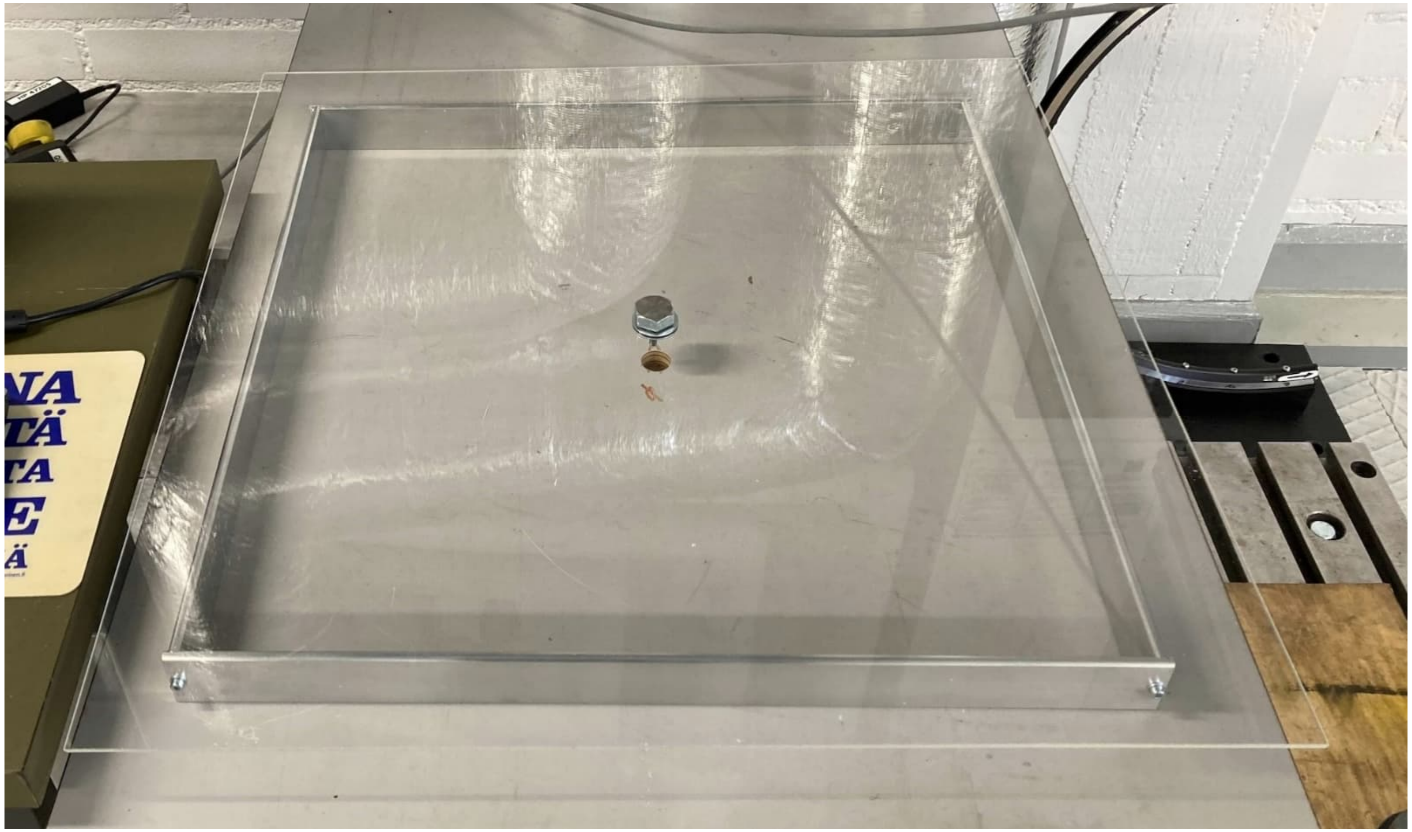
MEC-E1005

MODELLING IN APPLIED MECHANICS 2023

WEEKS 22: RECTANGULAR PLATE RIGIDITY

Thu 12:15-14:00 Non-linear analysis and design formula (JF)

ACRYLIC PLATE



ASSIGNMENT

According to linear theory, rigidity of a simply supported plate is constant whose value depends on the plate thickness, size of the plate, and the plate material. Experiments indicate, however, that rigidity increases rapidly in the transverse displacement. The actual boundary conditions at the support may also affect the setting. For example, in rectangle geometry, the contact between the plate and support may be lost at the corner regions (if the displacement at the support is constrained only downwards like in the figure).

