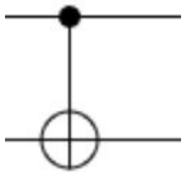
Practical Quantum Computing

Lecture 11
CHP, Surface Code

Using on *Fowler AG, Mariantoni M, Martinis JM, Cleland AN.* Surface codes: Towards practical large-scale quantum computation. Physical Review A. 2012 Sep 18;86(3):032324. and slides from Aaronson

Stabilizer Circuits

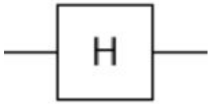
1. Controlled-NOT



|00〉□|00〉, |01〉□|01〉,
|10〉□|11〉, |11〉□|10〉

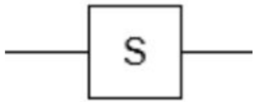
2. Hadamard

|0〉□(|0〉+|1〉)/√2
|1〉□ (|0〉-|1〉)/√2



3. Phase $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

|0〉□|0〉, |1〉□i|1〉



4. **Measurement** of a single qubit

Pauli Matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X^2=Y^2=Z^2=I$

$XY=iZ \quad YZ=iX \quad ZX=iY$

$XZ=-iY \quad ZY=-iX \quad YX=-iZ$

Unitary matrix U **stabilizes** a quantum state $|\psi\rangle$ if $U|\psi\rangle = |\psi\rangle$. Stabilizers of $|\psi\rangle$ form a group

X stabilizes $|0\rangle+|1\rangle$

Y stabilizes $|0\rangle+i|1\rangle$

Z stabilizes $|0\rangle$

-X stabilizes $|0\rangle-|1\rangle$

-Y stabilizes $|0\rangle-i|1\rangle$

-Z stabilizes $|1\rangle$

Gottesman-Knill Theorem

If |ψ⟩can be produced from the all |0> state of length n by just

- CNOT
- Hadamard
- and phase gates

then |ψ> is stabilized by 2^n tensor products:

- of Pauli matrices
- or their opposites
- where n = number of qubits

The **stabilizer** group is generated by $\log(2^n)$ = **n** such tensor products
Indeed, |ψ⟩is then **uniquely** determined by these generators
We call |ψ⟩a **stabilizer state**

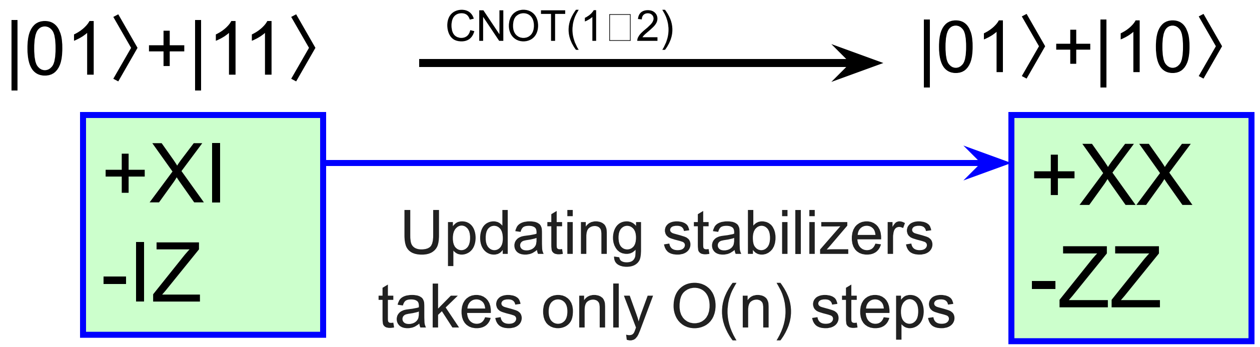
CHP Simulator

Goal: Using a classical computer, simulate an n-qubit CNOT/Hadamard/Phase computer.

