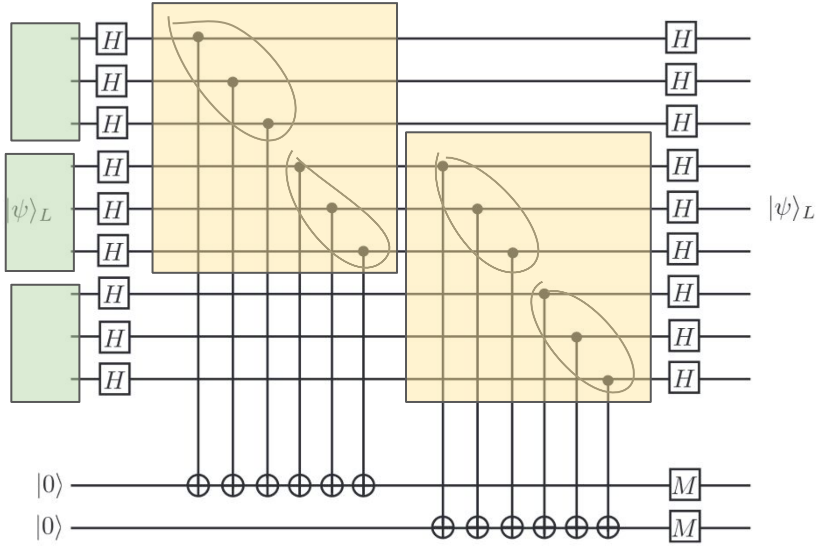


FIG. 5 Circuit required to perform Z-error correction for the 9-qubit code.

https://arxiv.org/pdf/0905.2794.pdf