In the modelling assignment, you will study the effects of geometrical and material parameters, and displacement on rigidity of a rectangular plate on a rectangular support. The starting point is a generic expression predicted by dimension analysis. First, a simplified linear model is used for a more specific relationship. After that, analysis by FEM is used for a more precise picture. The final outcome is a design formula for rigidity. The modelled rigidities are compared with that given by an experiment.

IDEALIZATION AND PARAMETERIZATION

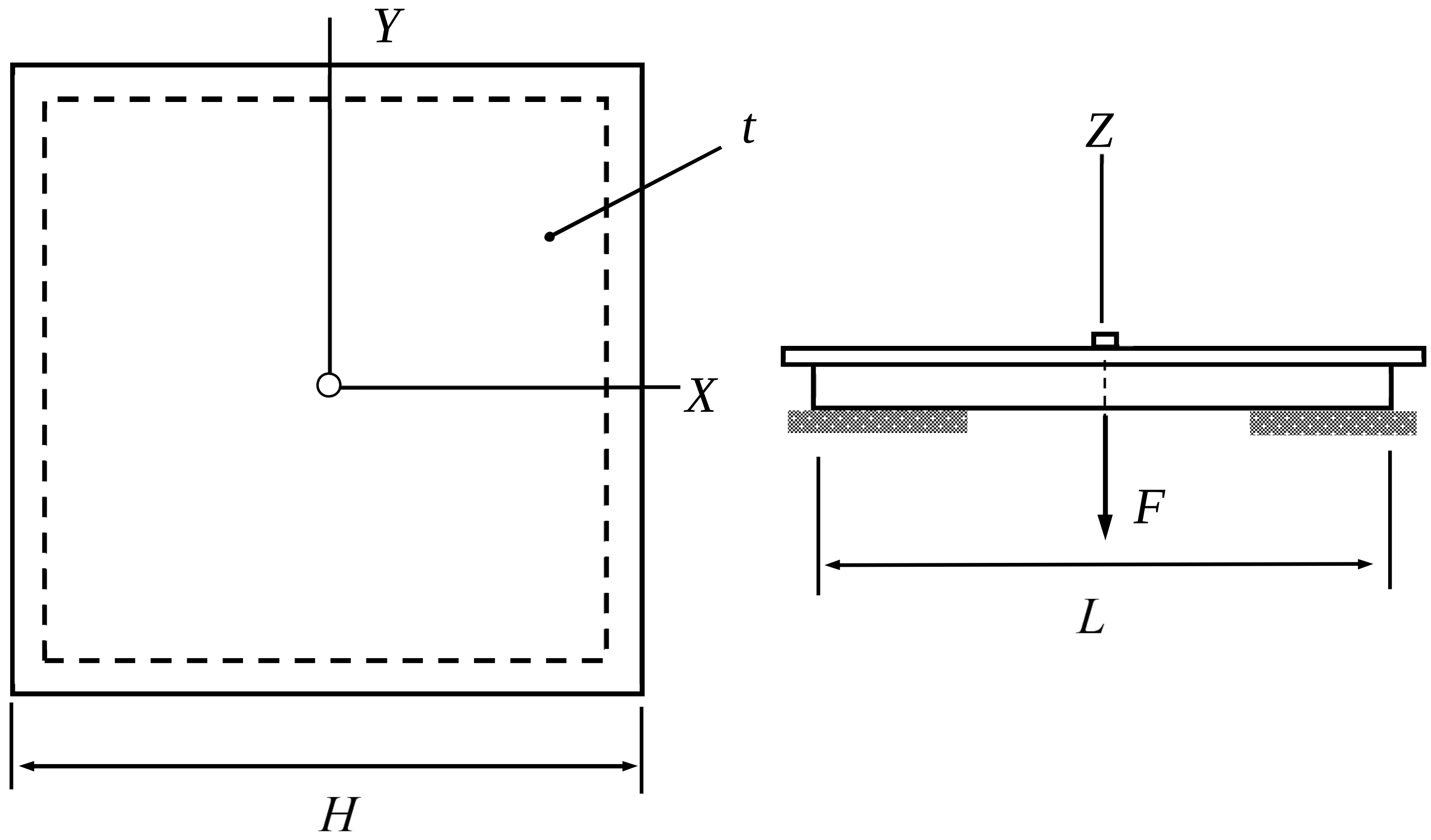


Table 1. Geometrical and material parameters.

Parameter	symbol	value
Plate size	H	0.6 m
Support size	L	0.5 m
Thickness	t	3 mm
Young's modulus	E	(2.7-3.5) GPa
Poisson's ratio	ν	3.70

SOURCES OF NON-LINEARITY

- **Geometry:** Equilibrium equations should be satisfied in deformed geometry depending on displacement. Strain measures of large displacements are always non-linear.