Gottesman & Knill's solution: Keep track of **n** generators of the stabilizer group
Each generator uses 2n+1 bits: 2 for each Pauli matrix and 1 for the sign.

So n(2n+1) bits total

Example:



But measurement takes $O(n^3)$ steps by Gaussian elimination

CHP Simulator

Idea: Instead of n(2n+1) bits, store 2n(2n+1) bits

- n stabilizers $S_1,\ldots,S_n$, 2n+1 bits each
- n "destabilizers" $D_1,\ldots,D_n$

Together generate full Pauli group

Maintain the following invariants:

- D_i's commute with each other
- S_i anticommutes with D_i
- S_i commutes with D_j for $i \neq j$

https://arxiv.org/pdf/quant-ph/0406196.pdf

Example

I: x_{ij}=0, z_{ij}=0 + phase: r_i=0
X: x_{ij}=1, z_{ij}=0 - phase: r_i=1
Y: x_{ij}=1, z_{ij}=1
Z: x_{ij}=0, z_{ij}=1

State: $|00\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	1	0	0	0	0	+XI
	D_2	0	1	0	0	0	+IX
Stabilizers	S_1	0	0	1	0	0	+ZI
	S_2	0	0	0	1	0	+IZ

Example

Hadamard on qubit a:

For all $i \in \{1,\ldots,2n\}$, swap x_{ia} with z_{ia}, and set $r_i := r_i \oplus x_{ia} z_{ia}$

State: $|00\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	1	0	0	0	0	+XI
	D_2	0	1	0	0	0	+IX
Stabilizers	S_1	0	0	1	0	0	+ZI
	S_2	0	0	0	1	0	+IZ

Example

Hadamard on qubit a:

For all $i \in \{1,\ldots,2n\}$, swap x_{ia} with z_{ia}, and set $r_i := r_i \oplus x_{ia} z_{ia}$

State: $|00\rangle + |10\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	0	0	1	0	0	+ZI
	D_2	0	1	0	0	0	+IX
Stabilizers	S_1	1	0	0	0	0	+XI
	S_2	0	0	0	1	0	+IZ

Example

CNOT from qubit a to qubit b:

For all $i \in \{1,\ldots,2n\}$, set $x_{ib} := x_{ib} \oplus x_{ia}$ and $z_{ia} := z_{ia} \oplus z_{ib}$

State: $|00\rangle + |10\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	0	0	1	0	0	+ZI
	D_2	0	1	0	0	0	+IX
Stabilizers	S_1	1	0	0	0	0	+XI
	S_2	0	0	0	1	0	+IZ

Example

CNOT from qubit a to qubit b:

For all $i \in \{1,\ldots,2n\}$, set $x_{ib} := x_{ib} \oplus x_{ia}$ and $z_{ia} := z_{ia} \oplus z_{ib}$

State: $|00\rangle + |11\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	0	0	1	0	0	+ZI
	D_2	0	1	0	0	0	+IX
Stabilizers	S_1	1	1	0	0	0	+XX
	S_2	0	0	1	1	0	+ZZ

Example

Phase on qubit a:

For all $i \in \{1,\ldots,2n\}$, set $r_i := r_i \oplus x_{ia} z_{ia}$, then set $z_{ia} := z_{ia} \oplus x_{ia}$

State: $|00\rangle + |11\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	0	0	1	0	0	+ZI
	D_2	0	1	0	0	0	+IX
Stabilizers	S_1	1	1	0	0	0	+XX
	S_2	0	0	1	1	0	+ZZ

Example

Phase on qubit a:

For all $i \in \{1,\ldots,2n\}$, set $r_i := r_i \oplus x_{ia} z_{ia}$, then set $z_{ia} := z_{ia} \oplus x_{ia}$

State: $|00\rangle + i|11\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	0	0	1	0	0	+ZI
	D_2	0	1	0	1	0	+IY
Stabilizers	S_1	1	1	0	1	0	+XY
	S_2	0	0	1	1	0	+ZZ

Example

Measurement of qubit a:

If x_{ia}=0 for all i∈{n+1,…,2n}, then outcome will be deterministic.
Otherwise 0 with ½ probability and 1 with ½ probability.