- **Material:** Constitutive equation $g(\sigma,u)=0$ may be non-linear. Near reference geometry, truncated Taylor series $g^\circ+(\partial g/\partial\sigma)^\circ\Delta\sigma+(\partial g/\partial u)^\circ\Delta u=0$ gives a useful approximation.

- **External forces:** External forces may be non-linear. A typical example is a follower force, whose direction depends on structure displacement.

- **Boundary conditions:** Even the simplest contact conditions are non-linear. The one-sided conditions in terms of inequalities are simple but rather tricky in a numerical solution method.

DIMENSION ANALYSIS

Assuming that the quantities related with the setting are E, ν, H, L, t, w, and F, dimension analysis implies the relationship (number of dimensionless groups being 7-2=5)

$$\frac{FL^2}{Et^4} = \alpha(\frac{w}{t}, \frac{H}{L}, \frac{t}{L}, \nu). \tag{1}$$

The dimensionless groups are based on plate theory. The expression on the right hand requires a more detailed analysis or additional assumptions. For example, assuming that the displacement vanishes without loading, and that the relationship is the same with the opposite force direction, the truncated Taylor expansion with respect to the first argument gives

$$\alpha(\frac{w}{t}, \frac{H}{L}, \frac{t}{L}, \nu) = \alpha_1(\frac{H}{L}, \frac{t}{L}, \nu)\frac{w}{t} + \alpha_3(\frac{H}{L}, \frac{t}{L}, \nu)(\frac{w}{t})^3.$$

If the force-displacement relationship is known to be linear, the second term can be omitted. In terms of dimensionless groups, the design formula to be considered (without or with the second terms on the right -hand side) will be of the form

$$\pi_1 = \pi_2 \alpha_1(\pi_3, \pi_4, \nu) + \pi_2^3 \alpha_3(\pi_3, \pi_4, \nu), \text{ where } \pi_1 = \frac{FL^2}{Et^4},\ \pi_2 = \frac{w}{t},\ \pi_3 = \frac{H}{L},\ \pi_4 = \frac{t}{L}. \quad (2)$$

Dependencies of α_1 and α_3 on π_3, π_4, and ν need to be found by an analytical model, experiments, or by simulation experiments using FEM (or by using all those methods).

EXPERIMENT

The set-up is located in Puumiehenkuja 5L (Konemiehentie side of the building). The hall is open for the experiment during the office hours (09:00-16:00) from Thu of week 21 to Thu of week 22.

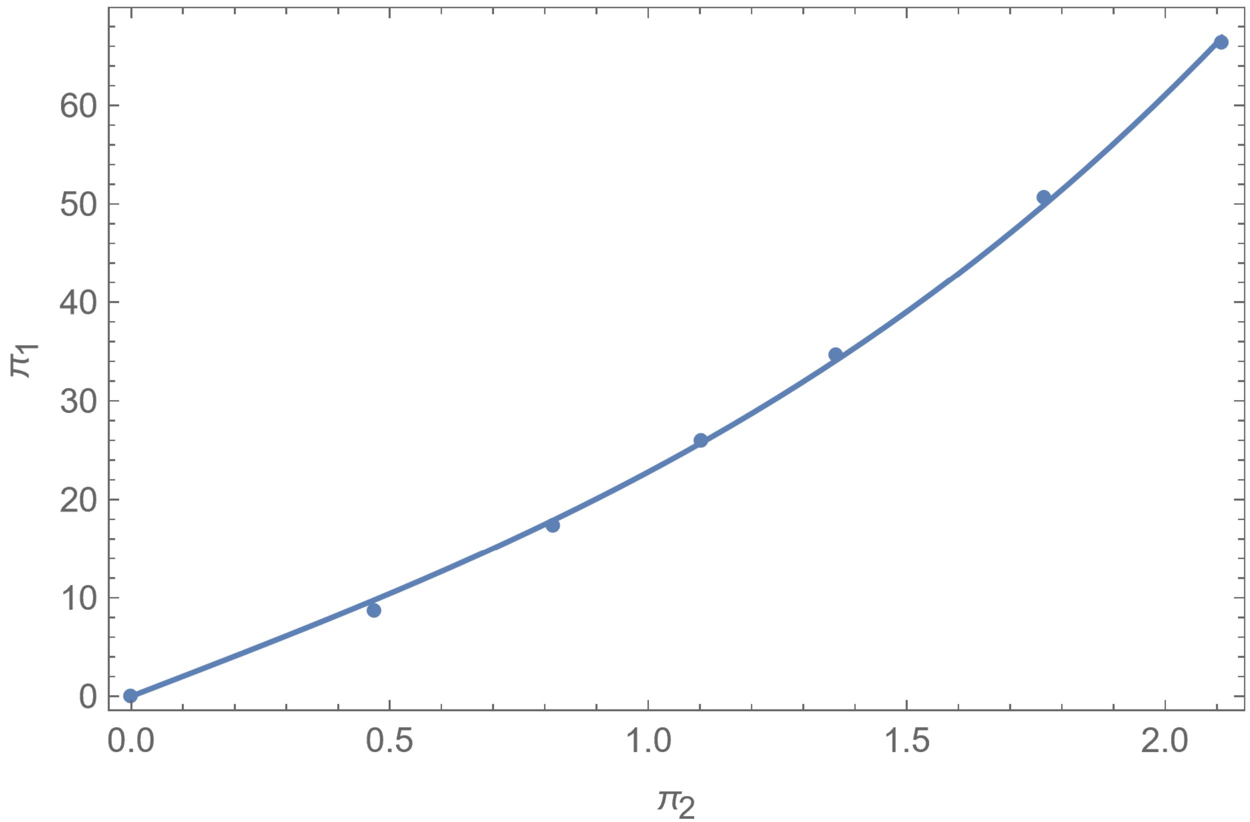
Place a mass on the loading tray and record the displacement shown on the laptop display. Disk material is not purely elastic so wait for the displacement reading to settle (almost). Gather enough mass-displacement data to find the displacement-force relationship reliably. For example, you may repeat a measurement with certain loading several times to reduce the effect of random error by averaging etc. You may also consider different loading sequences (like increasing and decreasing the mass) to minimize the effect of the viscous part of material response.