State: $|00\rangle + i|11\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	0	0	1	0	0	+ZI
	D_2	0	1	0	1	0	+IY
Stabilizers	S_1	1	1	0	1	0	+XY
	S_2	0	0	1	1	0	+ZZ

Example

Random outcome:

Pick a stabilizer S_i such that $x_{ia}=1$ and set $D_i:=S_i$. Then set $S_i:=Z_a$ and output 0 with ½ probability, and set $S_i:=-Z_a$ and output with ½ probability, where Z_a is Z on a^{th} qubit and I elsewhere. Finally, left-multiply whatever rows don't commute with S_i by D_{ixs}

State: $|11\rangle$

		x_{ij} bits		z_{ij} bits		r_i bits	
Destabilizers	D_1	1	1	0	1	0	+XY
	D_2	0	1	0	1	0	+IY
Stabilizers	S_1	0	0	1	0	1	- ZI
	S_2	0	0	1	1	0	+ZZ

Using destabilizers for $O(n^2)$ measurement

Obtain deterministic measurement outcomes in only $O(n^2)$ steps, without using Gaussian elimination

Z_a (Z on qubit *a* and I everywhere else) commutes with the stabilizers

$$\sum_{h=1}^{n} c_h R_{h+n} = \pm Z_a$$

and Z_a is a linear combination for a unique choice of $c_1,\ldots,c_n \in \{0,1\}$

Determine c_i's, then by summing corresponding S_h's we learn sign of Z_a

$$c_i \equiv \sum_{h=1}^{n} c_h \left(R_i \cdot R_{h+n}\right) \equiv R_i \cdot \sum_{h=1}^{n} c_h R_{h+n} \equiv R_i \cdot Z_a \pmod 2$$

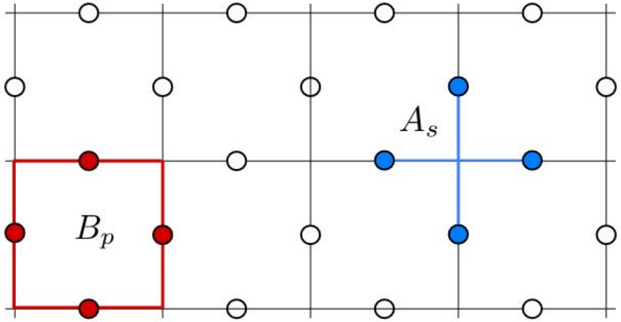
So just have to check **if D_i commutes with Z_a**, or equivalently **iff** $x_{ia}=1$