Table 2. Measured mass-displacement values

Mass [kg]	**Displacement** [mm]
0.0	0.00
1.0	1.41
2.0	2.45
3.0	3.31
4.0	4.09
5.8	5.30
7.7	6.33

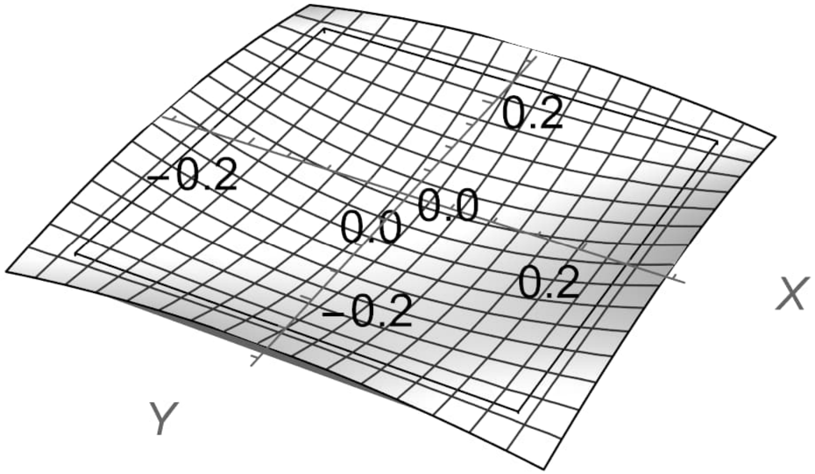
LEAST-SQUARES FIT

Using the least squares fit with the form predicted by dimension analysis gives $\alpha_0(\pi_3,\pi_4,\nu)=20.21$, $\alpha_3(\pi_3,\pi_4,\nu)=2.58$ when $\pi_3=1.2$, $\pi_4=0.006$ and $\nu=0.37$.

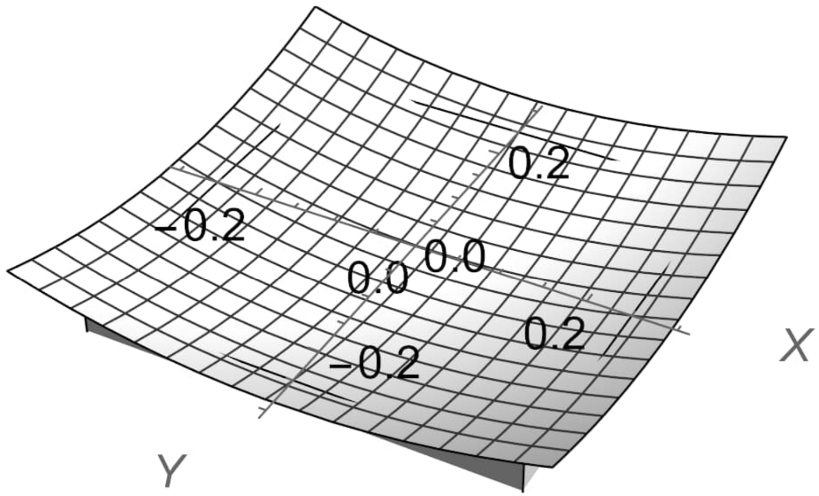


SIMPLIFIED ANALYSIS

Simplified small displacement analysis can be based on the double-sine series solution to a rectangular simply supported plate. One may also use the virtual work density of Kirchhoff plate and, e.g., a one-parameter displacement approximation (MEC-E1050, MEC-E8001, MEC-E8003).