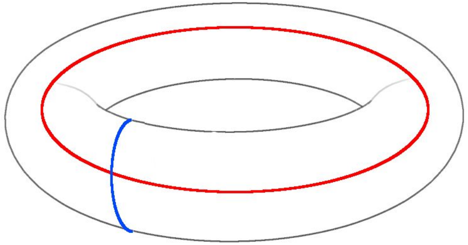
Surface Code

The surface codes

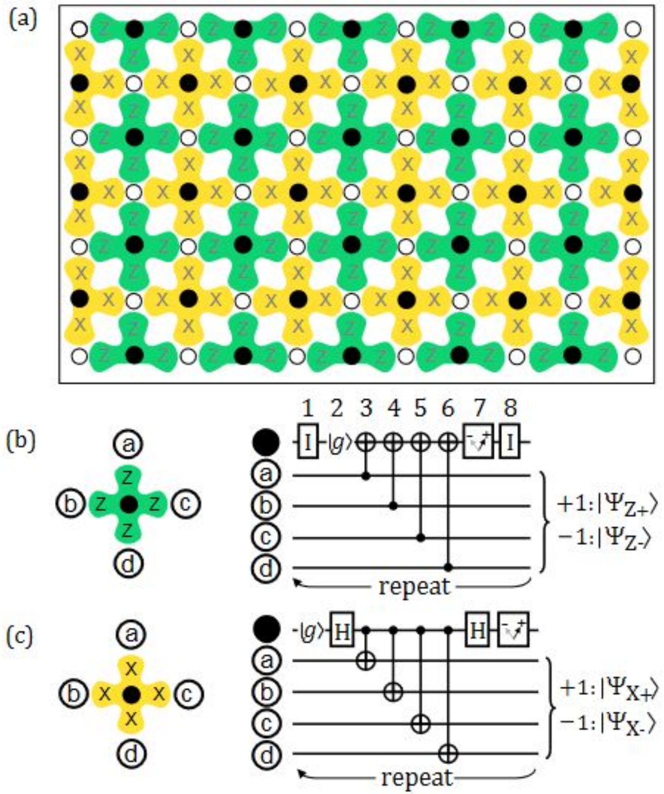
- evolved from an invention of Alexei Kitaev
- known as toric codes
- efforts to develop simple models for topological order
- using qubits distributed on the surface of a toroid

The toroidal geometry employed by Kitaev turned out to be unnecessary, and planar versions (thus “surface codes”) were developed



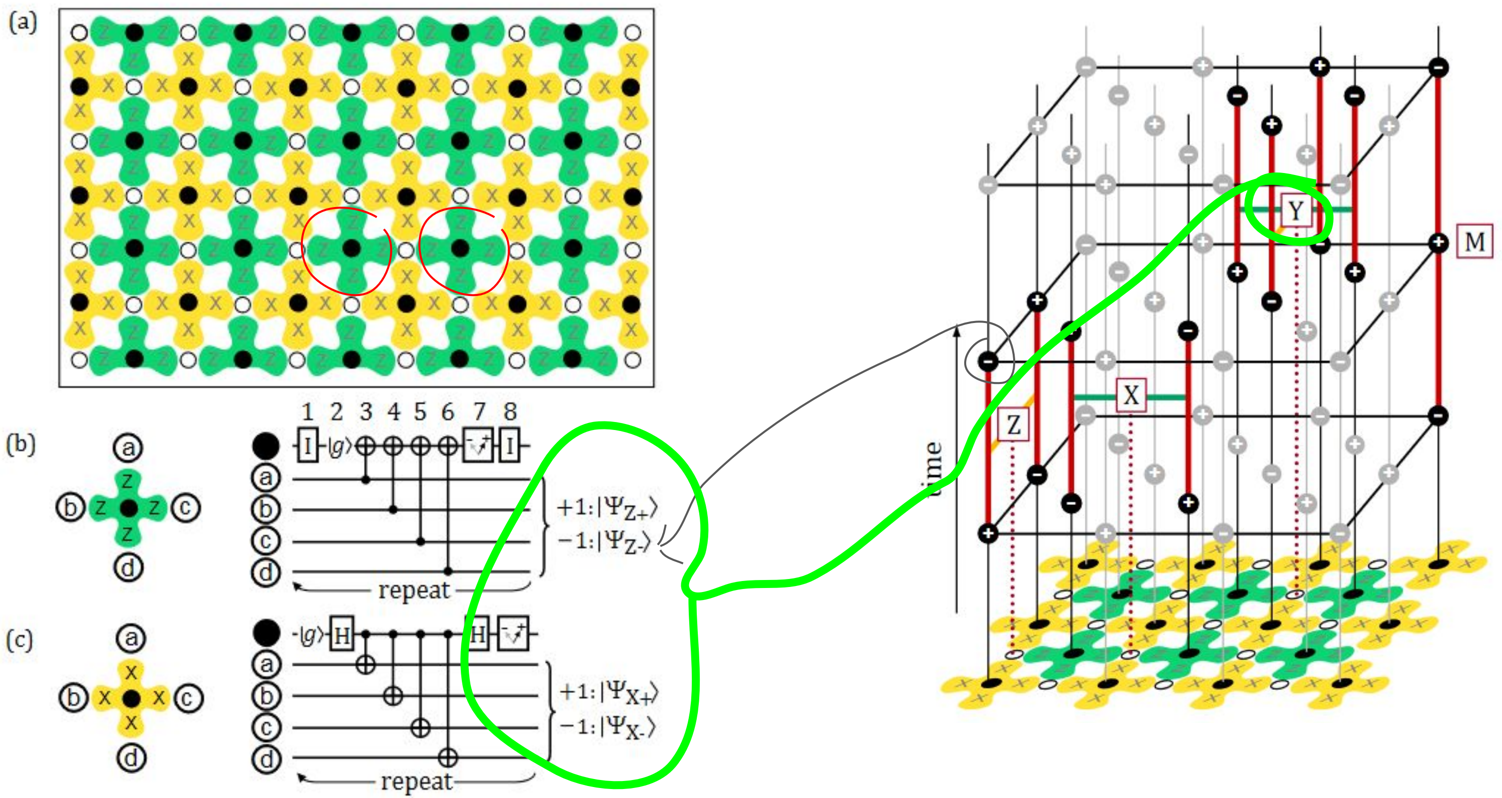


Surface Code



Eigenvalue	$\hat{Z}_a\hat{Z}_b\hat{Z}_c\hat{Z}_d$	$\hat{X}_a\hat{X}_b\hat{X}_c\hat{X}_d$
$+1$	$\lvert gggg\rangle$	$\lvert ++++\rangle$
	$\lvert ggee\rangle$	$\lvert ++--\rangle$
	$\lvert geeg\rangle$	$\lvert +--+\rangle$
	$\lvert eegg\rangle$	$\lvert --++\rangle$
	$\lvert egge\rangle$	$\lvert -++-\rangle$
	$\lvert gege\rangle$	$\lvert +-+-\rangle$
	$\lvert egeg\rangle$	$\lvert -+-+\rangle$
	$\lvert eeee\rangle$	$\lvert ----\rangle$
-1	$\lvert ggge\rangle$	$\lvert +++-\rangle$
	$\lvert ggeg\rangle$	$\lvert ++-+\rangle$
	$\lvert gegg\rangle$	$\lvert +-++\rangle$
	$\lvert eggg\rangle$	$\lvert -+++\rangle$
	$\lvert geee\rangle$	$\lvert +---\rangle$
	$\lvert egee\rangle$	$\lvert -+--\rangle$
	$\lvert eege\rangle$	$\lvert --+-\rangle$
	$\lvert eeeg\rangle$	$\lvert ---+\rangle$

Surface Code



Stabilizer measurement cycles

In the absence of errors, the same state is maintained by each subsequent cycle of the sequence, with each measure qubit yielding a measurement outcome Xabcd or Zabcd equal to that of the previous cycle:

- because all X and Z stabilizers commute with one another
- trivial for stabilizers that do not have any qubits in common
- X and Z operators on different qubits always commute.

Stabilizers that have qubits in common will **always share two such qubits**, For an X and Z stabilizer that measure data qubits a and b in common

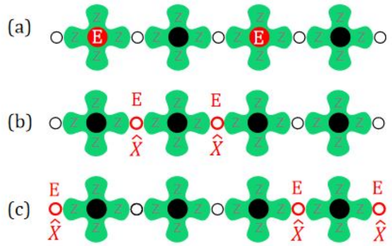


FIG. 5. (Color online) (a) An example where two measure-Z qubits report errors in a single row of a 2D array, marked by "E"s. This error report could be generated by (b) two $\hat{X}$ errors appearing in the same surface code cycle on the 2nd and 3rd data qubit from the left, or (c) three $\hat{X}$ errors appearing in the other three data qubits in the row.