Simply supported



Contact

Let us assume small displacements, contact only at the midpoints of the sides of the rectangular support, and the one-parameter displacement expression (w_F is the displacement at the loading/center point)

$$w(x,y) = w_F\,(1-4\frac{x^2+y^2}{L^2})\,.$$

The one-parameter approximation is clearly too simplistic for a precise force-displacement relationship but may be used to verify the form given by dimension analysis. Using the virtual work expression for the Kirchhoff plate model in bending with the approximation above

$$\frac{FL^2}{t^4E} = \frac{w_F}{t}\frac{32}{3}(\frac{H}{L})^2\frac{1}{1-\nu} \quad \text{so} \quad \alpha_1(\pi_3,\pi_4,\nu) = \frac{32}{3}\pi_3^2\frac{1}{1-\nu}$$

which is consistent (in the form) with the outcome of dimension analysis. Finding the expression of α_3 requires large displacement analysis.

Virtual work expression for small displacement analysis (just the bending mode contribution) is given by

$$\delta W = -\int_{\Omega} \delta \frac{1}{2}(\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{Bmatrix}) dA + \delta w(x_F, y_F) F,$$

where the curvature components and elasticity matrix are defined by

$$\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} \text{ and } [E]_{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}.$$

When the approximation is substituted there, virtual work expression simplifies to

$$\delta W = \delta w_F \left(F - \frac{32H^2t^3E}{3L^4(1-\nu)} w_F\right).$$

Then, principle of virtual work $\delta W = 0 \quad \forall \delta w_F$ and the fundamental lemma of variation calculus implies that

$$F - \frac{32H^2t^3E}{3L^4(1-\nu)} w_F = 0.$$

LARGE DISPLACEMENT ANALYSIS (MEC-E8001)

Virtual work expression for the large displacement analysis is given by

$$\delta W=-\int_{\Omega^\circ}\delta\frac{1}{2}\left(\begin{Bmatrix}\mathrm{E}_{xx}\\ \mathrm{E}_{yy}\\ 2\mathrm{E}_{xy}\end{Bmatrix}^{\mathrm{T}}t[E]_\sigma\begin{Bmatrix}\mathrm{E}_{xx}\\ \mathrm{E}_{yy}\\ 2\mathrm{E}_{xy}\end{Bmatrix}+\begin{Bmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{Bmatrix}\right)dA+\delta w(x_F,y_F)F\,,$$

where the Green-Lagrange strains and curvature components simplified for moderate displacements are

$$\begin{Bmatrix}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix}\text{ and }\begin{Bmatrix}\mathrm{E}_{xx}\\ \mathrm{E}_{yy}\\ 2\mathrm{E}_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix}+\frac{1}{2}\begin{Bmatrix}(\partial w/\partial x)(\partial w/\partial x)\\ (\partial w/\partial y)(\partial w/\partial y)\\ 2(\partial w/\partial x)(\partial w/\partial y)\end{Bmatrix}.$$

Transverse displacement is chosen to coincide with the one use in linear analysis and the in-plane components are assumed to vanish (actually, they do not vanish). When the approximation is substituted there, virtual work expression simplifies to (Mathematica is useful here)

$$\delta W = \delta w_F F - \delta w_F \frac{8H^2 tEw_F}{15L^8(1-\nu^2)}[5L^4H^2(1-\nu)+20L^4t^2(1+\nu)+96H^4(1+\nu)w_F^2].$$