$$\begin{aligned}
\hat{X}^2 &= -\hat{Y}^2 = \hat{Z}^2 = \hat{I}, \\
\hat{X}\hat{Z} &= -\hat{Z}\hat{X}, \\
\left[\hat{X},\hat{Y}\right] &\equiv \hat{X}\hat{Y} - \hat{Y}\hat{X} = -2\hat{Z},
\end{aligned}$$

$$\begin{aligned}
[\hat{X}_a\hat{X}_b\hat{X}_c\hat{X}_d \ , \ \hat{Z}_a\hat{Z}_b\hat{Z}_e\hat{Z}_f] \\
&= (\hat{X}_a\hat{Z}_a)(\hat{X}_b\hat{Z}_b)\hat{X}_c\hat{X}_d\hat{Z}_e\hat{Z}_f \\
&- (\hat{Z}_a\hat{X}_a)(\hat{Z}_b\hat{X}_b)\hat{X}_c\hat{X}_d\hat{Z}_e\hat{Z}_f \\
&= 0,
\end{aligned}$$

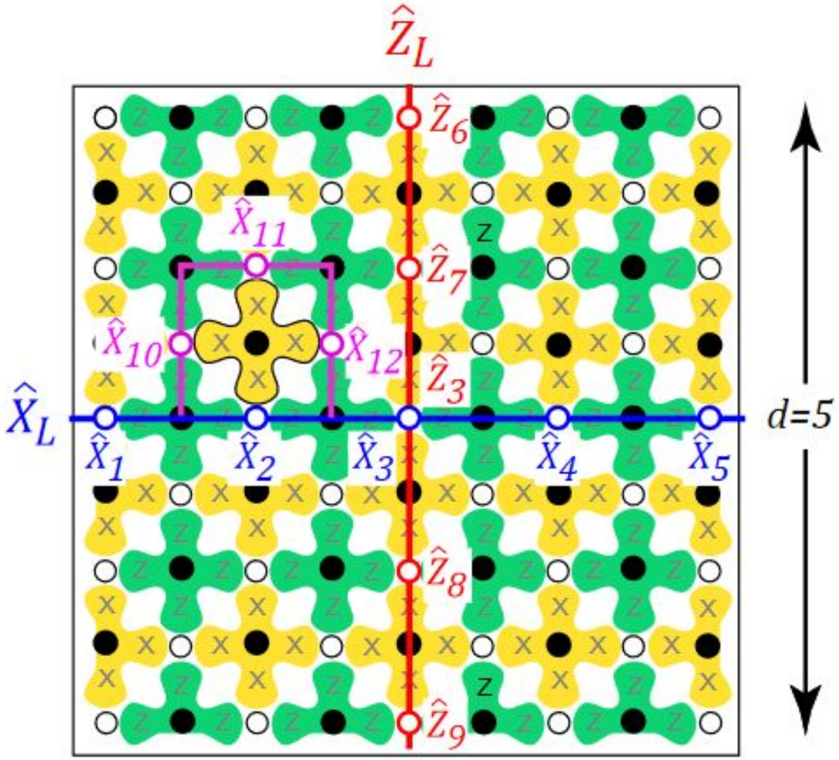
Logical Operators

Any two-level quantum system that satisfies the relations

$$\begin{aligned}\hat{X}^2 &= -\hat{Y}^2 = \hat{Z}^2 = \hat{I},\\ \hat{X}\hat{Z} &= -\hat{Z}\hat{X},\\ \left[\hat{X},\hat{Y}\right] &\equiv \hat{X}\hat{Y} - \hat{Y}\hat{X} = -2\hat{Z},\end{aligned}$$

can in principle be used as a qubit.

Any system in which one can define X and Z operators that satisfy the relations can be used as a qubit, even if the system has more than two degrees of freedom



Suppression of Errors - Decoding