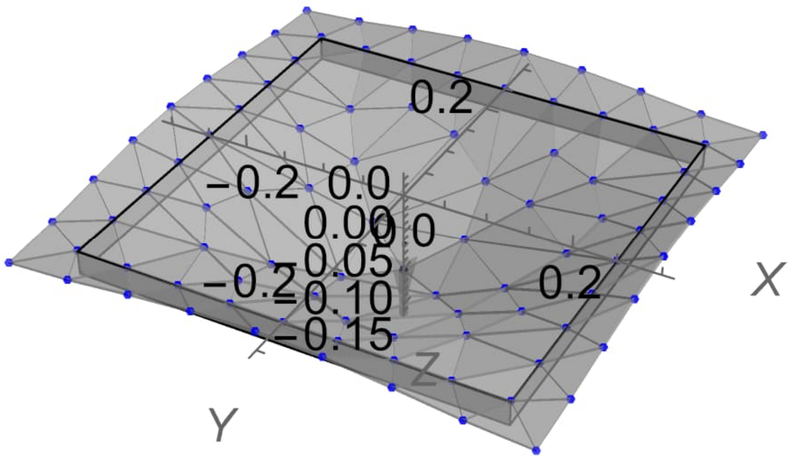
Then, principle of virtual work $\delta W = 0 \quad \forall \delta w_F$ and the fundamental lemma of variation calculus implies that in terms of the dimensionless groups

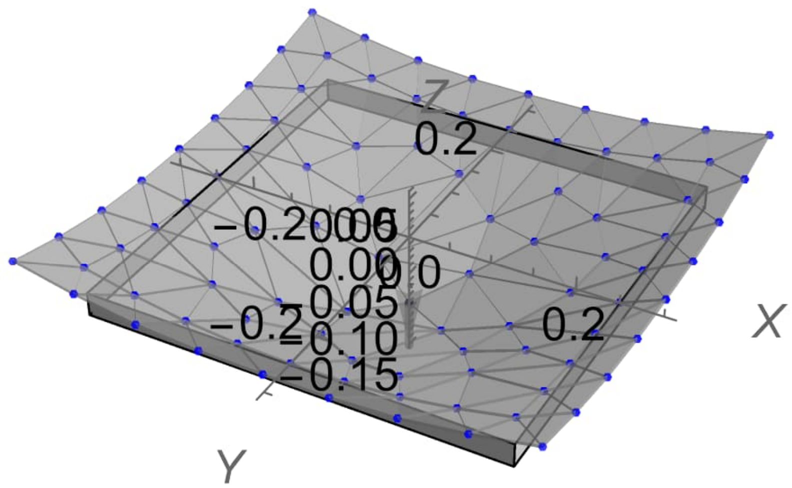
$$\pi_1 = [\frac{1}{3}\pi_3^2 \frac{32}{(1-\nu)} + \frac{1}{3}\frac{8}{(1+\nu)}\frac{\pi_3^3}{\pi_4}]\pi_2 + [24\frac{32}{15}\frac{\pi_3^6}{(1-\nu)}]\pi_2^3.$$

Expression is of the same form as that predicted by dimension analysis.

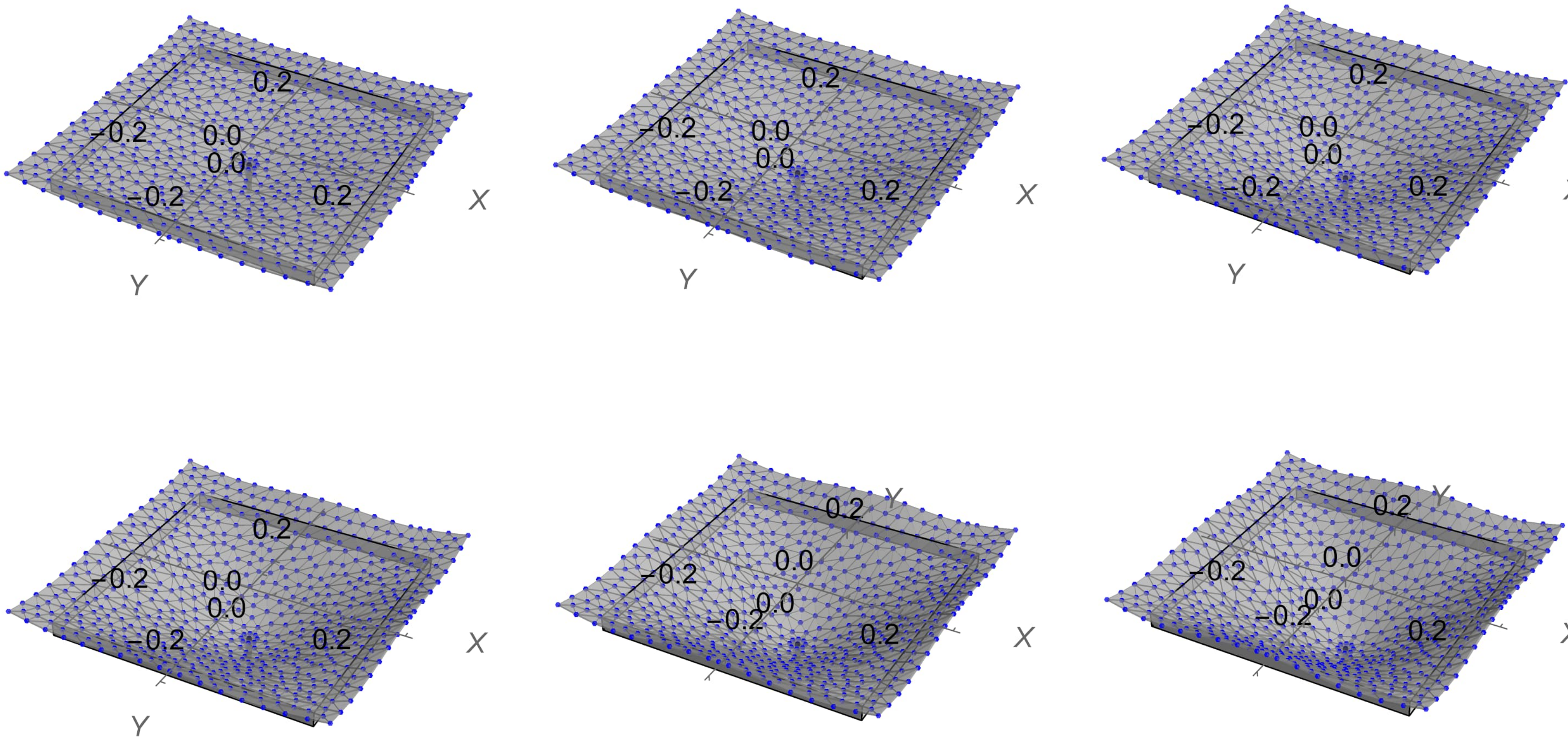
FINITE ELEMENT ANALYSIS

Analysis by the finite element method and solid or plate/shell elements gives the displacement without (too many) simplifying assumptions. Numerical method requires numerical values for all the problem parameters, but one may consider the effects of the washer at the loading point, non-linearity due to large displacement and one-sided boundary condition at the support etc.



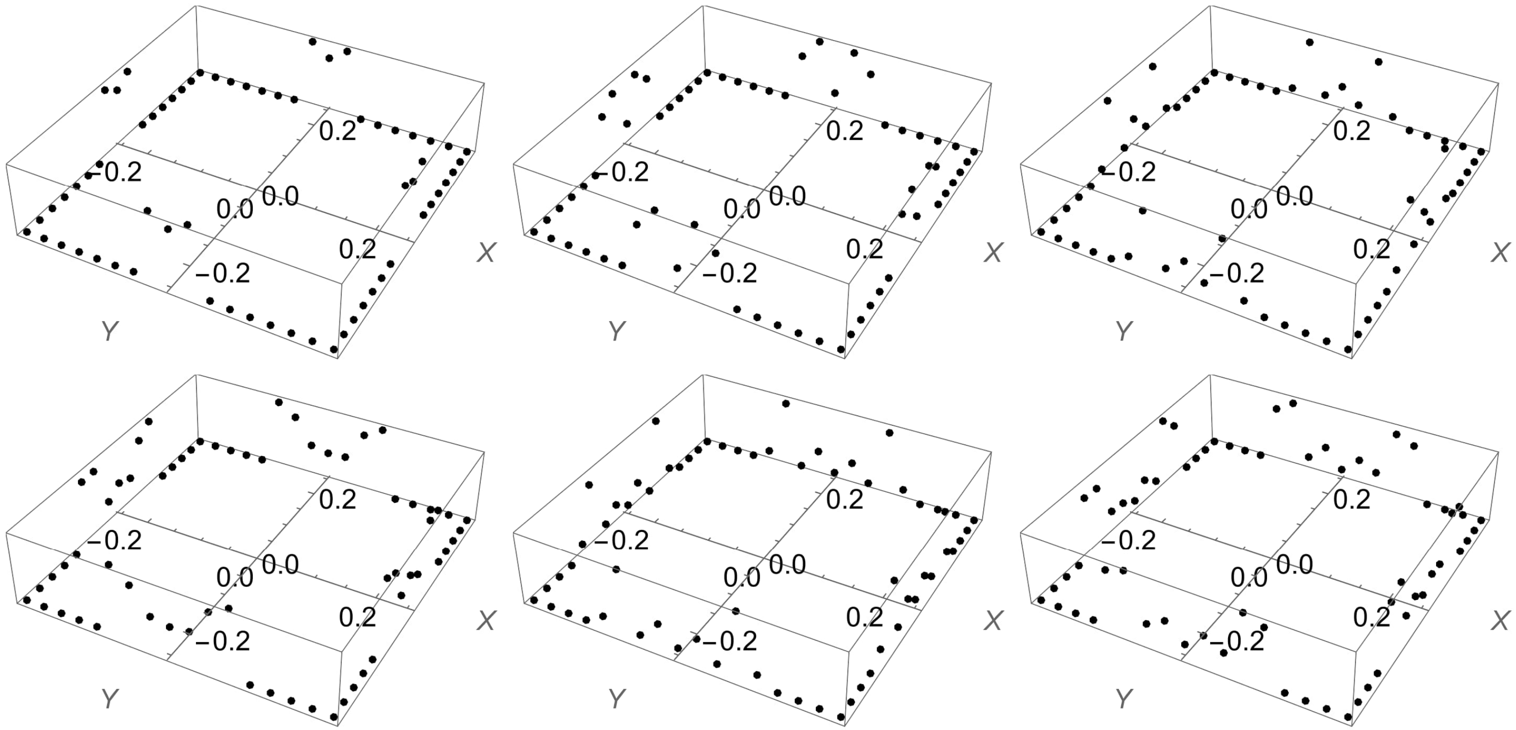


LARGE DISPLACEMENT CONTACT PROBLEM



Displacements when load $m \in \{1, 2, 3, 4, 5.8, 7.7\}$. The model considers the washer, contact at the support, and large displacements.

ONE-SIDED BOUNDARY CONDITIONS



At the nodes on the support $u_Z \leq 0$, $F_Z \geq 0$ (above), and $u_Z F_Z = 0$. The contact region increases in load (above from left to right and from first to the second row). The sum of the contact forces equals to the load.

In the simulation experiment, one decides the reasonable range of values for the parameters and uses FEM to find the right-hand side of equation (2)

$$\pi_1 = \alpha_1(\pi_3, \pi_4, \nu)\pi_2 + \alpha_3(\pi_3, \pi_4, \nu)\pi_2^3 \quad \text{where} \quad \pi_1 = \frac{FL^2}{Et^4},\ \pi_2 = \frac{w_F}{t},\ \pi_3 = \frac{H}{L},\ \pi_4 = \frac{t}{L}$$

for each parameter combination and thereby tabulated form of the expression. After that, one may try to compress the data with, e.g., a polynomial fit.

Table 3. Effects of π_3 (columns) and ν (rows) on α_1 when $\pi_4 = 0.006$.

	1.1	1.3	1.5	1.7	1.9
0.15	8.70919	9.38605	10.1054	10.6835	10.9413
0.3	9.28692	9.99052	10.7074	11.2786	11.5343
0.45	10.3497	11.154	11.881	12.4728	12.737

Table 4. Effects of π_3 (columns) and ν (rows) on α_3 when $\pi_4 = 0.006$.

	1.1	1.3	1.5	1.7	1.9
0.15	1.62887	2.32433	2.74466	2.94212	3.12889
0.3	1.63837	2.29055	2.71817	2.92288	3.12171
0.45	1.67452	2.28098	2.72797	2.94493	3.16397

One may use the design formula composed of

$$F = \frac{Et^4}{L^2}[\alpha_1 \frac{w}{t} + \alpha_3 (\frac{w}{t})^3]$$

and tabulated data for α_1 and α_3 to predict the displacement for a given load, load for a given displacement, thickness for a given load and displacement etc.

For example, assuming the displacement $w_F = 3\,\text{mm}$, the dimensionless groups for the plate of set-up are $\pi_2 = 1$, $\pi_3 = 1.2$, $\pi_4 = 0.006$ and $\nu = 0.37$. Tables 3 and 4 give $\alpha_1 \approx 10.2$ and $\alpha_3 \approx 1.97$ so

$$\frac{mgL^2}{Et^4} = \pi_1 = 10.1 \times 1 + 1.97 \times 1^3 = 12.1 \quad \Rightarrow \quad m = 12.1 \frac{Et^4}{gL^2} \approx 1.4\,\text{kg}.$$

Direct simulation with the model and load $m = 1.4\,\text{kg}$ gives $w_F = 2.9\,\text{mm}$.

DISPLACEMENTS

m[kg]	w_{exp}[mm]	w_{des}[mm]	w_{fem}[mm]
0.0	0.00	0.00	0.00
1.0	1.41	2.30	2.26
2.0	2.45	3.87	3.61
3.0	3.31	4.99	4.58
4.0	4.09	5.86	5.36
5.8	5.30	7.12	6.51
7.7	6.33	8.10	7.40